MEASURABLE MULTIFUNCTIONS AND THEIR APPLICATIONS TO CONVEX INTEGRAL FUNCTIONALS

NIKOLAOS S. PAPAGEORGIOU

University of California
1015 Department of Mathematics
Davis, California 95616

ABSTRACT. The purpose of this paper is to establish some new properties of set valued measurable functions and of their sets of integrable selectors and to use them to study convex integral functionals defined on Lebesgue-Bochner spaces. In this process we also obtain a characterization of separable dual Banach spaces using multifunctions and we present some generalizations of the classical "bang-bang" principle to infinite dimensional linear control systems with time dependent control constraints.

KEY WORD AND PHRASES. Measurable multifunction, integrably bounded, measurable selection, Souslin space, Weak Randon-Nikodym property, set valued conditional expectation, infinite dimensional linear control systems, bang-bang principle, convex normal integrand, subgradient, subdifferential, conjugate, effective domain, absolutely continuous functional, singular functional, Yosida-Hewitt theorem.
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1. INTRODUCTION.

In the last decade the study of measurable set valued functions has been developed extensively, both in the theoretical direction and the direction of applications. Many mathematicians have contributed significant results in this area, which combines challenging theoretical problems with important applications in a variety of fields, like optimization theory, optimal control, statistics and mathematical economics. In all those areas the systematic use of multifunctions has allowed people to make significant progress and solve many problems.

With a series of recent papers [18] ÷ [25] the author has started an effort to extend the general theory of Banach space valued multifunctions and the closely related theory of multimeasures. The present paper continues this effort and provides some applications of the theoretical results obtained.

Briefly this paper is organized as follows. In the next section we establish our notation and for the convenience of the reader we recall some basic definitions and

facts from the general theory of multifunctions and the theory of measurable integrands. In section 3 we have gathered some results, in which starting from properties of the set of integrable selectors of a multifunction we extract information about its pointwise properties, the structure of its conditional expectation and the properties of the underlying Banach space. Some other related observations of functional analytic nature are also included. In section 4, we proceed to a detailed study of the properties of the set of integrable selectors of a multifunction and we present an application to control theory ("bang-bang" type results). Finally in section 5, we use the results obtained earlier in order to study convex integral functionals that appear often in problems of optimization, optimal control and mathematical economics. With this combination of theoretical and applied results, we want to emphasize the importance and the versatility of the theory of multifunctions and attract the interest of mathematicians from different areas.

2. PRELIMINARIES.

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this work we will be using the following notations:

$$P_{f(c)} = \{ A \subseteq X: \text{ nonempty, closed, (convex)} \}$$

$$P_{(w)k(c)}(X) = \{ A \subseteq X: \text{ nonempty, (w-) compact, (convex)} \}$$

Also we will be using the following additional three pieces of notation. Let

$A \varepsilon 2^X \setminus \{\emptyset\}$. By $|A|$ we will denote the norm of A i.e., $|A| = \sup_{\alpha \varepsilon A} ||\alpha||$, by

$\sigma(.,A)$ the support function of A i.e., $\sigma(x^*,A) = \sup_{\alpha \varepsilon A} (x^*,\alpha)$, $x^* \varepsilon X^*$ and by $d(.,A)$ the

distance function from A i.e., $d(x,A) = \inf_{\alpha \varepsilon A} ||x-\alpha||$.

A multifunction $F:\Omega \to P_f(X)$ is said to be measurable if for every $x \varepsilon X$, the function $\omega \to d(x,F(\omega))$ is measurable. This definition is equivalent to saying that there exist $f_n: \Omega \to X$ measurable functions s.t. for every $\omega \varepsilon \Omega$

$F(\omega) = cl \{f_n(\omega)\}_{n \geqslant 1}$ ("Castaing's representation" - see Castaing - Valadier [5]).

A function $f: \Omega \to X$ s.t. $f(\omega) \varepsilon F(\omega)$ is said to be a selector of $F(.)$. The problem of existence of measurable selectors is central in the theory of multifunctions. In applications the most widely used selection theorem, is the following one which was first proved by Aumann [1] for Polish spaces and was later extended to Souslin spaces by Saint-Beuve [30]. By $\bar{\Sigma}$ we will denote the universal σ-field corresponding to Σ.

THEOREM 2.1 [30]. If X is a Souslin space and $F:\Omega \to 2^X \setminus \{\emptyset\}$ is a multifunction s.t. $GrF=\{(\omega,x) \varepsilon \Omega xX:x \varepsilon F(\omega)\} \varepsilon \Sigma xB(X)$, where B(X) is the Borel σ-field of X, then there exist $f_n: \Omega \to X$, $\bar{\Sigma}$-measurable selectors of $F(.)$ s.t. $F(\omega) \subseteq cl\{f_n(\omega)\}_{n \geqslant 1}$ for all $\omega \varepsilon \Omega$

REMARKS. a) If (Ω, Σ, μ) is a σ-finite complete measure space, then $\Sigma = \hat{\Sigma}$.
b) Recall that a Souslin space is always separable, but it need be metrizable (for
example a separable Banach space with the weak topology). c) If $F(.)$ is closed valued
and measurable in the sense defined earlier then $GrF \in \Sigma xB(X)$ (graph measurablity).
The converse is true if $\Sigma = \hat{\Sigma}$ (i.e. Σ is complete).

Let (Ω, Σ, μ) be a σ-finite measure space and let $S_F^1 = \{f(.) \in L_X^1 : f(\omega) \in F(\omega)$.
μ-a.e$\}$.
Using S_F^1 we can define a set valued integral for $F(.)$ by

$$\int_\Omega F(\omega) d\mu(\omega) = \{\int_\Omega f(\omega) d\mu(\omega) : f \in S_F^1\}.$$ We say that $F:\Omega \to P_f(X)$ is integrably bounded

if it is measurable and $\omega \to |F(\omega)| \in L_+^1$. Using theorem 2.1 we can see that if $F(.)$

is integrably bounded then S_F^1 and $\int_\Omega F$ are both nonempty.

Let $K \subseteq L_X^1$ be nonempty. We say that K is decomposable (also known as "convex
with respect to switching") if for all $A \in \Sigma$ and all $(f_1, f_2) \in KxK, \chi_A f_1 + \chi_{A^c} f_2 \in K$.
In [9], theorem 3.1., Hiai-Umegaki proved that if K is closed and decomposable, then
there exists $F:\Omega \to P_f(X)$ measurable s.t. $K = S_F^1$. Furthermore if K is bounded, then
$F(.)$ is integrably bounded. Using this fact, Hiai-Umegaki [9], went on and defined as
set valued conditional expectation for $F(.)$ as follows. Let Σ_0 be a
sub-σ-field of Σ let $F:\Omega \to P_f(X)$ be measurable with $S_F^1 \neq \emptyset$ Define
$K = cl\{E^{\Sigma_0}f : f \in S_F^1\}$. Then K is Σ_0-decomposable and so there exists

$E^{\Sigma_0}F : \Omega \to P_f(X) \Sigma_0$ -measurable s.t. $K = S_{E^{\Sigma_0}F}^1$. The multifunction $E^{\Sigma_0}F(.)$ is the
set valued conditional expectation of $F(.)$.

Next let (Ω, Σ, μ) be a complete, σ-finite measure space and X a separable Banach
space. Let $f:\Omega xX \to RU\{+\infty\}, f \not\equiv +\infty$. Following Rockafellar [28] we say that $f(.,.)$ is
a normal integrand if the following conditions are satisfied:

        i) $(\omega, x) \to f(\omega, x)$ is $\Sigma xB(X)$ -measurable

        ii) for all $\omega \in \Omega$, $x \to f(\omega, x)$ is lower semicontinuous
                        (l.s.c)
Using the celebrated "Von Neumann projection theorem" (see Castaing - Valadier
[5], theorem III - 23) we can show that the above two conditions are equivalent to the
following. Recall that if $g:X \to RU\{+\infty\}$, then $epig = \{(x, \lambda) \in XxR : f(x) \leq \lambda\}$.
        i') the multifunction $\omega \to epif(\omega, .)$ is graph measurable
        ii') for all $\omega \in \Omega$, $epif(\omega, .) \in P_f(XxR)$

Note that ii') immediately implies that $\omega \to epif(\omega, .)$ is measurable. Because of
condition i) a normal integrand $f(.,.)$ is superpositionally measurable i.e. if
$x:\Omega \to X$ is measurable, then $\omega \to f(\omega, x(\omega))$ is measurable. A well known example of
normal integrands are the Caratheodory integrands. A normal integrand $f(.,.)$ is said

to be convex, if for all $\omega\epsilon\Omega$, $f(\omega,.)$ is convex.

Let us also recall some notions from convex analysis. Let $f \epsilon \bar{R}^X$. We define $\partial f(x) = \{x^*\epsilon X^* : (x^*,y-x) \leqslant f(y)-f(x)$ for all $y\epsilon X\}$. This is called the subdifferential of $f(.)$ at x. Also we define $f^*\epsilon\bar{R}^{X^*}$ by $f^*(x^*)=\sup\{(x^*,x)-f(x); x\epsilon X\}$ and this is known as the conjugate of $f(.)$. The conjugate and the subdifferential are related by the Young-Fenchel equality, namely: $x^*\epsilon\partial f(x)$ if and only if $f^*(x^*)+f(x) = (x^*,x)$. Since we are dealing here with extended real valued functions we define the effective domain of f to be: domf = $\{x\epsilon X:f(x)<\infty\}$.

Finally, recall that X has the weak Randon-Nikodym property (WRNP) if it has the Radon-Nikodym property for the Pettis integral.

3. MEASURABLE MULTIFUNCTIONS.

We start with a result in which, knowing the structure of the set of integrable selectors of a multifunction, we deduce some pointwise properties of the multifunction. Another such result was obtained by the author in [25] (theorem 5.1).

Assume that (Ω,Σ,μ) is a complete, σ -finite measure space and X a weakly sequentially complete, separable Banach space.

THEOREM 3.1. If X* has the WRNP and S_F^1 nonempty, bounded, closed and convex.

then $F(\omega)\epsilon P_{wkc}(X)$ μ -a.e.

PROOF. From Hiai-Umegaki [9] (theorem 3.2) we know that $F(.)$ is integrably bounded and so for all $\omega\epsilon\Omega\backslash N, \mu(N)=0$, $F(\omega)$ is bounded.

Suppose that for some $\omega\epsilon\Omega\backslash N, F(\omega)$ is not w-compact. Then the Eberlein-Smulian theorem and the fact that X is w-sequentially complete give us a sequence $\{x_n\}_{n\geqslant 1}$ with no Cauchy subsequence. Recalling that $\{x_n\}_{n\geqslant 1}$ is bounded, we can apply the result of Rosenthal [29] and deduce that $\{x_n\}_{n\geqslant 1}$ is an ℓ^1 -sequence. Hence $\ell^1\hookrightarrow X$ a contradiction to the fact that X* has the WRNP (see Musial-Ryll Nardzewski [17]).

REMARK. If X* has the RNP, then the result is immediate, since X is reflexive (see Diestel -Uhl [7], corollary 11, p. 198). But we know that in general the WRNP does not imply the RNP.

In fact, when X is a Banach lattice we can have a partial converse of the above theorem. So assume that (Ω,Σ,μ) is a complete, σ -finite measure space and X a separable Banach lattice.

THEOREM 3.2. If the following implication holds: " S_F^1 is nonempty, convex and w-compact ==> $F(\omega)\epsilon P_{wkc}(X)$ μ -a.e.", then X* is separable and w-sequentially complete.

PROOF. We will show that $\ell_1 \not\hookrightarrow X$. Suppose not. Then from our hypothesis if

$S_F^1 \subseteq L^1_{\ell^1}(\Omega)$ is nonempty, convex and w-compact then $F(\omega)\epsilon P_{wkc}(\ell^1)$ μ -a.e. Set M(A)

= $\{\int_A f(\omega)d\mu(\omega): f\epsilon S_F^1\}$, A $\epsilon\Sigma$. Then M(.) is a $P_{wkc}(\ell^1)$ -valued multimeasure with a

$P_{wkc}(\ell^1)$ —valued density, a contradiction to example 2 of Costé [6]. So $\ell_1 \not\hookrightarrow X$ and

then from a result of Lotz (see Diestel-Uhl [7], p.95) we deduce that X^* has the

RNP. Since X is separable from corollary 8, p.98 of Diestel-Uhl [7], we deduce that

X* is separable. Also $c_o \not\hookrightarrow X*$ and so theorem 1, c.4 of Lindenstrauss – Tzafriri

[16] tells us that X* is w-sequentially complete.   Q.E.D.

Also we have a weak compactness result for the set of integrable selectors of a

multifunction. Another such result can be found in [25]. But first we will need a

property of decomposable subsets of L^1_X, which was proved by the author in [26]

(Proposition 5.1 - see also Diestel-Uhl [7], Theorem 4, p. 104). For the convenience

of the reader we recall the result here. Let (Ω,Σ,μ) be a σ-finite measure space

and X a Banach space.

PROPOSITION 3.1 [26]. If $K \subseteq L^1_X$ is nonempty, decomposable and bounded then K is

uniformly integrable.

Now assume that in addition X is weakly sequentially complete.

THEOREM 3.3. If $K \subseteq L^1_X$ is nonempty, decomposable, bounded and w-closed, with no

ℓ^1-sequence then K is w-compact.

PROOF. From Proposition 4.1 we know that K is uniformly integrable. Also since X

is weakly sequentially complete, so is L^1_X (see Talagrand [31]). Combining these

facts with Corollary 8 of Bourgain [4] and the Eberlein-Smulian theorem we get that K

is w-compact.   Q.E.D.

REMARK. If X is separable then $K = S^1_F$ where $F:\Omega \to P_f(X)$ is integrably

bounded. So indeed our result is a w-compactness result for the set of integrable

selectors of a multifunction.

4. PROPERTIES OF THE SET OF INTEGRABLE SELECTORS.

This section is devoted to a detailed study of the properties of the set of

integrable selectors of measurable (or graph measurable) multifunctions. Those

results are then applied to the analysis of a family of infinite dimensional linear

control systems with time dependent control constraints. The material of this section

will also be used in the next section, in the study of convex integral functionals.

We will start with an auxilliary result that we will need in what follows and

which also generalizes Theorem 1.5 of Hiai-Umegaki [9].

Assume that (Ω,Σ,μ) is a complete, σ-finite measure space and X a separable

Banach space.

LEMMA A. If $F:\Omega \to 2^X \setminus \{\emptyset\}$ is graph measurable and $S^1_F \neq \emptyset$ then $\overline{conv}\ S^1_F = S^1_{\overline{conv}\ F}$

PROOF. Clearly $\overline{conv}\ S^1_F \subseteq S^1_{\overline{conv}\ F}$. Suppose that the inclusion is strict so

we can find $f \in S^1_{\overline{conv}\ F}$ s.t. $f \notin \overline{conv}\ S^1_F$. From the strong separation theorem there

exists $u(.) \in L^\infty_{X^*_{w^*}} = [L^1_X]^*$ s.t. $\sigma(u,\overline{conv}S^1_F) < \langle u,f \rangle$

But from [23] we know that:

$$\sigma(u,\overline{conv}S_F^1) = \sigma(u,S_F^1) = \sup_{h \in S_F^1} \int_\Omega (u(\omega),h(\omega))d\mu(\omega)$$

$$\int_\Omega \sup_{x \in F(\omega)} (u(\omega),x)d\mu(\omega) = \int_\Omega \sigma(u(\omega),F(\omega))d\mu(\omega)$$

Hence $\sigma(u,\overline{conv}S_F^1) < \int_\Omega u(\omega),f(\omega))d\mu(\omega)$

On the other hand since $f(.) \in S_{\overline{convF}}^1$, we have $f(\omega) \in \overline{conv} F(\omega) \mu$-a.e.

$(u(\omega),f(\omega)) \leqslant \sigma(u(\omega),F(\omega))\mu$-a.e $\Longrightarrow \int_\Omega (u(\omega),f(\omega))d\mu(\omega) \leqslant \int_\Omega \sigma(u(\omega),F(\omega))d\mu(\omega) \Longrightarrow$

a contradiction.   Q.E.D.

Using this lemma we can have the following Lyapunov type result. So we assume that (Ω,Σ,μ) is a complete, σ-finite, nonatomic measure space and X a separable Banach space.

THEOREM 4.1. If $F:\Omega \to 2^X \setminus \{\emptyset\}$ is graph measurable and $S_F^1 \neq \emptyset$ then $\overline{S_F^1}^w = S_{\overline{conv} F}^1$ (here w indicates the weak topology on L_X^1).

PROOF. Note that $S_{\overline{convF}}^1$ is a closed, convex set. Hence we immediately have:

$$\overline{S_F^1}^w \subseteq S_{\overline{convF}}^1 = \overline{conv} \, S_F^1 \text{(lemma a)} \tag{4.1}$$

Next let $g \in conv \, S_F^1$, $g = \sum_{i=1}^n \lambda_i f_i, f_i \in S_F^1$. Let $V(g,\{u_k\}_{k=1}^m,\varepsilon)$ be a weak neighborhood of g. So

$$V = \{h \in L_X^1 : |\langle u_k,g-h \rangle| < \varepsilon, k=1,\ldots,m\}$$

where $u_k \in L_{X^*_{w^*}}^\infty$ and $\varepsilon > 0$. Let $L:L_X^1 \to R^m$ be defined by

$$L(h) = \langle u_k, h \rangle_{k=1}^m$$

Clearly $L(.)$ is a continuous linear operator. We claim that $L(S_F^1)$ is convex. Let $f_1, f_2 \in K$ and consider $A \to m(A)=L(\chi_A(f_1-f_2))$. It is easy to see that $m(.)$ is a vector measure of bounded variation which is μ-continuous. So applying Lyapunov's theorem we get that Range(m) is

convex $\Longrightarrow \bigcup_{A \in \Sigma} L(\chi_A(f_1-f_2))+L(f_2) = \bigcup_{A \in \Sigma} L(\chi_A f_1 + \chi_{A^c} f_2)$

is convex. But since K is decomposable.

$$\bigcup_{A \in \Sigma} L(\chi_A f_1 + \chi_{A^c} f_2) \subseteq L(S_F^1) \text{ and } L(f_1),L(f_2) \in \bigcup_{A \in \Sigma} L(\chi_A f_1 + \chi_{A^c} f_2).$$

So indeed $L(S_F^1)$ is convex.

Thus we can find $f \in S_F^1$ s.t.

$$L(g) = \{ \sum_{i=1}^{n} \lambda_i \langle u_k, f_i \rangle \}_{k=1}^{m} = L(f).$$

Therefore $V(g, \{u_k\}_{k=1}^{m}, \varepsilon\} \cap S_F^1 \neq \emptyset$ and so.

$$\overline{S_F^1}^w \quad \overline{\text{conv}} \; S_F^1 \qquad\qquad (4.2)$$

Combining (4.1) and (4.2) we finally have:

$$\overline{S_F^1}^w = \overline{\text{conv}} \; S_F^1 \qquad\qquad \underline{Q.E.D.}$$

An immediate important consequence of theorem 4.1 is the following result. The spaces remain as before.

THEORM 4.2. <u>If</u> $F:\Omega \to P_{wkc}(X)$ is measurable and S_F^1, $S_{extF}^1 \neq \emptyset$ then $\overline{S_{extF}^1}^w = S_F^1$.

PROOF. From Benamara [2] we know that $\omega \to extF(\omega)$ is graph measurable. So applying Theorem 4.1 we get that $\overline{S_{extF}^1}^w = S_{\overline{\text{convextF}}}^1 = S_F^1$ (Krein-Milman theorem).

Another interesting consequence is the following result about the set valued conditional expectation. Let Σ_0 be a sub-σ-field of Σ and assume that $\mu(.)$ has no Σ_0-atoms.

THEOREM 4.3. <u>If</u> $F:\Omega \to P_f(X)$ is integrably bounded and S_F^1 is w-compact in L_X^1 <u>then</u> $E^{\Sigma_0}F(\omega)$ is μ-a.e. convex.

PROOF. From Theorem 4.1 we have that $\cdot \overline{S_{E^{\Sigma_0}F}^1}^w$ is convex. Since $E^{\Sigma_0}: L_X^1 \to L_X^1(\Sigma_0)$ is continuous linear, $E^{\Sigma_0}(S_F^1)$ is w-compact $\Longrightarrow S_{E^{\Sigma_0}F}^1$ is w-compact in $L_X^1(\Sigma_0)$. Therefore $S_{E^{\Sigma_0}F}^1$ is convex $E^{\Sigma_0}F(\omega)$ is μ.a.e. convex.

Now we examine the strong closure of S_F^1. So let (Ω, Σ, μ) is a complete, σ-finite measure space and X a separable Banach space.

THEOREM 4.4. <u>If</u> $F:\Omega \to 2^X \setminus \{\emptyset\}$ is graph measurable and $S_F^1 \neq \emptyset$ <u>then</u> $\overline{S_F^1} = S_{\overline{F}}^1$

PROOF. Since $S_{\overline{F}}^1$ is closed in L_X^1, we get that $\overline{S_F^1} \subseteq S_{\overline{F}}^1$. Let $f(.)\epsilon S_{\overline{F}}^1$ For $n \geqslant 1$ let $L_n:\Omega \to 2^X \setminus \{\emptyset\}$ be defined by:

$$L_n(\omega) = \{x\epsilon F(\omega): ||x-f(\omega)|| < \frac{1}{n}\}$$

$$GrL_n = \{(\omega,x)\epsilon\Omega x X: ||x-f(\omega)|| < \frac{1}{n}\} \cap GrF$$

Note that $(\omega,x) \to ||x-f(\omega)||$ is a Caratheodory function and so it is jointly

measurable. Hence $\{(\omega,x)\in\Omega\times X: ||x-f(\omega)|| < \frac{1}{n}\} \in \Sigma\times B(X)$.   Also by hypothesis

$GrF\in\Sigma\times B(X)$.   Therefore for all $n\geq 1$, $GrL_n\in\Sigma\times B(X)$.   Apply Theorem 2.1 to find

$f_n:\Omega \to X$ measurable s.t. $f_n(\omega)\in L_n(\omega)$ for all $\omega\in\Omega$, $n\geq 1$.   Then clearly for all

$\omega\in\Omega$, $f_n(\omega) \overset{s}{\to} f(\omega)$.   Since $||f_n(\omega)|| < ||f(\omega)|| + 1$, applying   the   dominated

convergence theorem, we get that

$$f_n \xrightarrow{\ s-L_X^1\ } f \implies f\in\overline{S_F^1} \implies \overline{S_F^1} = S_F^1 .$$ Q.E.D.

The result has an interesting consequence. However before passing to it, we need
to have the following lemma. It generalizes a similar result of Hiai-Umegaki [9], who
required the multifunctions to be closed valued (see Corollary 1.2 of [9]). The
spaces remain as before.

LEMMA   B.   If $F_1,F_2:\Omega\to 2^{X}\smallsetminus\{\emptyset\}$ are   graph   measurable   and $S_{F_1}^1 = S_{F_2}^1 \neq \emptyset$ then
$F_1(\omega) = F_2(\omega)$ μ-a.e.

PROOF. Suppose not. Then there exists $A\in\Sigma$ with $\mu(A)>0$ s.t. $F_1(\omega)\smallsetminus F_2(\omega) \neq 0$ for
all $\omega\in A$. Let $R:A \to 2^{X}\smallsetminus\{0\}$ be defined by $R(\omega) = F_1(\omega)\smallsetminus F_2(\omega)$. Then $GrR\in\Sigma_A\times B(X)$
(where $\Sigma_A = \Sigma$ A). Apply theorem 2.1. to find $g:A \to X$ measurable s.t.
$g(\omega) \in R(\omega)$ for all $\omega\in A$. Let $\{\Omega_n\}_{n\geq 1}$ be a Σ-partition of Ω s.t. $\mu(\Omega_n)<\infty$.
Define

$$C_{mn} = \{\omega\in A: m-1 < ||g(\omega)|| < m\}\ \Omega_n$$

Then clearly $\{C_{mn}\}_{m,n\geq 1}$ is a Σ-partition of A. Since $\mu(A) > 0$ we can

find $m,n\geq 1$ s.t. $\mu(C_{mn}) > 0$.   Set

$$f(\omega) = \begin{cases} g(\omega) \text{ for } \omega\in C_{mn} \\ f(\omega) \text{ for } \omega\in\Omega\smallsetminus C_{mn} \end{cases}$$

where $f(.)\in S_{F_1}^1$.   Then because of the decomposability of $S_{F_1}^1$ $f\in S_{F_1}^1$, while $f\notin S_{F_2}^1$.
This produces the derived contradiction.   Q.E.D.

Now we are ready for the theorem.
THEOREM 4.5.   If $F:\Omega \to 2^{X}\smallsetminus\{\phi\}$ is graph measurable and S_F^1 is nonempty and closed
in L_X^1

then $F(\omega) \in P_f(X)$ μ-a.e.

PROOF. From Theorem 4.4 we have that $\overline{S_F^1} = S_{\overline{F}}^1 = S_F^1$. Apply lemma β to get
that $F(\omega) = \bar{F}(\omega)$ μ-a.e. $\implies F(.)$ is valued μ-a.e.   Q.E.D.

Now we will apply the results of this section to obtain versions of the "bang-bang
principle", for infinite dimensional linear control systems with time dependent
control constraints. So consider the following system:

$$\dot{x}(t) = A(t)x(t)+B(t)u(t)$$ (4.3)

$$x(o) = x_o, \ u(t)\in U(t) \text{ a.e, } u(.)\in L_X^1$$

Here $t \epsilon R_+$, $A(t)$ is an unbounded, linear operator and $B(.) \epsilon L_{loc}^{\infty}(R_+; L(X))$, where $L(X)$ is the space of all bounded linear operators from X into itself. We will assume that the linear evolution problem $\dot{x}(t) = A(t)x(t)$, $x(o) = x_0$ admits a fundamental solution $\Phi: \{(t,s): 0 \leqslant s \leqslant t\} \to L(X)$. Conditions on $A(.)$ that guarantee the existence of $\Phi(.,.)$ can be found in Kato [12]. Then if $B(.)u(.) \epsilon L_{loc}^1 (R_1, X)$, we know that (4.3) has a mild solution $x_u(.) \epsilon C_X(R_+)$ given by:

$$x_u(t) = \Phi(t,o)x_0 + \int_o^t \Phi(t,s)B(s)u(s)ds, \ t \epsilon R_+$$

Let $R(t)$ be the set of the attainable points of system (4.3) using all feasible controls and let $R_e(t)$ be the set of attainable points of (4.3) using extremal controls (i.e. $u(t) \epsilon extU(t)$ a.e.).

The next theorem establishes the relation between those two sets and can be viewed as an infinite dimensional generalization of the classical "bang-bang principle" (see Hermes - LaSalle [8]).

THEOREM 4.6. <u>If</u> $U: R_+ \to R_{wkc}(X)$ is measurable and S_U^1, $S_{extU}^1 \neq 0$

<u>then</u> for all $t \epsilon R_+$, $\overline{R_e(t)}^w = \overline{R(t)}^w$ = convex

PROOF. Clearly we need to show that $R(t) \underline{C} \overline{R_e(t)}^w$. So let $x \epsilon R(t)$. By definition there exists $u(.) \epsilon S_U^1$ s.t.

$$x(t) = \Phi(t,o)x_0 + \int_o^t \Phi(t,s)B(s)u(s)ds$$

From Theorem 4.2 we know that there exists net $u_b(.) \epsilon S_{extU}^1$ s.t. $u_b \xrightarrow{w-L_X^1} u$. Then for every $x^* \epsilon X^*$ we have:

$$\int_o^t (x^*, \Phi(t,s)B(s)u_b(s))ds = \int_o^t (B^*(s)\Phi^*(t,s)x^*, u_b(s))ds$$

Note that $||B^*(s)\Phi^*(t,s)x^*|| \leqslant ||B^*(s)|| \ ||\Phi^*(t,s)|| \ ||x^*||$.
Also from the properties of $\Phi(.,.)$ (see Kato [12]), we have
$||\Phi(t,s)|| = ||\Phi^*(t,s)|| \leqslant M$, while by hypothesis $||B(s)|| = ||B^*(s)||$

$\leqslant ||B||_{L^{\infty}[0,t]}$.

Hence $s \to B^*(s)\Phi^*(t,s)x^*$ belongs in $L_{X^*}^{\infty} [0,t]$ and so we have:

$$\int_0^t (B^*(s)\Phi^*(t,s)x^*, u_b(s))ds \xrightarrow{w^*} \int_0^t (B^*(s)\Phi^*(t,s)x^*,u(s))ds$$

$$\Longrightarrow \qquad\qquad = \int_o^t (x^*, \Phi(t,s)B(s)u(s))ds$$

$$\Longrightarrow \int_o^t \Phi(t,s)B(s)u_b(s)ds \xrightarrow{w} \int_o^t \Phi(t,s)B(s)u_b(s)ds$$

Set $x_b = \Phi(t,o)x_0 + \int_o^t \Phi(t,s)B(s)u(s)ds$

Then

$\Rightarrow x_b \xrightarrow{w} x$. Clearly $x_b \epsilon R_e(t) \Longrightarrow x(t) \overline{\epsilon R_e(t)}^w \Longrightarrow \overline{R_e(t)}^w = \overline{R(t)}^w$.

The convexity of the set is clear from the convexity of the values of U(.).   Q.E.D.

With an $L^1_{loc}(R_+)$- boundedness hypothesis on the control constraint multifunction

u(.), we can improve the conclusion of Theorem 4.6.

THEOREM 4.7.   If U:R$_+$ → R$_{wkc}$(X) is measurable and $\left|U(.)\right| \in L^1_{loc}(R_+)$

then for all t∈R$_+$, R(t) = $\overline{R_e(t)}^w \in P_{wkc}(X)$.

PROOF. By definition R(t)=Φ(t,o)x$_0$ + $\int_o^t$ Φ(t,s)B(s)U(s)ds.
Note that for all x*∈X*

$$\sigma(x^*, \Phi(t,s)B(s)U(s)) = \sigma(B^*(s)\Phi^*(t,s)x^*, U(s))$$

Also since U(.) is P$_{wkc}$(X)—valued and measurable,(s,z*) → σ(z*,u(s)) is a
Caratheodory function from Ω×X* into R.   Therefore it is jointly measurable and
so s → σ(B*(s)Φ*(t,s)x*,U(s)) is measurable.   Invoking theorem III-37 of Casting-
Valadier [5], we conclude that s → Φ(t,s)B(s)U(s) is a P$_{wkc}$(X)-valued, integrably
bounded multifunction.   So we can apply proposition 3.1 of [18] (see also [22]) and
get that

$$\int_o^t \Phi(t,s)B(s)U(s)ds \in P_{wkc}(X) \implies R(t) \in P_{wkc}(X)\ t \in R_+.$$

Q.E.D.

When X is a finite dimensiional Banach space, we obtain an extension of LaSalle's
"bang-bang principle" (see Hermes-LaSalle [8]),to linear control systems with time
dependent, nonconvex control constraints.   Recall that if F:Ω → 2^X\{∅} is graph
measurable, $S^1_F \neq \emptyset$ and μ(.) is nonatomic, then $\int_\Omega F$ is convex (see Klein-Thompson [13],

Theorem 17.1.6 and for a generalization to Banach spaces [23]).

THEOREM 4.8.   If U:R$_+$ → P$_f$(X) is measurable and $\left|U(.)\right| \in L^1_{loc}(R_+)$

then for all t∈R$_+$, $\overline{R_e(t)}$ = R(t)∈P$_{kc}$(X).

5.   CONVEX INTEGRAL FUNCTIONALS ON LEBESGUE-BOCHNER SPACES.

In this section we use the theoretical results obtained previously, to conduct a
study of convex integral functionals which are defined on Lebesgue-Bochner spaces.
Our work in this section extends earlier results of Rockafellar [27] (finite
dimensions) and Bismut [3] (finite dimensional or separable, reflexive Banach spaces).

It is well known that if X is finite dimensional space and an integral functional
is weakly lower semicontinuous on L^1_X, then the integrand is automatically convex in
the state variable (See Bismut [3], Theorem 1 and Rockafellar [27] Theorem 1). Here we
extend this result to separable Banach spaces and we present a different, simpler
proof using Theorem 4.1.

First we need a Lemma.   Assume that (Ω,Σ,μ) is a complete, σ-finite, nonatomic
measure space and X a separable Banach space.

LEMMÀ A.   If F:Ω → 2^X\{∅} is graph measurable and $S^1_F \neq 0$,

then S_F^1 is w-closed if an only if $F(\omega)\varepsilon P_{fc}(X)$ μ-a.e.

PROOF. First assume that S_F^1 is w-closed in L_X^1. Then from Theorem 4.1. we have $S_F^1 = S_{\overline{convF}}^1$. Using Lemma B of section 4 we conclude that

$$F(\omega) = \overline{convF}(\omega) \;\; \mu\text{-a.e.} \implies F(\omega)\varepsilon P_{fc}(X) \;\; \mu\text{-a.e.}$$

Now assume that $F(\omega)\varepsilon P_{fc}(X)$ μ-a.e. Then S_F^1 is closed and convex. So it is w-closed.   Q.E.D.

Now we are ready for our theorem. The spaces are as above.

Also if $f:\Omega \times X \to \overline{R}$ is an integrand for $x:\Omega \to X$ measurable we set $I_f(x) = \int_\Omega f(\omega,x(\omega))d\mu(\omega)$ (if the integral is not defined then we set $I_f(x) = +\infty$).

THEOREM 5.1. If $f:\Omega \times X \to \overline{R}$ is a normal integrand s.t.

1) there exist $x_0(.)\varepsilon L_X^1$ s.t. $I_f(x_0) < \infty$,

b) $I_f(.)$ is $w(L_X^1, L_{X^*_{w*}}^\infty)$ -lower semicontinuous,

then $f(\omega,.)$ is μ-a.e convex

PROOF. Let $E:\Omega \to 2^{X}\setminus\{\emptyset\}$ be the multifunction definted by $E(\omega) = epif(\omega,.)$. Since the integrand $f(.,.)$ is normal $E(.)$ is closed valued and measurable. Also note that $(x_0(.), f^+(.,x_0(.)))\varepsilon S_E^1 \neq /0$. We claim that S_E^1 is w-closed in $L_{X\times R}^1(\Omega)$. So let $(x_b,\lambda_b) \xrightarrow{w-L_{X\times R}^1} (x,\lambda)$. Then for all $A\varepsilon\Sigma$ we have:

$$I_f^A(x_b) = \int_A f(\omega,x_b(\omega))d\mu(\omega) < \int_A \lambda_b(\omega)d\mu(\omega)$$

Note that $I_f^A(z) = I_f(\chi_A z + \chi_{A_c}x_0) - \int_{A_c} f(\omega,x_0(\omega))d\mu(\omega)$

and $\int_{A_c} f(\omega,x_0(\omega))d\mu(\omega) < \infty$. Also $z(.) \to (\chi_A z + \chi_{A_c}x_0)$ is affine continuous.

So $z \to I_f^A(z)$ is w-l.s.c.   Hence

$$I_f^A(x) < \lim I_f^A(x_b) < \int_A \lambda(\omega) d\mu(\omega) \implies f(\omega,x(\omega)) < \lambda(\omega) \;\mu\text{-a.e.} \implies (x,\lambda)\varepsilon S_E^1$$

So indeed S_E^1 is w-closed in $L_{X\times R}^1$. Applying Lemma A we get that $E(\omega) \varepsilon P_{fc}(X)$ μ-a.e $\implies f(\omega,.)$ is μ-a.e. convex.   Q.E.D.

Now we pass to the subdifferential of $I_f(.)$. So assume that (Ω,Σ,μ) is a complete, σ-finite, measure space and X a separable Banach space. We will need the decomposition theorem for $[L_X^\infty]^*$. This result was first proved by Yosida-Hewitt [32] for X=R (see also Rockafellar [27]). Then it was extended to separable Banach spaces by Ioffe-Levin [10] and Rockafellar [28] and later to nonseparable Banach space by

Levin [15]. A functional $u(.) \epsilon [L_X^\infty]^*$ is said to be absolutely continuous with respect to μ, if there exists $g \epsilon L^1_{X^*_{w^*}}$ s.t.

$$< u,x > = \int_\Omega (g(\omega)), x(\omega))d\mu(\omega) \text{ for all } x(.) \epsilon L_X^\infty.$$

A functional $v \epsilon [L_X^\infty]^*$ is said to be singular with respect to μ if there exist $\{\Omega_n\}_{n \geqslant 1} \subseteq \Sigma$ s.t.

i) $\Omega_{n+1} \subseteq \Omega_n$ $n \geqslant 1$ ii) $\mu(\Omega_n \cap A) \to 0$ for all $A \epsilon \Sigma$, $\mu(A) < \infty$ and

iii) $< v,x > = 0$ for all $x \epsilon L_X^\infty$ s.t. $x|_{\Omega_n} = 0$ for some $n \geqslant 1$.

THEOREM 5.2. [15] Every functional $y \epsilon [L_X^\infty]^*$ admits a unique decomposition

$$y = y_a + y_s \text{ where } y_o \text{ is absolutely continuous and } y_s \text{ is singular with}$$

respect to μ. Furthermore $||y|| = ||y_0|| + ||y_0||$.

THEOREM 5.3. **If** $f:\Omega \times X \to \bar{R}$ is a convex, normal integrand and $I_f(.)$ is strongly continuous on L_X^∞ at $x_o(.)$

then $\partial I_f(x_o) \subseteq L^1_{X^*_{w^*}}$ is nonempty and $w(L^1_{X^*_{w^*}}, L_X^\infty)$-compact.

Furthermore **if** X^* is separable **then** $\omega \to \partial f(\omega, x_o(\omega))$ is a $P_{wkc}(X)$-valued integrably bounded multifunction.

PROOF. Let $x^* \epsilon [L_X^\infty]^*$ and let $x^* = x^*|_a + x^*|_s$ be its decomposition according to Theorem

5.2. From Levin [15], Theorem 6.4, we know that:

$$(I_f)^*(x^*) = I_{f*}(x_a^*) + \sigma(x_s^*, dom I_f)$$

Since $I_f(.)$ is s-continuous at x_o, $\partial I_f(x_o) \neq \emptyset$.

Let $x^* \epsilon \partial I_f(x_o)$. Then by definition we have:

$$<x_a^* + x_s^*, x_o> = I_f(x_o) + (I_f)^*(x^*) = I_f(x_o) + I_{f*}(x_a^*) + \sigma(x_s^*, dom I_f) \quad (4.4)$$

Note that because of the continuity of $I_f(.)$ at $x_o, x_o \epsilon$ intdom I_f and so if $x_s^* \neq 0$ then

$$< x_s^*, x_o > < \sigma(x_s^*, dom I_f)$$

which when used back in (4.4) produces a contradiction.

Therefore $x_s^* = 0 \Longrightarrow x^* = x_a^* \Longrightarrow \partial I_f(x_o) \subseteq L^1_{X^*_{w^*}}$.

Also the continuity at $x_0(.)$ tells us that $\partial I_f(x_0)$ is w^*-compact in $[L_X^\infty]^*$. But the restriction of the w^*-topology on $L_{X^*_{w^*}}^1$ coincides with the $w(L_{X^*_w}^{1*}, L_X^\infty)$-topology. Therefore $\partial I_f(x_0)$ is $w(L_{X^*_w}^{1*}, L_X^\infty)$-compact.

Now, if X^*, is separable, then $L_{X^*_w}^{1*} = L_{X^*}^1$ (see Ionescu-Tulcea [11]).

Also from Rockafellar [28], we know that $\partial I_f(x_0) = S_{\partial f(.,x_0(.))}^1$. So applying

proposition 5.1 of [25], we conclude that $\omega \to \partial f(\omega, x_0(\omega))$ is $P_{wkc}(X)$-valued integrably bounded multifunction.   Q.E.D.

REMARK. Our result generalizes Theorem 2 and its corollary in Bismut [3]. In this paper X was assumed to be separable, reflexive.

Next we look at some special type of subgradients namely extremal subgradients. This spaces are as above.

THEOREM 5.4. If $f:\Omega \times X \to \overline{R}$ is a convex, normal integrand and $I_f(.)$ is strongly

continuous on L_X^∞ at $x_0(.)$, then $ext \partial I_f(x_0) \neq \emptyset$ and for all $u \in ext \partial I_f(x_0)$,

$u(\omega) \in ext \partial f(\omega, x_0(\omega))$ μ-a.e.

PROOF. We know that $\partial I_f(x_0)$ is $w(L_{X^*_w}^{1*}, L_X^\infty)$-compact. So by the Krein-Milman

theorem we have that $ext \partial I_f(x_0) \neq \emptyset$. Also $\partial I_f(x_0) = S_{\partial f(.,x_0(.))}^1 \implies ext \partial I_f(x_0)$

$= ext S_{\partial f(.,x_0(.))}^1$. But from Benamara [2], $ext S_{\partial f(.,x_0(.))}^1 = S_{ext \partial f(.,x_0(.))}^1$.

Hence $u \in S_{ext \partial f(.,x_0(.))}^1 \implies u(\omega) \in ext \partial f(\omega, x_0(\omega)) \mu$-a.e.
                                                                            Q.E.D.

Now we turn our attention for the conjugate of the convex integral functional $I_f: L_X^1 \to \overline{R}$.

Assume that (Ω, Σ, μ) is a complete, finite, nonatomic measure space and X a separable Banach space. Recall that $f:X \to R \cup \{+\infty\}$ is w-inf-compact, if for all $\lambda \in R, \{x \in X: f(x) \leqslant \lambda\}$ is w-compact.

THEOREM 5.5. If $f:\Omega \times X \to \overline{R}$ is a convex, normal integrand which is w-inf compact in x for all $\omega \in \Omega$ and there exists $x(.) \in L_X^1$, s.t. $I_f(x(.)) < + \infty$,

then $I_{f*}(.)$ is $m(L_{X^*_{w^*}}^\infty, L_X^1)$-continuous (Here $m(.,.)$ denotes the Mackey topology).

PROOF. Since by hypothesis I_f is w-inf-compact, from Moreau's theorem (see Laurent [14]) we have that $(I_f)^*$ is m-continuous at 0. Also $[I_f]^* = I_{f*}$ (see Levin [15] or Rockafellar [28]). Since from convex analysis we know that $I_{f*}(.)$ is m-continuous in the interior of its domain, we have to show that $I_{f*}(.)$ is finite everywhere.

From the fact that $I_{f*}(.)$ is m-continuous at 0, we get that there exists

$V \epsilon N_m(0) = \{$filter of $m(L^\infty_{X^*_{w^*}}$, $L^1_X)$-neighborhoods of the origin$\}$ s.t. for all $x^* \epsilon V$ we have

$$I_{f*}(x^*) \leqslant I_{f*}(0) + 1$$

From the definition of the Mackey topology, we know that V is the polar set of a relatively w-compact set W in L^1_X. So we can write:

$$V = \{x^*(.)\epsilon L^\infty_{X^*_{w^*}} : \sup_{u\epsilon W} \int_\Omega (x^*(\omega),u(\omega))d\mu(\omega) \leqslant 1\}$$

Since W is relatively w-compact in L^1_X, from Theorem 4, p. 104 of Diestel-Uhl [7], we have that W is uniformly integrable. So for all $\epsilon > 0$ there exists $\delta > 0$ s.t. if $\mu(A) < \delta$, then $\sup_{u\epsilon W} \int_A ||u(\omega)|| d\mu(\omega) < \epsilon$. Let $x^* \epsilon L^\infty_{X^*_{w^*}}$, $x^* \neq \phi$.

Take $\epsilon = \dfrac{1}{||x^*||_\infty}$. Then $\sup_{u\epsilon W} \int_A (x^*(\omega),u(\omega))d\mu(\omega) < 1$, for $\mu(A)<\delta$.

Because (Ω,Σ,μ) is finite, nonatomic, from Saks lemma we known that we can find $\{A_k\}^n_{k=1} \subseteq \Sigma$.

$$\bigcup_{k=1}^n A_k = \Omega \text{ with } \mu(A_k) < \delta, \text{ for all } k=1,...,n. \quad \text{Then}$$

$\omega).\int_{A_k} f^*(\omega,x^*(\omega))d\mu(\omega) = \int_\Omega f^*(\omega,\chi_{A_k}(\omega)x^*(\omega))d\mu(\omega) - \int_{A_k^c}f^*(\omega,o)d\mu($

since $I_{f*}(.)$ is m-continuous at 0, $\int_{A_k} f^*(\omega,o)d\mu(\omega) < \infty$.

Also $\sup_{u\epsilon W} \int_\Omega (\chi_{A_k}(\omega)x^*(\omega), u(\omega))d\mu(\omega) = \sup_{u\epsilon W} \int_{A_k} (x^*(\omega),u(\omega))d\mu(\omega) < 1$.

Therefore $\chi_{A_k} x^*(.) \epsilon V$ and so:

$$\int_\Omega f^*(\omega,\chi_{A_k}(\omega)x^*(\omega))d\mu(\omega) = I_{f*}(\chi_{A_k} x^*) \leqslant I_{f*}(0) + 1$$

$$\Longrightarrow \int_{A_k} f^*(\omega,x^*(\omega))d\mu(\omega) < +\infty.$$

$$\Longrightarrow I_{f*}(x^*) < \infty$$

and since $x^*\epsilon L^\infty_{X^*_{w^*}}$ was arbitrary, we have that dom $I_{f*}=L^\infty_{X^*_{w^*}}$.   Q.E.D.

Now we will obtain a description for $\overline{domI_{f*}}$ So assume that (Ω,Σ,μ) is a complete, σ-finite measure space and X a separable Banach space. We recall that if $f:X \to RU\{+\infty\}$, $f \neq +\infty$ is convex, then the recession function

$f:X \to RU\{+\infty\}$ is defined by $f_\infty(h) = \sup\{f(x+h)-f(x): x\epsilon domf\}$. If $f(.)$ is in addition lower semicontinuous, then $f_\infty(h) = \sup_{\lambda>0} \dfrac{f(x+\lambda h)-f(x)}{\lambda}$

THEOREM 5.6. If $f:\Omega \times X \to \overline{R}$ is a convex, normal integrand, there exist

$x \in L_X^1$ and $x^* L_{X^*}^\infty{}_{w^*}$ s.t. $I_f(x) < \infty$ and $I_{f^*}(x^*) < +\infty$ and $I_f(\cdot)$ is lower semicontinuous

on L_X^1, then we have $\overline{domI}_{f^*} = S_{\overline{domf^*}(\cdot,\cdot)}^\infty$ and for all $x^*(\cdot) \in ext\overline{domI}_{f^*}$ we

have $x^*(\omega) \in extdomf^*(\omega,\cdot)$ μ-a.e.

PROOF. Since by hypothesis $I_f(\cdot)$ is l.s.c. on L_X^1 and convex, from Theorem 6.8.5.
of Laurent [14], we have:

$$(I_f)_\infty (x) = \sigma(x,\overline{dom}(I_f)^*).$$

Since by hypothesis $domI_f \neq \emptyset$, $(I_f)^* = I_{f^*}$, while from a simple application of
the monotone convergence theorem (see also Bismut [3], Proposition 1) we have that:

$$(I_f)_\infty(x) = I_{f_\infty}(x)$$

Therefore $\int_\Omega f_\infty(\omega,x(\omega))d\mu(\omega) = \sigma(x,\overline{dom}I_{f^*})$. But $f_\infty(\omega,x(\omega)) = \sigma(x(\omega,\overline{domf^*}(\omega,\cdot))$.

Observe that $domf^*(\omega,\cdot) = \bigcup_{n \geqslant 1} \{x^* \in X^*:f^*(\omega,x^*) \leqslant n\} \implies \omega \to domf^*(\omega,\cdot)$ is graph

measurable. Also $x^* \in S_{\overline{domf^*}(\cdot,\cdot)}^\infty$.   So applying Theorem 2.2 of Hiai-Umegaki [9] we get
that:

$$\int_\Omega \sigma(x(\omega),\overline{domf^*}(\omega,\cdot))d\mu(\omega) = \int_\Omega \sup_{x^* \in domf^*(\omega,\cdot)} (x^*,x(\omega))d\mu(\omega)$$

$$= \sup_{x^*(\cdot) \in S_{\overline{domf^*}(\cdot,\cdot)}} \int(x^*(\omega),x(\omega))d\mu(\omega)$$

$$\implies \sigma(x,\overline{dom}I_{f^*}) = \sigma(x,S_{\overline{domf^*}(\cdot,\cdot)}^\infty)$$

Since both sets are clearly convex, we conclude that:

$$\overline{dom}I_{f^*} = S_{\overline{domf^*}(\cdot,\cdot)}^\infty$$

The second part of the conclusion, follows from the following equalities (see
Benamara [2]):

$$ext\overline{dom}I_{f^*} = extS_{\overline{domf^*}(\cdot,)}^\infty = S_{ext\overline{domf^*}(\cdot,\cdot)}^\infty$$

Q.E.D.

REFERENCES

1. AUMANN, R. Measurable utility and the measurable choice theorem, La Decision,
    Colloque International du C.N.R.S.,Paris (1967), 15-26.

2. BENAMARA, M. Sections mesurables extremales d'une multi-application. C.R. Acad.
    Sc. Paris, t.278 (1974), 1249-1252.

3. BISMUT, J.M. Integrales Convexes et Probabilites, J. Math. Anal. Appl. 42
    (1973), 639-673.

4. BOURGAIN, J. An averaging result for ℓ^1-sequences and applications to weakly
    conditionally compact sets in L_X^1 ,Israel J. Math. 32 (1979), 289-298.

5. CASTAING, C. and VALADIER, M. Convex Analysis and Measurable Multifunctions.
    Lecture Notes in Math, 580 Springer, Berlin, (1977).

6. COSTE, A. La propriete de Radon-Nikodym en integration multivoque. C.R. Acad.
    Sc. Paris t.280 (1975), 1515-1518.

7. DIESTEL, J. and UHL, J.J. Vector Measures, Math. Surveys, 15, Providence, RI,
    (1977).

8. HERMES, H. and LASALLE, J. Functional Analysis and Time Optimal Control,
    Academic Press, New York (1969).

9. HIAI, F. and UMEGAKI, H. Integrals, conditional expectations and martingales
    of multivalued functions. J. Multiv. Anal. 7 (1977), 149-182.

10. IOFFE, A. and LEVIN, V. Subdifferentials of convex functions, Trans. Moscow
    Math.Soc. 26 (1972) 3-74.

11. IONESCU-TULCEA, A. and IONESCU-TULCEA, C. Topics in the theory of Liftings,
    Springer, Berlin (1969).

12. KATO, T. Linear evolution equations of hyperbolic type. I.Fac. Sc. Univ. Tokyo
    Sec. I. 17 (1970) 241-258.

13. KLEIN, E. and THOMPSON, A. Theory of Correspondences, John Wiley, New York
    (1984).

14. LAURENT, P.J. Approximation et Optimisation, Hermann, Paris (1972).

15. LEVIN, V. Convex Integral functionals and the theory of lifting, Russian Math.
    Surveys 30 (1975), 119-184.

16. LINDENSTRAUSS, J. and TZAFRIRI, L. Classical Banach Spaces II: Function Spaces
    Springer, Berlin (1979).

17. MUSIAL, K. and NARDZEWSKI, C.R. Liftings of vector measures and their
    applications to RNP and WRNP. Lecture Notes in Math. 645, Springer, Berlin
    (1977).

18. PAPAGEORGIOU, N.S. On the theory of Banach space valued multifunctions.
    Part 1: Integration and conditional expectation. J. Multiv.Anal. 17 (1986),
    185-206.

19. PAPAGEORGIOU, N.S.   On the theory of Banach space valued multifunctions.   Part 2: Set valued martingales and set valued measures. J. Multiv. Anal. 17 (1986), 207-227.

20. PAPAGEORGIOU, N.S.   Efficiency and optimality in economies described by coalitions. J. Math. Anal. Appl. 116, (1986), 497-512.

21. PAPAGEORGIOU, N.S.   Trajectories of set valued integrals. Bull. Austr. Math. Soc. 31 (1985), 389-411.

22. PAPAGEORGIOU, N.S.   Representation of set valued operators, Trans. Amer. Math. Soc. 292 (1985), 557-572.

23. PAPGEORGIOU, N.S.   On the properties of the Aumann integral, with applicattions to differential inclusions and control systems. Czechoslovak Math. Journ.   To appear.

24. PAPAGEORGIOU, N.S.   Random fixed point theorems for measurable multifunctions in Banach spaces. Proc. Amer. Soc. 97 (1986), 507-514.

25. PAPAGEORGIOU, N.S.   Contributions to the theory of set valued functions and set valued measures. Trans. Amer. Math. Soc. 304 (1987), 245-265.

26. PAPAGEORGIOU, N.S.   On the properties of multimeasures with values in a Banach space. submitted.

27. ROCKAFELLAR, R.T.   Weak compactness of level sets of integral functionals. Proc. of 3rd Coll. Anal. Fonct. (CBRM) H.G. Garnir ed, Vender, Lourain (1971), 85-98.

28. ROCKAFELLAR, R.T.   Convex integral functionals and duality in Contributions to Nonl. Anal. E. Zarantonello ed, Acad. Press, New York (1971), 215-236.

29. ROSENTHAL, H.   A characterization of Banach spaces containing ℓ^1. Proc.Nat. Acad. Sc. 71 (1974), 2411-2413.

30. SAINT-BEUVE, M.F.   On the extension of Von Neumann-Aumann's theorem. J. Funct. Anal. 17 (1974), 112-129.

31. TALAGRAND, M.   Weak Cauchy sequences in $L^1(E)$. Amer. Jour. of Math. 106 (1984), 703-724.

32. YOSIDA, K. and HEWITT, E.   Finctely additive measures. Trans. Amer. Math. Soc. 72 (1952), 46-66.

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