

## FURTHER RESULTS ON PRIMES IN SMALL INTERVALS

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**ABSTRACT.** In this paper we will deal with upper and lower bounds for  $\pi(x+y) - \pi(x)$ . In fact, given  $q$  with  $0 < q \leq 1$ , for sufficiently large integers  $m, n$  such that  $m \geq n \geq qm > 2$  we show that  $\pi(m+n) - \pi(m) < \ln(n)\pi(n)/\ln(m+1)$ . Moreover, explicit bounds are obtained and a wider range is given under the assumption of the Riemann hypothesis. Let  $m, n$  be positive integers with  $m > 2657$ . Let  $1 \leq \theta < 2$  and  $m \geq n \geq m^{1/\theta}$ . If the Riemann hypothesis holds, then  $\pi(m+n) - \pi(m) < n/\ln(m+1) + \sqrt{n^\theta} + n \ln(n^\theta + n)/4\pi$ . (Here  $\pi(x)$  = the number of primes  $\leq x$ .)

**KEY WORDS AND PHRASES.** Primes. Small intervals.  $\pi(x+y) \leq \pi(x) + \pi(y)$ .

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### 1. INTRODUCTION.

There are several accounts dealing with the validity of the conjecture that for  $x > 1$  and  $y > 1$ ,

$$\pi(x+y) \leq \pi(x) + \pi(y). \quad (1.1)$$

For example [1], [2], [3] deal with (1.1), whereas in [4] there is a discussion of the conjecture of the following form:

$$\pi(x+y) < \pi(x) + \pi(y) + cy/\ln^2(y). \quad (1.2)$$

(Here we let  $x \geq y \geq 1$  and  $c > 0$ .) In fact, one of the two authors of [4] believes that (1.2) is true, whereas the other one does not.

What is interesting to this author is a paper written by Hensley and Richards [5]; they proved that if the prime  $k$ -tuple conjecture is true then (1.1) is false. Furthermore, assuming that the  $k$ -tuple conjecture is true they have shown that  $\exists c > 0$  such that for sufficiently large  $y$  and infinitely many  $x$  we must have  $\pi(x+y) - \pi(x) - \pi(y) > cy/\ln^2(y)$ .

By using sophisticated techniques H.L. Montgomery and R.C. Vaughan [6] proved that if  $M > 0$  and  $N > 1$  are integers then  $\pi(M+N) - \pi(M) \leq 2N/\ln(N)$ . Now D.R. Heath-Brown and H. Iwaniec [7] show that if  $\theta > 11/20$  and  $x \geq x(\theta)$  then  $\pi(x) - \pi(x-y) > y/(212 \ln(x))$  in the range  $x^\theta \leq y \leq x/2$ . The methods used in this paper are elementary and give a different range of validity. The proofs of this paper use the following definitions and results.

$\pi(x)$  = the number of primes  $\leq x$

$$Li(x) = \int_2^x dt/\ln(t) \quad \text{for } x \geq 2$$

$$Ls(m) = \sum_{k=2}^m 1/\ln(k) \quad \text{for any integer } m \geq 2$$

$$\pi(x) = Li(x) + O(xe^{-a\sqrt{\ln(x)}}) \quad \text{for } x \geq 2, a > 0 \quad (1.3)$$

$$Li(x) = x(1 + \sum_{k=1}^{n-1} (k!/\ln^k(x)))/\ln(x) + O(x/\ln^{n+1}(x)) \quad \text{for } x \geq 2 \quad (1.4)$$

$$\pi(m) = Ls(m) + O(me^{-c\sqrt{\ln(m)}}) \quad \text{for integer } m \geq 2, c > 0 \quad (1.5)$$

$$|Li(m) - Ls(m)| < C \quad \text{for some constant } C \quad (1.6)$$

If the Riemann hypothesis holds, then (1.7) is true

$$|\pi(x) - Li(x)| < \sqrt{x} \ln(x) / 8\pi \quad \text{for } x \geq 2657 \quad (1.7)$$

$$x(1 + 1/(2 \ln(x))) / \ln(x) < \pi(x) \quad \text{for } 59 \leq x \quad (1.8)$$

$$\pi(x) < x(1 + 3/(2 \ln(x))) / \ln(x) \quad \text{for } 1 < x \quad (1.9)$$

Now (1.3), (1.4) can be found in Ayoub [8], whereas (1.5), (1.6) are found in T. Estermann [9]. Furthermore, the paper written by L. Schoenfeld [10] gives us (1.7). Finally (1.8), (1.9) were proven by J.B. Rosser and L. Schoenfeld [11].

## 2. THEOREMS, COROLLARIES AND THEIR PROOFS.

**THEOREM 1.** If  $0 < d \leq 1$  and  $x, y$  are sufficiently large with  $x \geq y \geq dx > 2$ , then  $\pi(x+y) - \pi(x) - \ln(y)\pi(y)/\ln(x+y) < O(y/\ln^{n+1}(y))$  for any natural number  $n \geq 2$ .

**PROOF.** We have from (1.3) and (1.4) the following:

$$\pi(x) = x/\ln(x) + x/\ln^2(x) + \dots + (n-1)!x/\ln^n(x) + O(x/\ln^{n+1}(x)). \quad (2.1)$$

Now it is obvious that

$$\begin{aligned} \pi(x+y) - \pi(x) &= x/\ln(x+y) - x/\ln(x) + \sum_{k=1}^{n-1} \left[ k!x/\ln^{k+1}(x+y) - k!x/\ln^{k+1}(x) \right] \\ &\quad + y \left[ 1 + \sum_{k=1}^{n-1} (k!/\ln^k(x+y)) \right] / \ln(x+y) + O \left[ (x+y)/\ln^{n+1}(x+y) \right]. \end{aligned} \quad (2.2)$$

Given that  $x \geq 2, y > 0$  then for  $0 \leq k \leq n-1$  we have

$$k!x / \ln^{k+1}(x+y) < k!x / \ln^{k+1}(x).$$

Hence (2.2) is replaced by

$$\pi(x+y) - \pi(x) < y \left[ 1 + \sum_{k=1}^{n-1} (k!/\ln^k(x+y)) \right] / \ln(x+y) + O \left[ (x+y)/\ln^{n+1}(x+y) \right] \quad (2.3)$$

For  $k \geq 1$ , we observe that  $\ln^k(x+y) \geq \ln^k(2y) > \ln^k(y)$ . Replacing  $\ln^k(x+y)$ , (2.3) now becomes

$$\pi(x+y) - \pi(x) < y \left[ 1 + \sum_{k=1}^{n-1} (k!/\ln^k(y)) \right] / \ln(x+y) + O \left[ (x+y)/\ln^{n+1}(y) \right]. \quad (2.4)$$

Multiplying the first term on the right hand side of (2.4) by  $\ln(y)/\ln(y)$  and using (2.1) we have replaced (2.4) by the following:

$$\pi(x+y) - \pi(x) - \ln(y)\pi(y)/\ln(x+y) < O\left[(x+y)/\ln^{n+1}(y)\right]. \quad (2.5)$$

It is obvious  $\exists$  a constant  $M > 0$  such that for  $x + y$  sufficiently large the left hand side of (2.5) is strictly less than

$$M(x+y)/\ln^{n+1}(y) \quad (2.6)$$

Since  $x \geq y \geq dx > 2$  for  $0 < d \leq 1$  then

$$M(x+y)/\ln^{n+1}(y) < M(y/d + y)/\ln^{n+1}(y) < M'(y)/\ln^{n+1}(y). \quad (2.7)$$

Hence by using (2.7) we conclude that

$$\pi(x+y) - \pi(x) - \ln(y)\pi(y)/\ln(x+y) < O(y/\ln^{n+1}(y)).$$

**THEOREM 2.** Let  $0 < q \leq 1$ . If  $m, n$  are sufficiently large positive integers satisfying  $m \geq n \geq qm > 2$ , then  $\pi(m+n) - \pi(m) < n/\ln(m+1) + Bne^{-a\sqrt{\ln(2n)}}$  for  $B, a > 0$ .

**PROOF.** By using (1.5) we see that

$$\pi(m+n) - \pi(m) = \sum_{k=m+1}^{m+n} (1/\ln(k)) + O\left[(m+n)e^{-a\sqrt{\ln(m+n)}}\right]. \quad (2.8)$$

It is obvious that we can replace (2.8) by

$$\pi(m+n) - \pi(m) - n/\ln(m+1) < O\left[(m+n)e^{-a\sqrt{\ln(m+n)}}\right]. \quad (2.9)$$

Now  $\exists$  a constant  $M > 0$  such that for  $m + n$  sufficiently large that the left hand side of (2.9) is strictly less than

$$M(m+n)e^{-a\sqrt{\ln(m+n)}}.$$

Since  $m \geq n \geq qm > 2$  and  $0 < q \leq 1$  then

$$M(m+n)e^{-a\sqrt{\ln(m+n)}} < M(n/q + n)e^{-a\sqrt{\ln(2n)}} = Bne^{-a\sqrt{\ln(2n)}}.$$

Hence  $\pi(m+n) - \pi(m) < n/\ln(m+1) + Bne^{-a\sqrt{\ln(2n)}}$ .

**COROLLARY 1.** Let  $0 < q \leq 1$ . If  $m, n$  are sufficiently large positive integers satisfying  $m \geq n \geq qm > 2$ , then  $\pi(m+n) - \pi(m) < \ln(n)\pi(n)/\ln(m+1)$ .

**PROOF.** By using the result of Theorem 2 with a slight modification we have

$$\pi(m+n) - \pi(m) < n\ln(n)/(\ln(n)\ln(m+1)) + Bne^{-a\sqrt{\ln(2n)}}. \quad (2.10)$$

We rearrange the terms in (2.1) so that one can give an upper bound to replace  $n/\ln(n)$ . With  $M > 0$ , we now incorporate an upper bound of  $n/\ln(n)$  into (2.10) to establish that

$$\pi(m+n) - \pi(m) < \ln(n) \left[ \pi(n) - \sum_{k=2}^{n-1} ((k-1)!n/\ln^k(n)) + Mn/\ln^2(n) \right] / \ln(m+1) + Bne^{-a\sqrt{\ln(2n)}}.$$

Hence for  $n$  sufficiently large we have

$$\pi(m+n) - \pi(m) < \ln(n)\pi(n)/\ln(m+1).$$

**THEOREM 3.** Let  $0 < q \leq 1$ . If  $m, n$  are sufficiently large positive integers satisfying  $m \geq n \geq qm > 2$ , then  $\pi(m+n) - \pi(m) > n/\ln(m+n) - Ane^{-a\sqrt{\ln(2n)}}$  for  $a > 0$  and  $A > 0$ . constant  $M$  we have

$$\pi(m+n) - \pi(m) > \sum_{k=m+1}^{m+n} (1/\ln(k)) - M(m+n)e^{-a\sqrt{\ln(m+n)}} - Mme^{-a\sqrt{\ln(m)}}. \quad (2.11)$$

With a slight modification in (2.11) and using another constant  $M' > 0$  we see that

$$\pi(m+n) - \pi(m) > n/\ln(m+n) - M'(m+n)e^{-\sqrt{\ln(m+n)}}. \quad (2.12)$$

By rearranging the terms in (2.12) this will now become

$$M'(m+n)e^{-\sqrt{\ln(m+n)}} > n/\ln(m+n) + \pi(m) - \pi(m+n). \quad (2.13)$$

Since  $m \geq n \geq qm > 2$  and  $0 < q \leq 1$  then

$$M'(m+n)e^{-\sqrt{\ln(m+n)}} < M'(n/q+n)e^{-\sqrt{\ln(2n)}} = Ane^{-\sqrt{\ln(2n)}}.$$

Hence  $\pi(m+n) - \pi(m) > n/\ln(m+n) - Ane^{-\sqrt{\ln(2n)}}$ .

**COROLLARY 2.** Let  $0 < q \leq 1$ ,  $\epsilon > 0$ . If  $m, n$  are sufficiently large positive integers satisfying  $m \geq n \geq qm > 2$ , then  $\pi(m+n) - \pi(m) > \ln(n)(\pi(n) - (1+\epsilon)n/\ln^2(n))/\ln(m+n)$ .

**PROOF.** By using the results of Theorem 3 with a slight modification we have

$$\pi(m+n) - \pi(m) > n \ln(n) / (\ln(n)\ln(m+n)) - Ane^{-\sqrt{\ln(2n)}}. \quad (2.14)$$

Using an argument similar to that found in Corollary 1, we rearrange the terms in (2.1) so that one can give a lower bound to replace  $n/\ln(n)$ . With  $D > 0$ , we now incorporate a lower bound of  $n/\ln(n)$  into (2.14) to establish the following

$$\pi(m+n) - \pi(m) > \ln(n) \left[ \pi(n) - \sum_{k=2}^{t-1} ((k-1)!n/\ln^k(n)) - Dn/\ln^t(n) \right] / \ln(m+n) - Ane^{-\sqrt{\ln(2n)}}.$$

Hence for sufficiently large  $n$

$$\pi(m+n) - \pi(m) > \ln(n)(\pi(n) - (1+\epsilon)n/\ln^2(n))/\ln(m+n).$$

**THEOREM 4.** Let  $1 \leq \theta < 2$ . Let  $m, n$  be positive integers with  $m > 2657$  and  $m \geq n \geq m^{1/\theta}$ . If the Riemann hypothesis holds, then  $\pi(m+n) - \pi(m) < n/\ln(m+1) + \sqrt{n^\theta + n} \ln(n^\theta + n)/4\pi$ .

**PROOF.** By using the upper and lower bounds of (1.7) we have

$$\pi(m+n) - \pi(m) < \text{Li}(m+n) - \text{Li}(m) + (\sqrt{m+n} \ln(m+n) + \sqrt{m} \ln(m))/8\pi. \quad (2.15)$$

Noting that  $\sqrt{m+n} \ln(m+n) > \sqrt{m} \ln(m)$  and using (1.6), then (2.15) will now become

$$\pi(m+n) - \pi(m) < \sum_{k=m+1}^{m+n} (1/\ln(k)) + \sqrt{m+n} \ln(m+n) / 4\pi. \quad (2.16)$$

It is obvious that we can replace (2.16) by

$$\pi(m+n) - \pi(m) < n/\ln(m+1) + \sqrt{m+n} \ln(m+n)/4\pi.$$

Given that  $m \geq n \geq m^{1/\theta}$  for  $1 \leq \theta < 2$  we may now conclude

$$\pi(m+n) - \pi(m) < n/\ln(m+1) + \sqrt{n^\theta + n} \ln(n^\theta + n)/4\pi.$$

**COROLLARY 3.** Let  $1 \leq \theta < 2$ . Let  $m, n$  be positive integers with  $m > 2657$ ,  $n > 59$ , and  $m \geq n \geq m^{1/\theta}$ . If the Riemann hypothesis holds, then

$$\pi(m+n) - \pi(m) < \ln(n) \left[ \pi(n) - n/(2 \ln^2(n)) \right] / \ln(m+1) + \sqrt{n^\theta + n} \ln(n^\theta + n)/4\pi.$$

**PROOF.** By using the result of Theorem 4 with a slight modification we have

$$\pi(m+n) - \pi(m) < n \ln(n) / (\ln(m+1)\ln(n)) + \sqrt{n^\theta + n} \ln(n^\theta + n)/4\pi. \quad (2.17)$$

By rearranging (1.8) and incorporating it into (2.17) we achieve the following:

$$\pi(m+n) - \pi(m) < \ln(n) \left[ \pi(n) - n/(2 \ln^2(n)) \right] / \ln(m+1) + \sqrt{n^\theta + n} \ln(n^\theta + n)/4\pi.$$

**THEOREM 5.** Let  $1 \leq \theta < 2$ . Let  $m, n$  be positive integers with  $m > 2657$  and  $m \geq n \geq m^{1/\theta}$ . If the Riemann hypothesis holds then  $\pi(m+n) - \pi(m) > n/\ln(m+n) - \sqrt{n^\theta + n} \ln(n^\theta + n)/4\pi$ .

**PROOF.** By using the upper and lower bounds of (1.7) we have

$$\pi(m+n) - \pi(m) > \text{Li}(m+n) - \text{Li}(m) - (\sqrt{m+n} \ln(m+n) + \sqrt{m} \ln(m))/8\pi. \quad (2.18)$$

Noting that  $\sqrt{m+n} \ln(m+n) > \sqrt{m} \ln(m)$  and using (1.6), then (2.18) will now become

$$\pi(m+n) - \pi(m) > \sum_{k=m+1}^{m+n} (1/\ln(k)) - \sqrt{m+n} \ln(m+n)/4\pi. \quad (2.19)$$

It is obvious that we can replace (2.19) by

$$\pi(m+n) - \pi(m) > n/\ln(m+n) - \sqrt{m+n} \ln(m+n)/4\pi.$$

Given that  $m \geq n \geq m^{1/\theta}$  for  $1 \leq \theta < 2$  we may conclude that

$$\pi(m+n) - \pi(m) > n/\ln(m+n) - \sqrt{n^\theta + n} \ln(n^\theta + n)/4\pi.$$

**COROLLARY 4.** Let  $1 \leq \theta < 2$ . Let  $m, n$  be positive integers with  $m > 2657$  and  $m \geq n \geq m^{1/\theta}$ . If the Riemann hypothesis holds, then

$$\pi(m+n) - \pi(m) > \ln(n)(\pi(n) - 3n/(2 \ln^2(n)))/\ln(m+n) - \sqrt{n^\theta + n} \ln(n^\theta + n)/4\pi.$$

**PROOF.** By using the result of Theorem 5 with a slight modification we have

$$\pi(m+n) - \pi(m) > n \ln(n)/(\ln(m+n) \ln(n)) - \sqrt{n^\theta + n} \ln(n^\theta + n)/4\pi. \quad (2.20)$$

By rearranging (1.9) and incorporating into (2.20) we achieve the following

$$\pi(m+n) - \pi(m) > \ln(n)(\pi(n) - 3n/(2 \ln^2(n)))/\ln(m+n) - \sqrt{n^\theta + n} \ln(n^\theta + n)/4\pi.$$

### 3. FINAL COMMENTS.

I feel that Theorem 1 and the Corollaries 1 and 3 are relevant to the disagreement between Erdős and Richards in their paper [4] dealing about whether the following conjecture is true.

$$\pi(x+y) - \pi(x) - \pi(y) < cy / \ln^2(y). \quad (3.1)$$

Of course, Theorem 1 states that (3.1) is true provided that for  $0 < d \leq 1$ ,  $x$  and  $y$  are sufficiently large and  $x \geq y \geq dx > 2$ . Under similar restrictions, Corollary 1 also states that (3.1) is true. Moreover, if we assume the conditions that are given in the Corollary 3 then we can give explicit bounds for which (3.1) is correct.

As for the mysterious person who told P. Erdős [12] that the "correct" conjecture should be  $\pi(x+y) \leq \pi(x) + 2\pi(y/2)$ , I claim to have made some progress in this direction. From Rosser, Schoenfeld and Yohe [13] we have  $\pi(2x) - \pi(x) < \pi(x)$ . If  $m \geq n$  then  $\ln(n) \pi(n)/\ln(m+1) < \pi(n) < 2\pi(n/2)$ . Hence with the restrictions found in the Corollary 1 we have  $\pi(m+n) \leq \pi(m) + 2\pi(n/2)$ .

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