

ANOTHER NOTE ON KEMPISTY'S GENERALIZED CONTINUITY

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ABSTRACT. Under a fairly mild completeness condition on spaces Y and Z we show that every x -continuous function $f: X \times Y \times Z \rightarrow M$ has a "substantial" set $C(f)$ of points of continuity. Some odds and ends concerning a related earlier result shown by the authors are presented. Further, a generalization of S. Kempisty's ideas of generalized continuity on products of finitely many spaces is offered. As a corollary from the above results, a partial answer to M. Talagrand's problem is provided.

KEYS WORDS AND PHRASES. quasi-continuity, x -continuity, separate and joint continuity, Baire spaces.

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1. x -CONTINUITY.

The notion of symmetric quasi-continuity introduced by S. Kempisty [1] has been generalized in Lee and Piotrowski [2], to x -continuity. In what follows let X , Y , Z and T be spaces. Following Lee and Piotrowski [2] a function $f: X \times Y \times Z \rightarrow T$ is x -continuous if for every $(p, q, r) \in X \times Y \times Z$, for every neighborhood $U \times V \times W$ of (p, q, r) and for every neighborhood N of $f(p, q, r)$ there exists a neighborhood U' of p with $U' \subset U$ and nonempty open sets V' and W' with $V' \subset V$ and $W' \subset W$ such that for all $(x, y, z) \in U' \times V' \times W'$ it follows that $f(x, y, z) \in N$.

We shall first show that under certain general assumptions concerning the spaces, x -continuous functions have "large" sets of points of joint continuity. In order to do this we first list some necessary definitions.

Let A be an open covering of a space X . Then a subset S of X is said to be A -small if S is contained in a member of A . A space X is called strongly countably complete if there exists a sequence $\{A_i: i=1,2,\dots\}$ of open coverings of X such that and sequence $\{F_i\}$ of A_i -small, closed subsets of X for which $F_i \supset F_{i+1}$ has a non-

empty intersection.

The class of strongly countably complete spaces include countably compact and complete metric spaces. This fact follows easily from a theorem due to A. Arhangel'skii [3] and Z. Frolík [4] which states that in the class of completely regular spaces, Čech-complete and strongly countably complete spaces coincide (Engelking [5]), see also Frolík [4], where some other properties of these spaces such as their invariance under taking closed, open subspaces or products are discussed.

A space X is called quasi-regular, (Oxtoby [6]) if for every nonempty open set u , there is a nonempty open set V such that $\text{cl}V \subset u$. Obviously, every regular space is quasi-regular.

Let us recall that a function $f: X \times Y \rightarrow Z$ is said to be quasi-continuous with respect to x , (Kempisty [1] p.188,) if for every $(p,q) \in X \times Y$, for every neighborhood N of $f(p,q)$ and every neighborhood $U \times V$ of (p,q) there exists a neighborhood U' of p with $U' \subset U$ and a nonempty open set $V' \subset V$ such that for all $(x,y) \in U' \times V'$ we have $f(x,y) \in N$. Quasi-continuity with respect to y can be defined similarly.

LEMMA 1. (Lee and Piotrowski [2], Lemma 3 p. 383). Let X, Y, Z and T be spaces and let $F: X \times Y \times Z \rightarrow T$ be a function. Then f is x -continuous if and only if $g: X \times S \rightarrow T$ is quasi-continuous with respect to x , where $S = Y \times Z$ and $g(x, (y, z)) = f(x, y, z)$.

THEOREM 2. Let X be a space, Y and Z be spaces such that $Y \times Z$ is quasi-regular, strongly countably complete and let M be metric. If $f: X \times Y \times Z \rightarrow M$ is x -continuous, then for every $x \in X$, the set $C(f)$ of continuity points of f is dense G_δ subset in $\{x\} \times Y \times Z$.

PROOF. In view of Lemma 1 it is sufficient to prove the following:

CLAIM. Let X be a space, Y be a quasi-regular, strongly countably complete and Z be metric. If $f: X \times Y \rightarrow Z$ is quasi-continuous with respect to x , then for all $x \in X$ the set of points of joint continuity of f is a dense G_δ subset of $\{x\} \times Y$.

PROOF. First we will prove that the set of points of joint continuity of f is dense in $\{x\} \times Y$. Let $x \in X$, $y \in Y$ and $U \times V$ be any neighborhood U of x , contained in U , and a nonempty open set $V^1 \subset V$ such that for all (x', y') and (x'', y'') in $U^1 \times V^1$, we have $\rho(f(x', y'), f(x'', y'')) < 1$. Without loss of generality we may assume that V^1 is contained in an element A_1 of the covering A_1 of Y . Let W^1 be a nonempty open set such that $\text{cl} W^1 \subset V^1$. So $\text{cl} W^1$ is A_1 -small. Then $U^1 \times W^1$ is a neighborhood of (x, y_1) , where $y_1 \in W^1$, and since f is quasi-continuous with respect to x at (x, y_1) , there is a neighborhood U^2 of x , contained in U^1 and a nonempty open set $V^2 \subset W^1$, such that for all (x', y') and (x'', y'') in $U^2 \times V^2$ we have $\rho(f(x', y'), f(x'', y'')) < \frac{1}{2}$. Similarly, we may assume that V^2 is contained in an element A_2 of the covering A_2 . Let W^2 be a nonempty open set such that $\text{cl} W^2 \subset V^2$. We see, that $\text{cl} W^2$ is A_2 -small.

Now, proceeding by induction we get a neighborhood $U^n \times V^n$ of (x, y_n) , $y_n \in V^n$, such that for all (x', y') and (x'', y'') in $U^n \times V^n$, we have $\rho(f(x', y'), f(x'', y'')) < \frac{1}{n}$ and that V^n is contained in an element A_n of the covering A_n of Y . Moreover, there is a nonempty open sets W^n such that $V^{n+1} \subset \text{cl} W^n \subset V^n$. Thus each $\text{cl} W^n$ is A_n -small, obviously $\text{cl} W^n \supset \text{cl} W^{n+1}$. Since Y is strongly countably complete $\bigcap_{n=1}^{\infty} \text{cl} W^n \neq \emptyset$. Let

$y^* \in \bigcap_{n=1}^{\infty} \text{cl } w^n$. Then

$$(x, y^*) \in \bigcap_{n=1}^{\infty} (U^n \times \text{cl } w^n) \subset \bigcap_{n=1}^{\infty} (U^n \times v^n) \subset U \times V.$$

Thus $(x, y^*) \in (U \times V) \cap (\{x\} \times Y)$ and (x, y^*) is a point of joint continuity of f . This shows the density of the set of points of joint continuity of f in the set $\{x\} \times Y$.

The proof that this set is G_δ subset of $\{x\} \times Y$ easily follows, when we recall that the function f takes values in the metric space Z . This completes the proof of Claim.

Thus, Theorem 2 is shown.

The forthcoming, Proposition 3 is contained in Lemma 5.1 of [6], since any quasi-regular strongly countably complete space is pseudo-complete; take $B(n) = \text{the class of all nonempty open sets that are } A_n\text{-small}$. Then $\{B(n)\}$ is a sequence of (pseudo-) bases that shows X to be pseudo-complete.) We would like to thank the referee who make the above observation.

PROPOSITION 3. (Oxtoby [6], Lemma 5.1) Every quasi-regular strongly countably complete space X is a Baire space.

REMARK 4. Observe that neither base countability nor metrizability assumptions are made on the considered spaces X , Y , Z in Theorem 1 while in Theorem 2 of [2] the same conclusion concerning the set of points of continuity is obtained under an extra assumption that X is first countable, Y is Baire, Z is second countable in a neighborhood of any of its points and such that $Y \times Z$ is Baire.

2. CONDITIONS IMPLYING x -CONTINUITY - COUNTER-EXAMPLES.

Given spaces X and Y ; a function $f: X \rightarrow Y$ is said to be quasi-continuous (Martin [8], compare Kempisty [1]) if for every $x \in X$ and for every neighborhood U of x and for every neighborhood V of $f(x)$ have: $U \cap \text{Int } f^{-1}(V) \neq \emptyset$.

The main result of Lee and Piotrowski [2] is the following:

THEOREM A. (Lee and Piotrowski [2], Theorem 1, p. 383). Let X be first countable, Y be Baire, Z be second countable such that $Y \times Z$ is Baire and let T be regular. If $f: X \times Y \times Z \rightarrow T$ is:

- (1) continuous at $X \times \{y\} \times \{z\}$, $y \in Y$, $z \in Z$, and
- (2) quasi-continuous at points of $\{x\} \times Y \times \{z\}$ for all $x \in X$ and $z \in Z$, and
- (3) quasi-continuous at points of $\{x\} \times \{y\} \times Z$ for all $x \in X$ and $y \in Y$

then f is x -continuous.

The first natural question which comes up is to check whether the converse of Theorem A is true. Apparently, the following Example 5 settles this question in the negative.

EXAMPLE 5. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by

$$f(x, y, z) = \begin{cases} \sin \frac{1}{x^2+y^2+z^2}, & \text{if } (x, y, z) \neq (0, 0, 0) \\ 0, & \text{otherwise} \end{cases}$$

The function f is x -continuous, however, fixing $y = 0 = z$ we obtain that $f(x,0,0)$ is not continuous.

Now we shall investigate the necessity of the assumptions in Theorem A, in particular:

- (*) - continuity of f at points of $X \times \{y\} \times \{z\}$
- (**) - quasi-continuity of f at points of $\{x\} \times Y \times \{z\}$, and
- (***) - quasi-continuity of f at points of $\{x\} \times \{y\} \times Z$.

In what follows (Examples 6 and 7) such constructions will be provided.

EXAMPLE 6. The assumption (*) is essential. In fact, let us consider a function $f: [-1,1]^3 \rightarrow \mathbb{R}^3$ given as follows

$$f(x,y,z) = \begin{cases} (x,y,z+1), & \text{if } (x,y,z) \in [0,1] \times [0,1] \times [0,1] \\ (x,y,z-1), & \text{if } (x,y,z) \in [-1,0] \times [-1,0] \times [-1,0] \\ (x,y,z), & \text{otherwise} \end{cases}$$

A standard verification that f has the required property, (namely f is not x -continuous at $(0,0,0)$) is left to the reader. Using somewhat more complex, but still elementary techniques we shall show that also (**) (as well as (***)) is essential. In fact, we have

EXAMPLE 7. Consider the function $g: [-1,1]^3 \rightarrow \mathbb{R}^3$ given as follows:

$$g(x,y,z) = \begin{cases} (x,y,z+1) & \text{if } (x,y,z) \in [-1,1] \times [-\frac{1}{2},1] \times \\ & \quad \times \{(-\frac{1}{2}, \frac{1}{2}) \cap \text{IQ} \cup [\frac{1}{2}, 1]\} \\ (x,y,z), & \text{otherwise} \end{cases}$$

Again, we leave to the interested reader a standard verification that f is not x -continuous at $(0,0,0)$.

3. ONE-PROMISING HYPOTHESIS.

Observe that the definition of x -continuity at (p,q,r) requires the existence of a "small" neighborhood U' of p and "small" nonempty open sets V' and W' such that q and r "clusters" to V' and W' respectively and such that the set $f(U' \times V' \times W')$ is contained in a "small", previously chosen, open set N . This observation prompts us to label this kind of product almost continuity as *1-3-continuity* - since we require the existence of only one "small" neighborhood U' (around p) of the three neighborhoods U , V , W .

The term "1-3-continuity" has been used already, in a different sense in Breckenridge and Nishiura [9].

So, now let us consider "2-3-continuity".

More precisely, given spaces X , Y , Z and T , we say that $f: X \times Y \times Z \rightarrow T$ is *2-3-continuous* or more specifically *xy-continuous*, if for every $(p,q,r) \in X \times Y \times Z$, for every neighborhood $U \times V \times W$ of (p,q,r) and for every neighborhood N of $f(p,q,r)$ there is a neighborhood U' of p , with $U' \subset U$, there is a neighborhood V' of q , with $V' \subset V$ and a nonempty open set W' , with $W' \subset W$ such that for all $(x,y,z) \in U' \times V' \times W'$ we have $f(x,y,z) \in N$.

Now, 3-3-continuity can be defined easily; the set W' in definition of 2-3-continuity is assumed to be a neighborhood of r - not just only a nonempty open subset of W .

Clearly, every 3-3-continuous (\equiv continuous) function is 2-3-continuous; 2-3-continuous functions are 1-3-continuous and the latter are in turn 0-3-continuous (\equiv quasi-continuous).

It now follows from a result of T. Neubrunn [10] that if X, Y, Z are "nice" (e.g. Baire, second countable), T -regular then if $f: X \times Y \times Z \rightarrow T$ is separately quasi-continuous then it is (jointly) quasi-continuous.

We can present this fact in the following symbolic equality:

$$"0 + 0 + 0 = 0",$$

where the numbers (0 or 1) on the left side of the equality stand for quasi-continuity (0) or continuity (1) of the corresponding sections and the numbers on the right ($i = 0, 1, 2$ or 3) denote the corresponding i -3-continuity of f as a function of three variables.

Theorem A implies that if X, Y, Z and T are as above and if $f: X \times Y \times Z \rightarrow T$ is continuous in x and is quasi-continuous in y and is quasi-continuous in z , then f is 1-3-continuous. Consequently, we get:

$$"1 + 0 + 0 = 1".$$

In view of the above considerations it is now natural to state the following:

HYPOTHESIS. Let X, Y and Z be Baire, second countable spaces and let T be regular. If $f: X \times Y \times Z \rightarrow T$ is:

- 1) continuous in x , and
- 2) continuous in y , and
- 3) quasi-continuous in z ,

Then f is 2-3-continuous;

In other words:

$$"1 + 1 + 0 = 2"$$

We shall resolve this Hypothesis in the *negative* in the forthcoming Example 8.

Now we shall exhibit two examples of i -3-continuous functions which are not $(i + 1)$ -3-continuous, $i = 1, 2$.

EXAMPLE 8. A 1-3-continuous function which is not 2-3-continuous. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be given by $f(x_1, x_2, x_3) = g(x_1, x_2)$ where g is an arbitrary separately continuous function which is discontinuous at $(0, 0)$.

EXAMPLE 9. A 2-3-continuous function which is not 3-3-continuous (\equiv continuous).

Take $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ to be $f(x_1, x_2, x_3) = h(x_3)$, where h is any function which is continuous except for 0.

Using the above pattern the reader will easily construct 0-3-continuous function (\equiv quasi-continuous) which is not 1-3-continuous.

Apparently, the above constructions can be illustrated with the following very specific formula-ready example.

EXAMPLE 10. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function.

$$f(x_1, x_2, x_3) = g_1^3(x_1, \dots, x_i), \quad i = 1, 2 \quad \text{where}$$

$$g_1^3(x_1, \dots, x_i) = \frac{\prod_{j=1}^i x_j}{\prod_{j=1}^i (x_j)^i}, \quad \text{if } \sum_{j=1}^i (x_j)^i \neq 0$$

$$0, \quad \text{otherwise}$$

Then f is i -3-continuous which is not $(i+1)$ -3-continuous, $i = 1, 2$.

4. FURTHER GENERALIZATION OF i -3-CONTINUITY.

Having defined 1-3 and 2-3-continuity for $f: X_1 \times X_2 \times X_3 \rightarrow T$, we shall now extend these ideas to a general case.

Namely, let n be an arbitrary natural number. We say that f function

$f: \prod_{i=1}^n X_i \rightarrow T$ is A - n -continuous if for every $(p_1, p_2, \dots, p_n) \in \prod_{i=1}^n X_i$ and for every neighborhood $U_1 \times U_2 \times \dots \times U_n$ of (p_1, p_2, \dots, p_n) and for every neighborhood N of $f(p_1, p_2, \dots, p_n)$ there are neighborhoods $U'_{i,s}$ ($1 \leq s \leq k$) of the first k out of n points p_1, p_2, \dots, p_n with $U'_{i,s} \subset U_i$ and there are $(n-k)$ nonempty open sets $V'_{i,m}$ with $V'_{i,m} \subset U_i$ $1 \leq m \leq n-k$ such that for all $(x_1, x_2, \dots, x_n) \in \prod_{s=1}^k U'_{i,s} \times \prod_{m=1}^{n-k} V'_{i,m}$ we have $f(x_1, x_2, \dots, x_n) \in N$.

An interested reader will easily observe that the formula

$$g_k^n(x_1, \dots, x_k) = \frac{\prod_{i=1}^k x_i}{\prod_{i=1}^k (x_i)^k}, \quad \text{if } \sum_{i=1}^k (x_i)^k \neq 0$$

$$0, \quad \text{otherwise}$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}$ describes a k - n -continuous function f given by

$$f(x_1, \dots, x_n) = g_k^n(x_1, \dots, x_k), \quad k = 1, 2, 3, \dots, n-1.$$

One can also give analogues of Example 8 and 9 for k - n -continuity.

Studies of $C(f)$ in hyperspaces for separately continuous functions and related ones were done also in Bögel [11] and Hahn [12].

5. A PARTIAL SOLUTION TO A PROBLEM OF M. TALAGRAND.

M. Talagrand ([13] Problem 3 p. 160) asked whether if X is Baire, Y is compact and $f: X \times Y \rightarrow \mathbb{R}$ is any separately continuous function, is there the set $C(f)$ of points of continuity of f nonempty.

We shall answer this question in the positive if a compact space Y is additionally first countable.

In fact, we have shown the following result:

LEMMA 11. (Lee and Piotrowski [2], Lemma 2 p. 381). Let X be Baire, Y be first countable and Z be regular. If $f: X \times Y \rightarrow Z$ is a function such that all its x -sections f_X are continuous with the exception of a first category set, and all its y -sections f_Y are quasi-continuous, then f is quasi-continuous with respect to y .

It follows from the definition that

REMARK 12. Every quasi-continuous function with respect to y is quasi-continuous.

LEMMA 13. (Marcus [14]). Let X be a Baire, M be metric. If $f: X \rightarrow M$ is quasi-continuous, then $C(f)$, the set of point of continuity of f is dense G_δ subset of X .

PROPOSITION 14. Let X be Baire, Y be compact first countable and let $f: X \times Y \rightarrow \mathbb{R}$ be any separately continuous function. Then $C(f) \neq 0$.

PROOF. By Lemma 11 and Remark 12 such f is quasi-continuous. Now, since the Cartesian product of a compact space and a Baire space is Baire, we are done by Lemma 13.

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