

## ANOTHER NOTE ON KEMPISTY'S GENERALIZED CONTINUITY

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**ABSTRACT.** Under a fairly mild completeness condition on spaces  $Y$  and  $Z$  we show that every  $x$ -continuous function  $f: X \times Y \times Z \rightarrow M$  has a "substantial" set  $C(f)$  of points of continuity. Some odds and ends concerning a related earlier result shown by the authors are presented. Further, a generalization of S. Kempisty's ideas of generalized continuity on products of finitely many spaces is offered. As a corollary from the above results, a partial answer to M. Talagrand's problem is provided.

**KEYS WORDS AND PHRASES.** quasi-continuity,  $x$ -continuity, separate and joint continuity, Baire spaces.

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### 1. $x$ -CONTINUITY.

The notion of symmetric quasi-continuity introduced by S. Kempisty [1] has been generalized in Lee and Piotrowski [2], to  $x$ -continuity. In what follows let  $X$ ,  $Y$ ,  $Z$  and  $T$  be spaces. Following Lee and Piotrowski [2] a function  $f: X \times Y \times Z \rightarrow T$  is  $x$ -continuous if for every  $(p, q, r) \in X \times Y \times Z$ , for every neighborhood  $U \times V \times W$  of  $(p, q, r)$  and for every neighborhood  $N$  of  $f(p, q, r)$  there exists a neighborhood  $U'$  of  $p$  with  $U' \subset U$  and nonempty open sets  $V'$  and  $W'$  with  $V' \subset V$  and  $W' \subset W$  such that for all  $(x, y, z) \in U' \times V' \times W'$  it follows that  $f(x, y, z) \in N$ .

We shall first show that under certain general assumptions concerning the spaces,  $x$ -continuous functions have "large" sets of points of joint continuity. In order to do this we first list some necessary definitions.

Let  $\mathcal{A}$  be an open covering of a space  $X$ . Then a subset  $S$  of  $X$  is said to be  $\mathcal{A}$ -small if  $S$  is contained in a member of  $\mathcal{A}$ . A space  $X$  is called strongly countably complete if there exists a sequence  $\{\mathcal{A}_i: i=1, 2, \dots\}$  of open coverings of  $X$  such that and sequence  $\{F_i\}$  of  $\mathcal{A}_i$ -small, closed subsets of  $X$  for which  $F_i \supset F_{i+1}$  has a non-

empty intersection.

The class of strongly countably complete spaces include countably compact and complete metric spaces. This fact follows easily from a theorem due to A. Arhangel'skii [3] and Z. Frolík [4] which states that in the class of completely regular spaces, Čech-complete and strongly countably complete spaces coincide (Engelking [5]), see also Frolík [4], where some other properties of these spaces such as their invariance under taking closed, open subspaces or products are discussed.

A space  $X$  is called quasi-regular, (Oxtoby [6]) if for every nonempty open set  $u$ , there is a nonempty open set  $V$  such that  $\text{cl} V \subset u$ . Obviously, every regular space is quasi-regular.

Let us recall that a function  $f: X \times Y \rightarrow Z$  is said to be quasi-continuous with respect to  $x$ , (Kempisty [1] p.188,) if for every  $(p,q) \in X \times Y$ , for every neighborhood  $N$  of  $f(p,q)$  and every neighborhood  $U \times V$  of  $(p,q)$  there exists a neighborhood  $U'$  of  $p$  with  $U' \subset U$  and a nonempty open set  $V' \subset V$  such that for all  $(x,y) \in U' \times V'$  we have  $f(x,y) \in N$ . Quasi-continuity with respect to  $y$  can be defined similarly.

LEMMA 1. (Lee and Piotrowski [2], Lemma 3 p. 383). Let  $X, Y, Z$  and  $T$  be spaces and let  $F: X \times Y \times Z \rightarrow T$  be a function. Then  $f$  is  $x$ -continuous if and only if  $g: X \times S \rightarrow T$  is quasi-continuous with respect to  $x$ , where  $S = Y \times Z$  and  $g(x, (y,z)) = f(x,y,z)$ .

THEOREM 2. Let  $X$  be a space,  $Y$  and  $Z$  be spaces such that  $Y \times Z$  is quasi-regular, strongly countably complete and let  $M$  be metric. If  $f: X \times Y \times Z \rightarrow M$  is  $x$ -continuous, then for every  $x \in X$ , the set  $C(f)$  of continuity points of  $f$  is dense  $G_\delta$  subset in  $\{x\} \times Y \times Z$ .

PROOF. In view of Lemma 1 it is sufficient to prove the following:

CLAIM. Let  $X$  be a space,  $Y$  be a quasi-regular, strongly countably complete and  $Z$  be metric. If  $f: X \times Y \rightarrow Z$  is quasi-continuous with respect to  $x$ , then for all  $x \in X$  the set of points of joint continuity of  $f$  is a dense  $G_\delta$  subset of  $\{x\} \times Y$ .

PROOF. First we will prove that the set of points of joint continuity of  $f$  is dense in  $\{x\} \times Y$ . Let  $x \in X$ ,  $y \in Y$  and  $U \times V$  be any neighborhood  $U$  of  $x$ , contained in  $U$ , and a nonempty open set  $V^1 \subset V$  such that for all  $(x',y')$  and  $(x'',y'')$  in  $U^1 \times V^1$ , we have  $\rho(f(x',y'), f(x'',y'')) < 1$ . Without loss of generality we may assume that  $V^1$  is contained in an element  $A_1$  of the covering  $\mathcal{A}_1$  of  $Y$ . Let  $W^1$  be a nonempty open set such that  $\text{cl } W^1 \subset V^1$ . So  $\text{cl } W^1$  is  $A_1$ -small. Then  $U^1 \times W^1$  is a neighborhood of  $(x,y_1)$ , where  $y_1 \in W^1$ , and since  $f$  is quasi-continuous with respect to  $x$  at  $(x,y_1)$ , there is a neighborhood  $U^2$  of  $x$ , contained in  $U^1$  and a nonempty open set  $V^2 \subset W^1$ , such that for all  $(x',y')$  and  $(x'',y'')$  in  $U^2 \times V^2$  we have  $\rho(f(x',y'), f(x'',y'')) < \frac{1}{2}$ . Similarly, we may assume that  $V^2$  is contained in an element  $A_2$  of the covering  $\mathcal{A}_2$ . Let  $W^2$  be a nonempty open set such that  $\text{cl } W^2 \subset V^2$ . We see, that  $\text{cl } W^2$  is  $A_2$ -small.

Now, proceeding by induction we get a neighborhood  $U^n \times V^n$  of  $(x,y_n)$ ,  $y_n \in V^n$ , such that for all  $(x',y')$  and  $(x'',y'')$  in  $U^n \times V^n$ , we have  $\rho(f(x',y'), f(x'',y'')) < \frac{1}{n}$  and that  $V^n$  is contained in an element  $A_n$  of the covering  $\mathcal{A}_n$  of  $Y$ . Moreover, there is a nonempty open sets  $W^n$  such that  $V^{n+1} \subset \text{cl } W^n \subset V^n$ . Thus each  $\text{cl } W^n$  is  $A_n$ -small, obviously  $\text{cl } W^n \supset \text{cl } W^{n+1}$ . Since  $Y$  is strongly countably complete  $\bigcap_{n=1}^{\infty} \text{cl } W^n \neq \emptyset$ . Let

$y^* \in \bigcap_{n=1}^{\infty} c_1 W^n$ . Then

$$(x, y^*) \in \bigcap_{n=1}^{\infty} (U^n \times c_1 W^n) \subset \bigcap_{n=1}^{\infty} (U^n \times V^n) \subset U \times V.$$

Thus  $(x, y^*) \in (U \times V) \cap (\{x\} \times Y)$  and  $(x, y^*)$  is a point of joint continuity of  $f$ . This shows the density of the set of points of joint continuity of  $f$  in the set  $\{x\} \times Y$ .

The proof that this set is  $G_\delta$  subset of  $\{x\} \times Y$  easily follows, when we recall that the function  $f$  takes values in the metric space  $Z$ . This completes the proof of Claim.

Thus, Theorem 2 is shown.

The forthcoming, Proposition 3 is contained in Lemma 5.1 of [6], since any quasi-regular strongly countably complete space is pseudo-complete; take  $B(n)$  = the class of all nonempty open sets that are  $A_n$ -small. Then  $\{B(n)\}$  is a sequence of (pseudo-) bases that shows  $X$  to be pseudo-complete.) We would like to thank the referee who make the above observation.

PROPOSITION 3. (Oxtoby [6], Lemma 5.1) Every quasi-regular strongly countably complete space  $X$  is a Baire space.

REMARK 4. Observe that neither base countability nor metrizability assumptions are made on the considered spaces  $X, Y, Z$  in Theorem 1 while in Theorem 2 of [2] the same conclusion concerning the set of points of continuity is obtained under an *extra* assumption that  $X$  is first countable,  $Y$  is Baire,  $Z$  is second countable in a neighborhood of any of its points and such that  $Y \times Z$  is Baire.

## 2. CONDITIONS IMPLYING $x$ -CONTINUITY - COUNTER-EXAMPLES.

Given spaces  $X$  and  $Y$ ; a function  $f: X \rightarrow Y$  is said to be quasi-continuous (Martin [8], compare Kempisty [1]) if for every  $x \in X$  and for every neighborhood  $U$  of  $x$  and for every neighborhood  $V$  of  $f(x)$  have:  $U \cap \text{Int } f^{-1}(V) \neq \emptyset$ .

The main result of Lee and Piotrowski [2] is the following:

THEOREM A. (Lee and Piotrowski [2], Theorem 1, p. 383). Let  $X$  be first countable,  $Y$  be Baire,  $Z$  be second countable such that  $Y \times Z$  is Baire and let  $T$  be regular. If  $f: X \times Y \times Z \rightarrow T$  is:

- (1) continuous at  $X \times \{y\} \times \{z\}$ ,  $y \in Y$ ,  $z \in Z$ , and
- (2) quasi-continuous at points of  $\{x\} \times Y \times \{z\}$  for all  $x \in X$  and  $z \in Z$ , and
- (3) quasi-continuous at points of  $\{x\} \times \{y\} \times Z$  for all  $x \in X$  and  $y \in Y$

then  $f$  is  $x$ -continuous.

The first natural question which comes up is to check whether the converse of Theorem A is true. Apparently, the following Example 5 settles this question in the negative.

EXAMPLE 5. Let  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$  be defined by

$$f(x, y, z) = \begin{cases} \sin \frac{1}{x^2 + y^2 + z^2} & , \text{ if } (x, y, z) \neq (0, 0, 0) \\ 0, & \text{ otherwise} \end{cases}$$

The function  $f$  is  $x$ -continuous, however, fixing  $y = 0 = z$  we obtain that  $f(x, 0, 0)$  is not continuous.

Now we shall investigate the necessity of the assumptions in Theorem A, in particular:

- (\*) - continuity of  $f$  at points of  $X \times \{y\} \times \{z\}$
- (\*\*) - quasi-continuity of  $f$  at points of  $\{x\} \times Y \times \{z\}$ , and
- (\*\*\*) - quasi-continuity of  $f$  at points of  $\{x\} \times \{y\} \times Z$ .

In what follows (Examples 6 and 7) such constructions will be provided.

EXAMPLE 6. The assumption (\*) is essential. In fact, let us consider a function  $f: [-1, 1]^3 \rightarrow \mathbb{R}^3$  given as follows

$$f(x, y, z) = \begin{cases} (x, y, z+1), & \text{if } (x, y, z) \in [0, 1] \times [0, 1] \times [0, 1] \\ (x, y, z-1), & \text{if } (x, y, z) \in [-1, 0] \times [-1, 0] \times [-1, 0] \\ (x, y, z), & \text{otherwise} \end{cases}$$

A standard verification that  $f$  has the required property, (namely  $f$  is not  $x$ -continuous at  $(0, 0, 0)$ ) is left to the reader. Using somewhat more complex, but still elementary techniques we shall show that also (\*\*) (as well as (\*\*\*)) is essential. In fact, we have

EXAMPLE 7. Consider the function  $g: [-1, 1]^3 \rightarrow \mathbb{R}^3$  given as follows:

$$g(x, y, z) = \begin{cases} (x, y, z + 1) & \text{if } (x, y, z) \in [-1, 1] \times [-\frac{1}{2}, 1] \times \\ & \times \{(-\frac{1}{2}, \frac{1}{2}) \cap \mathbb{Q}\} \cup [\frac{1}{2}, 1] \\ (x, y, z), & \text{otherwise} \end{cases}$$

Again, we leave to the interested reader a standard verification that  $f$  is not  $x$ -continuous at  $(0, 0, 0)$ .

### 3. ONE-PROMISING HYPOTHESIS.

Observe that the definition of  $x$ -continuity at  $(p, q, r)$  requires the existence of a "small" neighborhood  $U'$  of  $p$  and "small" nonempty open sets  $V'$  and  $W'$  such that  $q$  and  $r$  "clusters" to  $V'$  and  $W'$  respectively and such that the set  $f(U' \times V' \times W')$  is contained in a "small", previously chosen, open set  $N$ . This observation prompts us to label this kind of product almost continuity as *1-3-continuity* - since we require the existence of only one "small" neighborhood  $U'$  (around  $p$ ) of the three neighborhoods  $U, V, W$ .

The term "1-3-continuity" has been used already, in a different sense in Breckenridge and Nishiura [9].

So, now let us consider "2-3-continuity".

More precisely, given spaces  $X, Y, Z$  and  $T$ , we say that  $f: X \times Y \times Z \rightarrow T$  is *2-3-continuous* or more specifically *xy-continuous*, if for every  $(p, q, r) \in X \times Y \times Z$ , for every neighborhood  $U \times V \times W$  of  $(p, q, r)$  and for every neighborhood  $N$  of  $f(p, q, r)$  there is a neighborhood  $U'$  of  $p$ , with  $U' \subset U$ , there is a neighborhood  $V^1$  of  $q$ , with  $V^1 \subset V$  and a nonempty open set  $W^1$ , with  $W^1 \subset W$  such that for all  $(x, y, z) \in U' \times V^1 \times W^1$  we have  $f(x, y, z) \in N$ .

Now, 3-3-continuity can be defined easily; the set  $W^1$  in definition of 2-3-continuity is assumed to be a neighborhood of  $r$  - not just only a nonempty open subset of  $W$ .

Clearly, every 3-3-continuous ( $\equiv$  continuous) function is 2-3-continuous; 2-3-continuous functions are 1-3-continuous and the latter are in turn 0-3-continuous ( $\equiv$  quasi-continuous).

It now follows from a result of T. Neubrunn [10] that if  $X, Y, Z$  are "nice" (e.g. Baire, second countable),  $T$ -regular then if  $f: X \times Y \times Z \rightarrow T$  is separately quasi-continuous then it is (jointly) quasi-continuous.

We can present this fact in the following symbolic equality:

$$0 + 0 + 0 = 0,$$

where the numbers (0 or 1) on the left side of the equality stand for quasi-continuity (0) or continuity (1) of the corresponding sections and the numbers on the right ( $i = 0, 1, 2$  or 3) denote the corresponding  $i$ -3-continuity of  $f$  as a function of three variables.

Theorem A implies that if  $X, Y, Z$  and  $T$  are as above and if  $f: X \times Y \times Z \rightarrow T$  is continuous in  $x$  and is quasi-continuous in  $y$  and is quasi-continuous in  $z$ , then  $f$  is 1-3-continuous. Consequently, we get:

$$1 + 0 + 0 = 1.$$

In view of the above considerations it is now natural to state the following:

HYPOTHESIS. Let  $X, Y$  and  $Z$  be Baire, second countable spaces and let  $T$  be regular. If  $f: X \times Y \rightarrow Z$   $T$  is:

- 1) continuous in  $x$ , and
- 2) continuous in  $y$ , and
- 3) quasi-continuous in  $z$ ,

Then  $f$  is 2-3-continuous;

In other words:

$$1 + 1 + 0 = 2$$

We shall resolve this Hypothesis in the *negative* in the forthcoming Example 8.

Now we shall exhibit two examples of  $i$ -3-continuous functions which are not  $(i + 1)$ -3-continuous,  $i = 1, 2$ .

EXAMPLE 8. A 1-3-continuous function which is not 2-3-continuous. Let  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$  be given by  $f(x_1, x_2, x_3) = g(x_1, x_2)$  where  $g$  is an arbitrary separately continuous function which is discontinuous at  $(0, 0)$ .

EXAMPLE 9. A 2-3-continuous function which is not 3-3-continuous ( $\equiv$  continuous).

Take  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$  to be  $f(x_1, x_2, x_3) = h(x_3)$ , where  $h$  is any function which is continuous except for 0.

Using the above pattern the reader will easily construct 0-3-continuous function ( $\equiv$  quasi-continuous) which is not 1-3-continuous.

Apparently, the above constructions can be illustrated with the following very specific formula-ready example.

EXAMPLE 10. Let  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$  be a function.

$$f(x_1, x_2, x_3) = g_i^3(x_1, \dots, x_i), \quad i = 1, 2 \quad \text{where}$$

$$g_i^3(x_1, \dots, x_i) = \frac{\prod_{j=1}^i x_j}{\prod_{j=1}^i (x_j)^i}, \quad \text{if } \sum_{j=1}^i (x_j)^i \neq 0$$

$$0, \text{ otherwise}$$

Then  $f$  is  $i$ -3-continuous which is not  $(i+1)$ -3-continuous,  $i = 1, 2$ .

#### 4. FURTHER GENERALIZATION OF $i$ -3-CONTINUITY.

Having defined 1-3 and 2-3-continuity for  $f: X_1 \times X_2 \times X_3 \rightarrow T$ , we shall now extend these ideas to a general case.

Namely, let  $n$  be an arbitrary natural number. We say that  $f$  function

$f: \prod_{i=1}^n X_i \rightarrow T$  is  $A$ - $n$ -continuous if for every  $(p_1, p_2, \dots, p_n) \in \prod_{i=1}^n X_i$  and for every neighborhood  $U_1 \times U_2 \times \dots \times U_n$  of  $(p_1, p_2, \dots, p_n)$  and for every neighborhood  $N$  of  $f(p_1, p_2, \dots, p_n)$  there are neighborhoods  $U'_{i,s}$  ( $1 \leq s \leq k$ ) of the first  $k$  out of  $n$  points  $p_1, p_2, \dots, p_n$  with  $U'_{i,s} \subset U_i$  and there are  $(n-k)$  nonempty open sets  $V'_{i,m}$  with  $V'_{i,m} \subset U_i$   $1 \leq m \leq n-k$  such that for all  $(x_1, x_2, \dots, x_n) \in \prod_{s=1}^k U'_{i,s} \times \prod_{m=1}^{n-k} V'_{i,m}$  we have  $f(x_1, x_2, \dots, x_n) \in N$ .

An interested reader will easily observe that the formula

$$g_k^n(x_1, \dots, x_k) = \frac{\prod_{i=1}^k x_i}{\prod_{i=1}^k (x_i)^k}, \quad \text{if } \sum_{i=1}^k (x_i)^k \neq 0$$

$$0, \text{ otherwise}$$

where  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  describes a  $k$ - $n$ -continuous function  $f$  given by

$$f(x_1, \dots, x_n) = g_k^n(x_1, \dots, x_k), \quad k = 1, 2, 3, \dots, n-1.$$

One can also give analogues of Example 8 and 9 for  $k$ - $n$ -continuity.

Studies of  $C(f)$  in hyperspaces for separately continuous functions and related ones were done also in Bögel [11] and Hahn [12].

#### 5. A PARTIAL SOLUTION TO A PROBLEM OF M. TALAGRAND.

M. Talagrand ([13] Problem 3 p. 160) asked whether if  $X$  is Baire,  $Y$  is compact and  $f: X \times Y \rightarrow \mathbb{R}$  is any separately continuous function, is there the set  $C(f)$  of points of continuity of  $f$  nonempty.

We shall answer this question in the positive if a compact space  $Y$  is additionally *first countable*.

In fact, we have shown the following result:

LEMMA 11. (Lee and Piotrowski [2], Lemma 2 p. 381). Let  $X$  be Baire,  $Y$  be first countable and  $Z$  be regular. If  $f: X \times Y \rightarrow Z$  is a function such that all its  $x$ -sections  $f_X$  are continuous with the exception of a first category set, and all its  $y$ -sections  $f_Y$  are quasi-continuous, then  $f$  is quasi-continuous with respect to  $y$ .

It follows from the definition that

REMARK 12. Every quasi-continuous function with respect to  $y$  is quasi-continuous.

LEMMA 13. (Marcus [14]). Let  $X$  be a Baire,  $M$  be metric. If  $f: X \rightarrow M$  is quasi-continuous, then  $C(f)$ , the set of point of continuity of  $f$  is dense  $G_\delta$  subset of  $X$ .

PROPOSITION 14. Let  $X$  be Baire,  $Y$  be compact first countable and let  $f: X \rightarrow Y \rightarrow \mathbb{R}$  be any separately continuous function. Then  $C(f) \neq 0$ .

PROOF. By Lemma 11 and Remark 12 such  $f$  is quasi-continuous. Now, since the Cartesian product of a compact space and a Baire space is Baire, we are done by Lemma 13.

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