

ON THE EXPONENTIAL GROWTH OF SOLUTIONS TO NON-LINEAR HYPERBOLIC EQUATIONS

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ABSTRACT. Existence-uniqueness theorems are proved for continuous solutions of some classes of non-linear hyperbolic equations in bounded and unbounded regions. In case of unbounded region, certain conditions ensure that the solution cannot grow to infinity faster than exponentially.

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1. INTRODUCTION.

In this paper we study the existence of a unique solution to some non-linear partial differential equations of hyperbolic type. These equations appear in a mathematical model for the dynamics of gas absorption [1], and the main interest is to find solutions of exponential growth to a non-linear hyperbolic equation with characteristic data. It is possible to investigate such problems by the method of successive approximations, after reducing the differential equation to a Volterra integral equation in two variables. However, here we use the method of equivalent (weighted) norms, which considerably reduces the volume of computations. It should be noticed that in [1], an asymptotic investigation of corresponding linear equations has been conducted as $t \rightarrow \infty$. Periodic and almost-periodic solutions of a similar class of non-linear hyperbolic equations have been studied in [2]. The method of successive approximations has been applied in [3] and [4] to find bounded solutions of non-linear hyperbolic equations with time delay, which arise in control theory and in certain biomedical models.

We consider the equation

$$u_{xt}(x,t) = F(x,t,u(x,t),u_x(x,t)), \quad (1)$$

and pose for (1) the following initial and boundary conditions:

$u(0, t) = u_0(t); \quad 0 \leq t \leq T$
 $u(x, 0) = \varphi(x); \quad 0 \leq x \leq \ell$ (2)
 where $u_0(t)$ and $\varphi(x)$ are given functions in the domain
 $\Delta : [0, \ell] \times [0, T]$, and we are interested in existence-uniqueness
 to problem (1)-(2).

Two norms $\|x\|$, $\|x\|_*$ on a Banach space are called equivalent if there exist two positive numbers p and q such that

$$p \|x\| \leq \|x\|_* \leq q \|x\|.$$

For example, if the function $x(t)$ belongs to the space of continuous functions on $[0, T]$, it is easy to see that the norms

$$\|x\| = \max_{0 \leq t \leq T} |x(t)|,$$

and

$$\|x\|_* = \max_{0 \leq t \leq T} e^{-L_1 t} |x(t)|, \quad L_1 > 0 \quad (3)$$

are equivalent. In order to prove the existence of a unique continuous solution to our problem, we use a norm similar to (3) and choose L_1 so that a certain integral operator becomes a contraction.

2. MAIN RESULTS

We prove our first result for equation (1) with the initial and boundary conditions (2) as follows.

THEOREM 1. Assume the hypotheses:

- (i) The function $u_0(t)$ is continuously differentiable on $[0, T]$ and $\varphi(x)$ is continuously differentiable on $[0, \ell]$.
- (ii) The function $F(x, t, u, v)$ is continuous in $\Delta \times \mathbb{R}^2$ and satisfies the Lipschitz condition

$$|F(x, t, u, v) - F(x, t, \bar{u}, \bar{v})| \leq L \left[|u - \bar{u}| + |v - \bar{v}| \right] \quad (L)$$

for $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ uniformly with respect to x, t .

Then problem (1)-(2) has a unique continuous solution in Δ .

Proof. We change equation (1) to

$$u(x, t) = u_0(t) + \varphi(x) - \varphi(0) + \int_0^x \int_0^t F(\xi, \eta, u(\xi, \eta), u_\xi(\xi, \eta)) d\eta d\xi \quad (4)$$

and introduce the operator

$$Aw(x, t) = u_0(t) + \varphi(x) - \varphi(0) + \int_0^x \int_0^t F(\xi, \eta, w(\xi, \eta), w_\xi(\xi, \eta)) d\eta d\xi \quad (4')$$

on the space $C^1(\Delta)$ of all functions $w(x, t)$ continuously differentiable in Δ .

We define a weighted norm in $C^1(\Delta)$ by the formula:

$$\|w\|_* = \max_{\Delta} e^{-L_1 t} \left[|w(x, t)| + |w_x(x, t)| \right] \quad (5)$$

where the constant $L_1 > 0$ will be chosen later. Since $u_0(t)$, $\varphi(x)$, are continuously differentiable and $F(x, t, u, v)$ is a continuous

function of its variables, operator (4') maps $C^1(\Delta)$ into $C^1(\Delta)$.

Now, we want to show that A is a contraction on $C^1(\Delta)$. Consider the difference

$$Aw(x, t) - \bar{Aw}(x, t) = \int_0^x \int_0^t \left[F(\xi, \eta, w(\xi, \eta), w_\xi(\xi, \eta)) - F(\xi, \eta, \bar{w}(\xi, \eta), \bar{w}_\xi(\xi, \eta)) \right] d\eta d\xi$$

for $w, \bar{w} \in C^1(\Delta)$ and apply the Lipschitz condition, then

$$|Aw(x, t) - \bar{Aw}(x, t)| \leq L \int_0^x \int_0^t \left[|w(\xi, \eta) - \bar{w}(\xi, \eta)| + |w_\xi(\xi, \eta) - \bar{w}_\xi(\xi, \eta)| \right] d\eta d\xi$$

Consider the derivative of $Aw(x, t)$ and $\bar{Aw}(x, t)$ with respect to x , then

$$\begin{aligned} & \left| (Aw(x, t))_x - (\bar{Aw}(x, t))_x \right| \\ & \leq \int_0^t \left| F(x, \eta, w(x, \eta), w_x(x, \eta)) - F(x, \eta, \bar{w}(x, \eta), \bar{w}_x(x, \eta)) \right| d\eta \\ & \leq L \int_0^t \left[|w(x, \eta) - \bar{w}(x, \eta)| + |w_x(x, \eta) - \bar{w}_x(x, \eta)| \right] d\eta \end{aligned}$$

From here,

$$\begin{aligned} & e^{-L_1 t} \left(|Aw(x, t) - \bar{Aw}(x, t)| + \left| (Aw(x, t))_x - (\bar{Aw}(x, t))_x \right| \right) \\ & \leq L \int_0^x \int_0^t e^{-L_1(t-\eta)} e^{-L_1 \eta} \left[|w(\xi, \eta) - \bar{w}(\xi, \eta)| + |w_\xi(\xi, \eta) - \bar{w}_\xi(\xi, \eta)| \right] d\eta d\xi \\ & \quad + L \int_0^t e^{-L_1(t-\eta)} e^{-L_1 \eta} \left[|w(x, \eta) - \bar{w}(x, \eta)| + |w_x(x, \eta) - \bar{w}_x(x, \eta)| \right] d\eta \\ & \leq L \|w - \bar{w}\|_* \int_0^x \int_0^t e^{-L_1(t-\eta)} d\eta d\xi + L \|w - \bar{w}\|_* \int_0^t e^{-L_1(t-\eta)} d\eta \\ & \leq L \left[\frac{\ell}{L_1} \|w - \bar{w}\|_* + \frac{1}{L_1} \|w - \bar{w}\|_* \right] = \frac{L(\ell+1)}{L_1} \|w - \bar{w}\|_* . \end{aligned}$$

If we pick $L_1 > L(\ell+1)$ and define $q = \frac{L(\ell+1)}{L_1}$, then

$$\|Aw - \bar{Aw}\|_* \leq q \|w - \bar{w}\|_*$$

with $0 < q < 1$. This shows that the operator A is a contraction and proves the theorem.

The following proposition concerns the solution behaviour of an equation linear with respect to $u_x(x, t)$ in an unbounded region as $t \rightarrow \infty$. Although this result is generalized in Theorem 3, its proof is given for instructive purposes.

THEOREM 2. For equation

$$u_{xt}(x,t) + a(x,t)u_x(x,t) = f(x,t,u(x,t)), \quad (6)$$

and the initial and boundary conditions

$$\begin{aligned} u(0,t) &= u_0(t); & 0 \leq t < \infty \\ u(x,0) &= \varphi(x); & 0 \leq x \leq \ell \end{aligned} \quad (7)$$

assume:

- (i) $a(x,t)$ is continuous in $\Omega : [0,\ell] \times [0,\infty)$ and satisfies the condition $a(x,t) \geq m$, where m is a constant, the function $\varphi(x)$ is continuously differentiable on $[0,\ell]$.
- (ii) The function $f(x,t,u)$ is continuous in $\Omega \times \mathbb{R}$ and satisfies the Lipschitz condition

$$|f(x,t,u) - f(x,t,v)| \leq L |u - v|$$

for $u, v \in \mathbb{R}$, uniformly with respect to x, t ; the function $f(x,t,0)$ satisfies the inequality

$$|f(x,t,0)| \leq K_1 e^{L_1 t}$$

where $(x,t) \in \Omega$, K_1 is a constant, and

$$L_1 > L\ell - m. \quad (8)$$

- (iii) The function $u_0(t)$ is continuously differentiable on $[0,\infty)$ and satisfies

$$|u_0(t)| \leq K_2 e^{L_1 t}$$

for $t \in [0,\infty)$, K_2 is constant, and L_1 satisfies (8).

Then problem (6)-(7) has a unique continuous solution $u(x,t)$ in Ω and

$$\sup_{\Omega} e^{-L_1 t} |u(x,t)| < \infty.$$

Proof. First, transform equation (6) to

$$\begin{aligned} u(x,t) &= u_0(t) + \int_0^x \varphi(\xi) e^{-\int_0^t a(\xi,s) ds} d\xi \\ &+ \int_0^x \int_0^t e^{-\int_\eta^t a(\xi,s) ds} f(\xi, \eta, u(\xi, \eta)) d\eta d\xi \end{aligned} \quad (9)$$

and introduce the operator

$$\begin{aligned} Aw(x,t) &= u_0(t) + \int_0^x \varphi(\xi) e^{-\int_0^t a(\xi,s) ds} d\xi \\ &+ \int_0^x \int_0^t e^{-\int_\eta^t a(\xi,s) ds} f(\xi, \eta, w(\xi, \eta)) d\eta d\xi \end{aligned} \quad (10)$$

on the space $B(\Omega)$ of all functions $w(x,t)$ continuous in Ω , with the norm

$$\| w \|_{**} = \sup_{\Omega} e^{-L_1 t} |w(x, t)|, \quad L_1 > 0. \quad (11)$$

Now we prove that the operator (10) maps $B(\Omega)$ into $B(\Omega)$. Indeed,

$$\begin{aligned} Aw(x, t) &= u_0(t) + \int_0^x \varphi(\xi) e^{-\int_0^\xi a(\xi, \omega) d\omega} d\xi \\ &+ \int_0^x \int_0^t e^{-\int_\eta^\xi a(\xi, \omega) d\omega} [f(\xi, \eta, w(\xi, \eta)) - f(\xi, \eta, 0)] d\eta d\xi \\ &+ \int_0^x \int_0^t e^{-\int_\eta^\xi a(\xi, \omega) d\omega} f(\xi, \eta, 0) d\eta d\xi \end{aligned}$$

and from the hypotheses of the theorem, we can write

$$|\varphi(x)| \leq K, \quad e^{-L_1 t} |f(x, t, 0)| \leq K_1, \quad e^{-L_1 t} |u_0(t)| \leq K_2.$$

Hence, by virtue of Lipschitz condition (L), we obtain

$$\begin{aligned} e^{-L_1 t} |Aw(x, t)| &\leq K_2 + K \ell e^{-(L_1 + m)t} \\ &+ L \int_0^x \int_0^t e^{-(m+L_1)(t-\eta)} e^{-L_1 \eta} |w(\xi, \eta)| d\eta d\xi + \frac{K_1 \ell}{m + L_1}. \end{aligned}$$

Taking into account (11), this implies

$$e^{-L_1 t} |Aw(x, t)| \leq \left[K_2 + K \ell + \frac{K_1 \ell}{m + L_1} \right] + L \| w \|_{**} \int_0^x \int_0^t e^{-(m+L_1)(t-\eta)} d\eta d\xi.$$

Therefore,

$$\| Aw \|_{**} \leq K_2 + \left(K + \frac{K_1 \ell}{m + L_1} \right) \ell + \frac{L \ell}{m + L_1} \| w \|_{**}.$$

From here we see that if $\| w \|_{**}$ is bounded, then $\| Aw \|_{**}$ is bounded, which proves that A maps the space $B(\Omega)$ into itself.

Now, we evaluate $Au - Av$ for $u, v \in B(\Omega)$,

$$\| Au - Av \|_{**} \leq \frac{L \ell}{m + L_1} \lim_{T \rightarrow \infty} [1 - e^{-(m+L_1)T}] \| u - v \|_{**}.$$

Since the above limit is 1, one can write

$$\| Au - Av \|_{**} \leq \frac{L \ell}{m + L_1} \| u - v \|_{**},$$

which shows that A is a contraction on Ω and proves that problem (6)-(7) has a unique continuous solution in Ω which is bounded in the sense of norm (11).

THEOREM 3. Assume for problem (1)-(7) the following hypotheses:

- (i) The function $u_0(t)$ is continuously differentiable and satisfies $|u_0(t)| \leq K_2 e^{L_1 t}$ for $t \in [0, \infty)$, where K_2 is a constant, and $\varphi(x)$ is continuously differentiable on $[0, \ell]$.
- (ii) The function $F(x, t, u, v)$ is continuous in $\Omega \times \mathbb{R}^2$ and satisfies Lipschitz condition (L), uniformly in x, t .

(iii) The function $F(x, t, 0, 0)$ satisfies

$$|F(x, t, 0, 0)| \leq K_3 e^{L_1 t},$$

K_3 is a constant and $L_1 > L(\ell+1)$, where L is Lipschitz constant.

Then problem (1)-(7) has a unique continuous solution $u(x, t)$ in Ω and

$$\sup_{\Omega} e^{-L_1 t} |u(x, t)| < \infty.$$

Proof. We reduce (1) to (4) and introduce the operator (4') on the space $C^1(\Omega)$ of all functions $w(x, t)$ continuously differentiable in Ω , with the norm

$$\|w\|_* = \sup_{\Omega} e^{-L_1 t} \left(|w(x, t)| + |w_x(x, t)| \right) \quad (12)$$

First, we prove that the operator maps $C^1(\Omega)$ into $C^1(\Omega)$. Indeed,

$$\begin{aligned} Aw(x, t) &= u_0(t) + \varphi(x) - \varphi(0) + \int_0^x \int_0^t [F(\xi, \eta, w(\xi, \eta), w_\xi(\xi, \eta)) - F(\xi, \eta, 0, 0)] d\eta d\xi \\ &\quad + \int_0^x \int_0^t F(\xi, \eta, 0, 0) d\eta d\xi \end{aligned}$$

and

$$\begin{aligned} [Aw(x, t)]_x &= \varphi'(x) + \int_0^t [F(x, \eta, w(x, \eta), w_x(x, \eta)) - F(x, \eta, 0, 0)] d\eta \\ &\quad + \int_0^t F(x, \eta, 0, 0) d\eta. \end{aligned}$$

Hence,

$$\begin{aligned} |Aw(x, t)| + |[Aw(x, t)]_x| &\leq |u_0(t)| + |\varphi(x) - \varphi(0)| + |\varphi'(x)| \\ &\quad + L \int_0^x \int_0^t [|w(\xi, \eta)| + |w_\xi(\xi, \eta)|] d\eta d\xi + \int_0^x \int_0^t |F(\xi, \eta, 0, 0)| d\eta d\xi \\ &\quad + L \int_0^t [|w(x, \eta)| + |w_x(x, \eta)|] d\eta + \int_0^t |F(x, \eta, 0, 0)| d\eta. \end{aligned}$$

Multiplying the previous expression by $e^{-L_1 t}$, we have

$$\begin{aligned} e^{-L_1 t} \left(|Aw(x, t)| + |[Aw(x, t)]_x| \right) &\leq e^{-L_1 t} |u_0(t)| + e^{-L_1 t} |\varphi(x) - \varphi(0)| \\ &\quad + e^{-L_1 t} |\varphi'(x)| + L \int_0^x \int_0^t e^{-L_1(t-\eta)} e^{-L_1 \eta} \left(|w(\xi, \eta)| + |w_\xi(\xi, \eta)| \right) d\eta d\xi \\ &\quad + \int_0^x \int_0^t e^{-L_1(t-\eta)} e^{-L_1 \eta} |F(\xi, \eta, 0, 0)| d\eta d\xi \end{aligned}$$

$$\begin{aligned}
& + \int_0^t e^{-L_1(t-\eta)} e^{-L_1\eta} \left(|w(x, \eta) + w_x(x, \eta)| \right) d\eta \\
& + \int_0^t e^{-L_1(t-\eta)} e^{-L_1\eta} |F(x, \eta, 0, 0)| d\eta .
\end{aligned}$$

If we let $e^{-L_1 t} \left(|\varphi(x) - \varphi(0)| + |\varphi'(x)| \right) \leq K_1$, and take into account

$$\int_0^t e^{-L_1(t-\eta)} d\eta \leq \frac{1}{L_1} ,$$

then

$$\begin{aligned}
e^{-L_1 t} \left(\left| A(x, t) \right| + \left| (Aw(x, t))_x \right| \right) & \leq K_2 + K_1 + \frac{L\ell}{L_1} \|w\|_* + \frac{K_3\ell}{L_1} \\
& + \frac{L}{L_1} \|w\|_* + \frac{K_3}{L_1} .
\end{aligned}$$

Therefore,

$$\|Aw(x, t)\|_* \leq K_2 + K_1 + \frac{K_3(\ell+1)}{L_1} + \frac{L(\ell+1)}{L_1} \|w\|_* ,$$

which proves that the operator (4) maps $C^1(\Omega)$ into $C^1(\Omega)$. For the proof of contraction, we simply repeat the corresponding computations of Theorem 1.

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