Internat. J. Math. & Math. Sci.
VOL. 12 NO. 3 (1989) 579-582

FOURIER SERIES AND THE MAXIMAL OPERATOR ON THE WEIGHTED SPECIAL ATOM SPACES

GERALDO SOARES DE SOUZA

Department of Mathematics
Auburn University
Auburn University, AL 36849-5310

(Received November 24, 1987 and in revised form July 5, 1988)

ABSTRACT. For an interval I in $[0,2\pi]$ with halves L and R, a weighted special atom looks like $b(t) = \frac{1}{\rho(|I|)}[\chi_L(t) - \chi_R(t)]$, where ρ is a non-negative function satisfying some properties.

We consider the weighted special atom space $B(\rho)$ formed by ℓ^1 linear combinations of these weighted atoms.

We showed that if $f \in B(\rho)$ then its Fourier series converges almost everywhere, using the Carleson-Hunt idea on their famous result about the almost everywhere convergence on L_p-spaces.

KEY WORDS AND PHRASES. Maximal operator, Lorentz Spaces and Fourier Series.
1980 AMS SUBJECT CLASSIFICATION CODE. Primary 42A20.

1. INTRODUCTION.

In [2] the following space was introduced.
$B(\rho) = \{f: [0,2\pi] \to R,\ f(t) = \sum_{n=0}^{\infty} c_n b_n(t),\ \sum_{n=0}^{\infty} |c_n| < \infty\}$. Each b_n is a weighted special atom, that is, a real valued function b, defined on $[0,2\pi]$, which is either $b(t) = \frac{1}{2\pi}$ or $b(t) = \frac{1}{\rho(|I|)}[\chi_R(t) - \chi_L(t)]$, where I is an interval in $[0,2\pi]$, L is the left half of I and R is the right half. $|I|$ denotes the length of I, χ_E the characteristic function of E and ρ is a non-negative, real valued function which is increasing, and $\rho(0) = 0$. $B(\rho)$ is endowed with the norm $\|f\|_{B(\rho)} = \text{Inf} \sum_{n=0}^{\infty} |c_n|$, where the infimum is taken over all possible representations of f. $B(\rho)$ is a Banach space. For more details about this space $B(\rho)$ the reader is referred to [2].

The Carleson-Hunt theorem on the almost everywhere convergence of a function f in L_p for $1 < p < \infty$, asserts that if $f \in L_p$, then the Fourier series of f, denoted by $S(f,x)$, converges to f almost everywhere. (See [1] and [5].)

In this note, using basically the idea of Carleson-Hunt, we will prove that with some additional condition on ρ if $f \in B(\rho)$ then the operator

$Tf(x) = \sup_n |S_n(f,x)|$ where $S_n(f,x)$ is the nth partial sum of the Fourier series of f, is bounded into some Banach space $L(\phi)$.

We would like to point out that other direct proofs of the almost everywhere convergence for functions in $B(\rho)$ are also available. For example, for some ρ any function in $B(\rho)$ satisfies Dini's condition. However we prefer this one, because it is a consequence of the boundedness of the maximal operator and this boundedness could be useful in other contexts, for example in the interpolation of operators.

One of the most important features of the space $B(\rho)$ is that under some additional conditions on ρ, it can be identified with the space of analytic functions F on the disc $D = \{z \in C ; |z| < 1\}$ satisfying

$$\int_0^1 \int_0^{2\pi} |F'(re^{i\theta})| \frac{\rho(1-r)}{1-r} d\theta dr < \infty$$

where the dash means the derivative of F with respect to z. For this characterization see [2].

2. PRELIMINARIES.

DEFINITION 2.1. A function $\rho: [0,\infty) \to [0,\infty)$ is said to be in the class b_1 if it satisfies

i) $\rho(0) = 0$, ii) ρ is non-decreasing, iii) $\frac{\rho(t)}{t}$ is decreasing,

iv) $\int_0^h \frac{\rho(s)}{s} ds \le C\rho(h)$ C an absolute constant

v) $\int_h^{2\pi} \frac{\rho(t)}{t^2} dt \le C \frac{\rho(h)}{h}$ with C independent of h and ρ.

Example of functions in the class b_1 are $\rho(t) = t^\alpha$, $\rho(t) = t^\alpha \log \frac{2\pi e^{1/\alpha}}{t}$ for $0 < \alpha < 1$.

DEFINITION 2.2. We define the space $L(\phi)$ by $L(\phi) = \{f:[0,2\pi] \to R$, $\|f\|_\phi = \int_0^{2\pi} f^*(t)\phi(t)dt < \infty\}$, where f^* is the decreasing rearrangement of f, defined by $f^*(t) = \text{Inf}\{y: \text{meas } \{x; |f(x)| > y\} \le t\}$, where meas means Lebesgue measure and ϕ is a non-negative, decreasing function.

In [6], G. G. Lorentz showed that $\| \ \|_\phi$ is a norm if and only if ϕ is a non-negative, decreasing function. Also he proved that $L(\phi)$ is a Banach space.

Recall that if $S_n(f,x)$ is the nth partial sum of the Fourier series of f and $\omega_f(x) = \lim_{n\to\infty} \sup_{\substack{k\ge n \\ \ell \ge n}} |S_k(f,x) - S_\ell(f,x)|$, then $S_n(f,x)$ converges to f almost everywhere if and only if $w_f(x) = 0$ almost everywhere.

3. MAIN RESULT

The main result can be stated as follows.

THEOREM. Let ρ be a function in the class b_1. If $f \in B(\rho)$ then the maximal operator defined by $Tf(x) = \sup_n |S_n(f,x)|$ maps $B(\rho)$ into $L(\phi)$ for $\phi(t) = \frac{\rho(t)}{t}$ boundedly, that is, $\|Tf\|_\phi \le M\|f\|_{B(\rho)}$, where M is a positive, absolute constant and $S_n(f,x)$ is the nth partial sum of the Fourier series of f.

PROOF. First of all we notice that the operator $T_a f = f^a$ where $f^a(x) = f(x-a)$ maps $B(\rho)$ into $B(\rho)$ continuously. Consequently we just need to prove the result for $f_h(t) = \frac{1}{\rho(2h)}[\chi_{[-h,0)}(t) - \chi_{[0,h]}(t)]$, $h > 0$ which will follow from the estimate for $g(t) = \chi_{[0,h]}(t)$. In fact let $S_n(g,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)D_n(x-t)dt$ where $D_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$ is the Dirichlet kernel. Therefore we have $S_n(f,x) = \frac{1}{\pi}\int_{x-h}^{x} D_n(t)dt$.
We now use the elementary inequality $\left|D_n(t)\right| \leq \frac{C}{|t|}$, $0 < |t| \leq \pi$, satisfied by the Dirichlet kernels D_n, where C is an absolute constant. Thus

$$\left|S_n(g,x)\right| \leq \int_{x-h}^{x} \left|D_n(t)\right|dt \leq \int_{x-h}^{x} \frac{1}{|t|}\,dt \leq \frac{Ch}{x-h} \leq 2C\frac{h}{x} \quad \text{for } x > 2h$$

and $\left|S_n(g,x)\right| \leq 2C\frac{h}{-x}$ for $x < -2h$.

Recall that $\int_0^x D_n(t)dt$ is uniformly bounded in n and x; that is, $\left|\int_0^x D_n(t)dt\right| \leq A$ where A is an absolute constant. (See [7], volume 1, page 57.)

Consequently we have

1) $Tg(x) \leq A$ for all x
2) $Tg(x) \leq 2C\frac{h}{|x|}$ for $|x| > 2h$.

We now evaluate $\|Tg\|_\phi$, for $\phi(t) = \frac{\rho(t)}{t}$. In fact, using 1) and 2) we get

$$\|Tg\|_\phi = \int_0^{2\pi}(Tg)^*(x)\frac{\rho(x)}{x}\,dx = \int_0^{2h}(Tg)^*(x)\frac{\rho(x)}{x}\,dx + \int_{2h}^{2\pi}(Tg)^*(x)\frac{\rho(x)}{x}\,dx$$

$$\leq A\int_0^{2h}\frac{\rho(x)}{x}\,dx + 2Ch\int_{2h}^{2\pi}\frac{\rho(x)}{x^2}\,dx.$$

Now using the conditions on ρ we get

$$\|Tg\|_\phi \leq AC\,\rho(2h) + 2Ch\,\frac{\rho(2h)}{2h} = M\rho(2h)$$

The constant C may not be the same at every occurence.

Consequently $\|Tf_h\|_\phi \leq M$ and so if $f(t) = \sum_{n=0}^{\infty} c_n b_n(t)$ where $b_n(t) = \frac{1}{\rho(|I_n|)}[\chi_{R_n}(t) - \chi_{L_n}(t)]$ and $\sum_{n=0}^{\infty}|c_n| < \infty$, we get $\|Tf\|_\phi \leq M\sum_{n=0}^{\infty}|c_n|$ which implies $\|Tf\|_\phi \leq M\|f\|_{B(\rho)}$. The proof is complete.

COROLLARY. Let ρ be in the class b_1. If $f \in B(\rho)$, then $\{S_n(f,x)\}$ converges to $f(x)$ almost everywhere.

PROOF. Let $f \in B(\rho)$ then $f(x) = \sum_{n=0}^{\infty} c_n b_n(x)$ where $\sum_{n=0}^{\infty}|c_n| < \infty$ and the b_n are weighted special atoms. Now observe that

$w_f(x) = \limsup_{n\to\infty} \sup_{k>n,\ \ell>n} \left|S_k(f,x) - S_\ell(f,x)\right|$ satisfies $w_f(x) < 2Tf(x)$. Therefore

$$\int_0^{2\pi} w_f^*(x)\,\frac{\rho(t)}{t}\,dt \leq 2\int_0^{2\pi}\left[Tf(x)\right]^*\frac{\rho(t)}{t}\,dt$$ and so $w_f \in L(\phi)$. On the other hand we see that $w_f = w_{f-f_k}$ where $f_k(x) = \sum_{n=0}^{k} c_n b_n(x)$ and

$\|f_k - f\|_{B(\rho)} \to 0$ as $k \to \infty$. Then $w_f(x) = w_{f-f_k}(x) \leq 2T(f - f_k)(x)$ and consequently the theorem, that is, the boundedness of T, implies that $\|w_f\|_\phi \leq 2\|T(f - f_k)\|_\phi \leq 2M\|f - f_k\|_{B(\rho)}$. So letting $k \to \infty$ we get $\|w_f\|_\phi = 0$. Thus $w_f(x) = 0$ almost everywhere, which implies by the comment right after definition 2.2 that $S_n(f,x) \to f(x)$ almost everywhere. The proof is complete.

We emphasize that for $\rho(t) = t^{1/p}$, $1 < p < \infty$ the space $B(\rho)$ is the space B^p and $L(\phi)$ is the space $L(p,1)$ in [3], and so the results in this paper on $B(\rho)$ is a generalization of that result.

We also point out that for $\rho(t) = t$, one can indeed show that there is a $f \in B(\rho)$ so that the Fourier series of f, diverges almost everywhere. (See [4]). This answers a question asked of the author by Professor Guido Weiss concerning almost everywhere convergence of functions in $B(\rho)$ for $\rho(t) = t$.

REFERENCES

[1] CARLESON, L., On the convergence and growth of partial sums of Fourier series, Acta. Math. 116, (1966) 135-157.

[2] BLOOM, S. and DE SOUZA, G. S., Atomic decomposition of the generalized Lipschitz spaces - To appear, Illinois Journal of Mathematics.

[3] DE SOUZA, G. S., On the convergence of Fourier series, Internat. J. Math. and Math. Sci. 7(4), (1984) 817-820.

[4] DE SOUZA, G. S. and SAMPSON, G., A function in Dirichlet space B^1 such that its Fourier series diverges almost everywhere. Preprint.

[5] HUNT, R. A., On the convergence of Fourier series, Proc. Conference Southern Illinois University, Edwardville, Ill. 1967.

[6] LORENTZ, G. G., On the theory of spaces Λ, Pacific Journal of Math. 1(1951), 411-429.

[7] ZYGMUND, A., Trigonometric Series, Cambridge University Press, London and New York, 1959.

Division of Mathematics
Foundations, Analysis and Topology
Auburn University
Auburn, AL 36849-5310

Journal of Applied Mathematics and Decision Sciences

Special Issue on Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk