SOME FURTHER RESULTS ON LEGENDRE NUMBERS

PAUL W. HAGGARD

Department of Mathematics, Last Carolina University
Greenville, North Carolina 27858 U.S.A.

(Received February 12, 1987)

ABSTRACT. The Legendre numbers, P_n^m, are expressed in terms of those numbers, P_k^{m-1}, in the previous column down to P_n^m and in terms of those, P_k^m, above but in the same column. Other results are given for numbers close to a given number. The limit of the quotient of two consecutive non-zero numbers in any one column is shown to be -1. Bounds for the Legendre numbers are described by circles centered at the origin. A connection between Legendre numbers and Pascal numbers is exhibited by expressing the Legendre numbers in terms of combinations.

1. INTRODUCTION.

The Legendre numbers were introduced in [1] and several elementary properties were given. In [2], some applications of the numbers were presented. Further applications are needed. In this note some relationships between the numbers are shown, bounds are given for the numbers, and the numbers are described in terms of combinations. For reference, we give (from [1]) the definition that we use, a general formula for the numbers, and a partial table of them.

Definition 1. The Legendre numbers, P_n^m, are the values of the associated Legendre functions, $P_n^m(x)$, for $x = 0$ and m, n non-negative integers.

A general formula for the Legendre numbers is

$$P_n^m = \begin{cases} 0, & m + n \ \text{odd} \\ 0, & m > n \\ \dfrac{(-1)^{\frac{n-m}{2}}(n + m)!}{2^n (\frac{n+m}{2})!\,(\frac{n-m}{2})!}, & m + n \ \text{even}, \ m \le n \ . \end{cases} \tag{1.1}$$

Another result needed is that

$$P_n^m = P_n^{(m)}(0), \tag{1.2}$$

where $P_n^{(m)}(0)$ is the mth derivative of the Legendre polynomial, $P_n(x)$, evaluated at $x = 0$.

2. SOME RELATIONSHIPS BETWEEN LEGENDRE NUMBERS.

Many relationships between Legendre numbers have been shown in [1], and [2].

Here, each Legendre number is expressed in terms of the non-zero entries in the previous column (see Table 1) down to this entry in two ways. Further, each is expressed in terms of the non-zero entries in the same column but above the entry.

n \ P_n^n	$P_n = P_n^0$	P_n^1	P_n^2	P_n^3	P_n^4	P_n^5	P_n^6	P_n^7	P_n^8 ...
0	1								
1	0	1							
2	$-\dfrac{1}{2}$	0	3						
3	0	$-\dfrac{3}{2}$	0	15					
4	$\dfrac{3}{8}$	0	$-\dfrac{15}{2}$	0	105				
5	0	$\dfrac{15}{8}$	0	$-\dfrac{105}{2}$	0	945			
6	$-\dfrac{15}{48}$	0	$\dfrac{105}{8}$	0	$-\dfrac{945}{2}$	0	10,395		
7	0	$-\dfrac{105}{48}$	0	$\dfrac{945}{8}$	0	$\dfrac{10,395}{2}$	0	135,135	
8	$\dfrac{105}{384}$	0	$-\dfrac{945}{48}$	0	$\dfrac{10,395}{8}$	0	$-\dfrac{135,135}{2}$	0	2,027,025
$\vdots$	$\cdots$								

TABLE 1.   LEGENDRE NUMBERS

From the known result, see [3],

$$P_n'(x) = \sum_{k=1}^{[\frac{n}{2}]} (2n - 4k + 3)P_{n-2k+1}(x) \; , \qquad (2.1)$$

where $[\frac{n}{2}] = \frac{n}{2}$ is n is even and $\frac{n+1}{2}$ if n is odd, it follows that by taking $m - 1$ derivatives then using (1.2), one has

$$P_n^m = \sum_{k=1}^{[\frac{n-m+1}{2}]} (2n - 4k + 3)P_{n-2k+1}^{m-1}, \quad m, \; n \geq 1 \; . \qquad (2.2)$$

This gives each Legendre number, P_n^m, as a sum of products involving the Legendre numbers in the preceeding column of Table 1 and above P_n^m. Other such formulas are possible. Four that can be proved (mathematical induction, inducting on n is one way) are:

$$P_{2n}^{2m} = (-1)^{n-m}(4m-1) \sum_{k=m}^{n} \left| P_{2k-1}^{2m-1} \right|, \qquad (2.3)$$

$$P_{2n+1}^{2m+1} = (-1)^{n-m}(4m + 1) \sum_{k=m}^{n} \left| P_{2k}^{2m} \right|, \qquad (2.4)$$

$$P_{2n}^{2m} = \frac{(-1)^{n-m}(4m+1)}{2(n-m)} \sum_{k=m}^{n-1} \left| P_{2k}^{2m} \right|, \quad n > m \; , \qquad (2.5)$$

$$P_{2n+1}^{2m+1} = \frac{(-1)^{n-m}(4m+3)}{2(n-m)} \sum_{k=m}^{n-1} \left| P_{2k+1}^{2m+1} \right|, \quad n > m. \qquad (2.6)$$

Note that (2.3) and (2.4) give each Legendre number as a product that involves the sum of the absolute values of the entries in the previous column and above the entry of Table 1 specified. Similarly, (2.5) and (2.6) involve the entries in the same

column but above the entry. Equations (2.4) and (2.6) can be obtained from (2.3) and (2.5), repectively, by replacing 2n with 2n + 1 and 2m with 2m + 1. In fact, (2.3) and (2.4) can be combined as

$$P_n^m = (-1)^{\frac{n-m}{2}}(2m-1) \sum_{k=1}^{[\frac{n-m+1}{2}]} \left| P_{n-2k+1}^{m-1} \right|, \tag{2.7}$$

while (2.5) and (2.6) can be combined as

$$P_n^m = \frac{(-1)^{\frac{n-m}{2}}(2m+1)}{n-m} \sum_{k=1}^{[\frac{n-m}{2}]} \left| P_{n-2k}^m \right|, \tag{2.8}$$

for m and n of the same parity.

There are several results concerning entries in Table 1 that are near each other. These can be easily proved using properties of Legendre numbers or by using (1.1). For example,

$$P_n^m = \frac{P_{n-1}^{m+1} + P_{n+1}^{m+1}}{2m + 1}, \quad n \geq m + 2 \tag{2.9}$$

gives each entry in terms of the entries in the next column and just above and below. Each entry in terms of the entries in the previous column and just above and below is given by

$$P_n^m = - \frac{(n+m-1)(n-m+2)}{2m - 3} (P_{n-1}^{m-1} + P_{n+1}^{m-1}), \quad n, \; m \geq 1. \tag{2.10}$$

Considering P_n^m and the nearest entries on slant lines through P_n^m leads to a determinant type result,

$$P_{n-1}^{m-1}P_{n+1}^{m+1} - P_{n-1}^{m+1}P_{n+1}^{m-1} = \frac{2(2n+1)}{(n+m-1)(n-m+2)} (P_n^m)^2. \tag{2.11}$$

Next, if we look at a particular non-zero entry and consider the first four non-zero entries above, below, to the left, and to the right, we have

$$P_{n-2}^m P_{n+2}^m - P_n^{m-2}P_n^{m+2} = 0, \quad n \geq 4, \quad 2 \leq m \leq n - 2 \tag{2.12}$$

which one can express as above · below = left · right.

In [1], it was shown that the sum of the non-zero entries in any column of Table 1 converges. The limit of the ratio of consecutive entries is somewhat surprising. Choose the mth column of Table 1. For n + m even and using (1.1), we have

$$\frac{P_{n+2}^m}{P_n^m} = - \frac{n+m+1}{n-m+2} = - \frac{1 + \frac{m}{n} + \frac{1}{n}}{1 - \frac{m}{n} + \frac{2}{n}}. \tag{2.13}$$

Therefore,

$$\lim_{n \to \infty} \frac{P_{n+2}^m}{P_n^m} = -1 . \tag{2.14}$$

From (2.13) it is clear that the limit approaches -1 from the right for m = 0 and from the left for m ≥ 1. It is clear that the limit of the absolute value of the ratios is 1.

3. BOUNDS FOR THE LEGENDRE NUMBERS.

From the known bound from [3],

$$|P_n(x)| < \left[\frac{\pi}{2n(1-x)}\right]^{\frac{1}{2}}, \tag{3.1}$$

for the Legendre polynomials, $P_n(x)$, one has, for $x = 0$

$$|P_n| < \sqrt{\frac{\pi}{2n}} = \frac{\sqrt{2\pi n}}{2n} = \frac{\sqrt{C}}{D} = \frac{\pi}{\sqrt{C}}, \quad n \geq 1 \tag{3.2}$$

where $C = 2\pi n$ is the circumference of a circle of radius n centered at the origin with $D = 2n$ the diameter of the circle.

In [1], the relationship

$$P_n^m = (n+m-1)(n+m-3)\cdots(n-m+3)(n-m+1)P_{n-m}, \quad m \geq 1 \tag{3.3}$$

was given where P_{n-m} is in the first column of Table 1. Using (3.2) in (3.3) we have the more general result

$$|P_n^m| < (n+m-1)(n+m-3)\cdots(n-m+3)(n-m+1)\frac{\sqrt{C}}{D}, \quad m \geq 1, \; n > m, \tag{3.4}$$

where $C = 2\pi(n-m)$ is the circumference of a circle of radius $n - m$ centered at the origin with $D = 2(n-m)$ the diameter of the circle.

4. LEGENDRE NUMBERS IN TERMS OF COMBINATIONS.

In [2], combinations were expressed in terms of the Legendre numbers. Here, we express the Legendre numbers as combinations. The equation

$$C(q,i) = \frac{(-1)^i q! \, 2^i P_{q+i}^{q-i}}{(q-i)! P_q^q}, \quad i = 0 \text{ to } q \tag{4.1}$$

from [2] becomes

$$P_n^m = \frac{(-1)^{\frac{n-m}{2}} m! \, P_{\frac{n+m}{2}}^{\frac{n+m}{2}} C(\frac{n+m}{2}, \frac{n-m}{2})}{(\frac{n+m}{2})! \, 2^{\frac{n-m}{2}}}, \tag{4.2}$$

after solving for P_{q+i}^{q-i} then letting $n = q + i$ and $m = q - i$. Notice that $\frac{n-m}{2} = i$ and $\frac{n+m}{2} = q$. Since $P_n^n = 1\cdot3\cdot5\cdots(2n-1)$, see [1],

$$P_{\frac{n+m}{2}}^{\frac{n+m}{2}} = 1\cdot3\cdot5\cdots(n+m-1)$$

$$= \frac{(n+m)!}{2^{\frac{n+m}{2}} (\frac{n+m}{2})!}. \tag{4.3}$$

Substituting (4.3) into (4.2) gives

$$P_n^m = \frac{(-1)^{\frac{n-m}{2}} m! \, (n+m)!}{2^n [(\frac{n+m}{2})!]^2} C(\frac{n+m}{2}, \frac{n-m}{2}) \tag{4.4}$$

for $n > 0$, m and n of the same parity, and $m \leq n$. The remaining values of P_n^m are given in [1] as $P_0^0 = 1$ and $P_n^m = 0$ for m and n of different parity.

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