

## A VARIATIONAL PRINCIPLE FOR COMPLEX BOUNDARY VALUE PROBLEMS

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**ABSTRACT** This paper provides a variational formalism for boundary value problems which arise in certain fields of research such as that of electricity, where the associated boundary conditions contain complex periodic conditions. A functional is provided which embodies the boundary conditions of the problem and hence the expansion (trial) functions need not satisfy any of them.

**KEY WORDS AND PHRASES** Variational principle, functional, stationary, boundary conditions complex functions, line integral.

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### 1. INTRODUCTION.

Motivated by complex periodic boundary conditions which arise in certain problems such as those of modelling the stator of a turbogenerator (see next section for detail), we give in this paper a variational formalism which takes into consideration such boundary conditions. We produce a functional which is stationary at the solution of a given boundary value problem for a class of expansion functions which do not satisfy any of the boundary conditions; these are satisfied only at the solution point. Three types of conditions are considered: 1) Dirichlet conditions, 2) Neumann or mixed conditions and 3) periodic conditions on parallel segments of the boundary.

Let  $R$  be a given complex domain with boundary  $\Gamma$ . Following the work of Delves and Hall [1], we split the boundary into four non-overlapping segments  $\Gamma_i$   $i = 1, 2, 3, 4$  and assume that periodicity conditions are imposed on the segments  $\Gamma_3$  and  $\Gamma_4$  such that for some fixed  $\underline{a}$ ,  $\Gamma_4 = \{ \underline{y} = \underline{x} + \underline{a} \mid \underline{x} \in \Gamma_3 \}$ . In this case we have the relations:

$$\underline{n}_4(\underline{x} + \underline{a}) = (\underline{n}_3 \cdot \underline{n}_4) \cdot \underline{n}_3(\underline{x}) \quad (1.1)$$

and

$$\int_{\Gamma_4} I(\underline{y}) \, ds = \int_{\Gamma_3} I(\underline{x} + \underline{a}) \, ds$$

where  $\underline{n}_3$  and  $\underline{n}_4$  are the unit outward normals to  $\Gamma_3$  and  $\Gamma_4$  respectively and  $\int ds$  is a line integral along the boundary with positive direction taken counterclockwise.

### 2. THE PROBLEM

Let the problem whose solution is sought be of the following form:

$$-\nabla^2 u + b(\underline{x})u = g(\underline{x}), \quad \underline{x} \in \Omega \quad (2.1.a)$$

with the prescribed boundary conditions:

$$\begin{aligned} u(\underline{x}) &= g_1(\underline{x}), \quad \underline{x} \in \Gamma_1 \\ \nabla u \cdot \underline{n}(\underline{x}) &= qu(\underline{x}) + g_2(\underline{x}), \quad \underline{x} \in \Gamma_2 \\ u(\underline{x}) &= e^{i\theta} u(\underline{x} + \underline{a}), \quad \underline{x} \in \Gamma_3 \\ \nabla u \cdot \underline{n}(\underline{x}) &= -e^{-i\theta} \nabla u \cdot \underline{n}(\underline{x} + \underline{a}), \quad \underline{x} \in \Gamma_3 \end{aligned} \quad (2.1.b)$$

where  $\Gamma_2$  and/or  $\Gamma_3$  may be void.

In modelling the stator of a turbogenerator where the rotor rotates at angular frequency and is effectively a bar magnet generating a rotating magnetic field, periodic boundary conditions of the form:

$$u(\underline{x}) = e^{i\theta} u(\underline{x} + \underline{a})$$

arise for the first harmonic component; and the normal gradient condition has:

$$\nabla u \cdot \underline{n}(\underline{x}) = -e^{-i\theta} \nabla u \cdot \underline{n}(\underline{x} + \underline{a})$$

where  $\theta$  is the sector angle. These two conditions are exactly the last two conditions of (2.1.b).

### 3. A FUNCTIONAL EMBODYING THE BOUNDARY CONDITIONS.

In this section we produce a functional which is stationary at the solution of (2.1) for a class of functions which do not satisfy any of the boundary conditions since these conditions are incorporated via suitable terms in the functional  $J$  given as:

$$\begin{aligned} J(V) &= \int_{\Omega} [\nabla^2 V + BV^2 - 2gV] d\underline{x} \\ &+ 2 \int_{\Gamma_1} (g_1 - V)(\nabla V \cdot \underline{n}) ds \\ &- 2 \int_{\Gamma_2} [q/2 V^2 + g_2 V] ds \\ &- \int_{\Gamma_3} [V(\underline{x}) - e^{i\theta} V(\underline{x} + \underline{a})][\nabla V(\underline{x}) - (\underline{n}_3 \cdot \underline{n}_4)e^{-i\theta} \nabla V(\underline{x} + \underline{a})] \cdot \underline{n} ds \end{aligned} \quad (3.1)$$

Next, it will be shown that if we expand the trial function  $V$  about the true solution  $u$ , of (2.1):  $V = u + \epsilon w$ , where  $\epsilon$  is a scalar and  $w$  is an arbitrary variation, then  $J(V)$  is stationary.

Define

$$G(\epsilon) = J(u + \epsilon w), \text{ then}$$

$$\begin{aligned} \frac{dG(0)}{d\epsilon} &= 2 \int_{\Omega} [\nabla w \cdot \nabla u + Bwu - gw] \cdot \underline{n} d\underline{x} \\ &+ 2 \int_{\Gamma_1} [(g_1 - u)\nabla w - w\nabla u] \cdot \underline{n} ds \\ &- 2 \int_{\Gamma_2} (qu + g_2)w ds \\ &- \int_{\Gamma_3} [u(\underline{x}) - e^{i\theta} u(\underline{x} + \underline{a})][\nabla w(\underline{x}) - (\underline{n}_3 \cdot \underline{n}_4)e^{-i\theta} \nabla w(\underline{x} + \underline{a})] \cdot \underline{n} ds \\ &- \int_{\Gamma_3} [w(\underline{x}) - e^{i\theta} w(\underline{x} + \underline{a})][\nabla u(\underline{x}) - (\underline{n}_3 \cdot \underline{n}_4)e^{-i\theta} \nabla u(\underline{x} + \underline{a})] \cdot \underline{n} ds \end{aligned} \quad (3.2)$$

The first line integral in (3.2) reduces by Green's theorem and (2.1.a) to:

$$2 \int_R [\nabla w \cdot \nabla u + B_{uw} - g w] d\underline{x} = 2 \int_{\Gamma} w \nabla u \cdot \underline{n} ds \quad (3.3)$$

$$= \left( \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} \right) (2w \nabla u \cdot \underline{n}) ds \quad (3.4)$$

where we have written the line integral of (3.3) as the sum of four line integrals along the boundaries into which  $\Gamma$  has been decomposed. The integrals over  $\Gamma_1$  and  $\Gamma_2$  of (3.4) cancel the corresponding integrals over  $\Gamma_1$  and  $\Gamma_2$  in (3.2) taking into consideration the boundary conditions in (2.1.b). Also from (2.1.b), it is obvious that the first of the two line integrals over  $\Gamma_3$  in (3.2) is equal to zero. What is left is to show that the last integral in (3.2) (hereafter referred to as LI) cancels the line integrals over  $\Gamma_3$  and  $\Gamma_4$  in (3.4). But

$$\begin{aligned} \text{LI} &= - \int_{\Gamma_3} w(\underline{x}) \nabla u(\underline{x}) \cdot \underline{n} ds \\ &\quad - \int_{\Gamma_3} w(\underline{x}) [-e^{-i\theta} \nabla u(\underline{x} + \underline{a})] (\underline{n}_3 \cdot \underline{n}_4) \cdot \underline{n}_3 ds \\ &\quad - \int_{\Gamma_3} w(\underline{x} + \underline{a}) [-e^{i\theta} \nabla u(\underline{x}) \cdot \underline{n}] ds \\ &\quad - \int_{\Gamma_3} w(\underline{x} + \underline{a}) [\nabla u(\underline{x} + \underline{a})] (\underline{n}_3 \cdot \underline{n}_4) \cdot \underline{n}_3 ds \end{aligned}$$

Using the relations (1.1) and the boundary conditions (2.1.b), we get:

$$\text{LI} = -2 \int_{\Gamma_3} w(\underline{x}) \nabla u(\underline{x}) \cdot \underline{n} ds - 2 \int_{\Gamma_4} w(\underline{x}) \nabla u(\underline{x}) \cdot \underline{n} ds \quad (3.5)$$

These line integrals over  $\Gamma_3$  and  $\Gamma_4$  cancel the corresponding ones in (3.4). Hence the functional  $J$  is stationary at the solution  $u$ .

#### 4. MATRIX SET-UP.

To describe the matrix set-up stage, we consider for convenience and simplicity the solution of the following one dimensional problem:

$$\left[ \frac{d^2}{dx^2} + B(x) \right] f(x) = G(x), \quad -1 \leq x \leq 1 \quad (4.1.a)$$

together with the boundary conditions:

$$f(-1) = \alpha, \quad f(1) = \beta \quad (4.1.b)$$

where  $z$  is regarded as a parameter that takes any complex value.

We seek an approximate solution  $f_N(x)$  to  $f(x)$  of the form:

$$f_N(x) = \sum_{n=1}^N a_n(z) h_n(x), \quad -1 \leq x \leq 1 \quad (4.2)$$

Then the problem represents a one-dimensional form of (2.1); and the functional  $J$  given in (3.1) reduces to:

$$J(V) = \int_{-1}^1 [(V')^2 + BV^2 - 2GV] dx - 2[\alpha - V(-1)]V'(-1) + 2[\beta - V(1)]V'(1) \quad (4.3)$$

The coefficients  $a_n(z)$  are defined by the stationary point of  $J$  (at the solution where  $V = f$ ); that is, by the equations:

$$L \underline{a} = [A + B + S] \underline{a} = \underline{G} + \underline{H} \quad (4.4.a)$$

where  $A$ ,  $B$ , and  $S$  are  $N \times N$  matrices;  $\underline{u}$  and  $\underline{h}$  are  $N$ -vectors, with components:

$$A_{ij} = \int_{-1}^1 h_i' h_j' dx, \quad B_{ij} = \int_{-1}^1 h_i B(x) h_j dx, \quad G_i = \int_{-1}^1 h_i G(x) dx,$$

$$S_{ij} = h_i(-1)h_j'(-1) + h_j(-1)h_i'(-1) - h_i(1)h_j'(1) - h_j(1)h_i'(1), \quad (4.4.b)$$

$$H_i = \alpha h_i'(-1) - \beta h_i'(1), \quad i, j = 1, 2, \dots, N.$$

When using global expansion functions, it is desirable for stability reasons to use orthogonal polynomials (see Mikhlin [2]). Accordingly, in (4.2) we take

$$h_{-2} = 1; h_{-1} = x; h_n = (1-x^2)^{1/2} T_n(x), \quad n=0,1,2,\dots,r \quad (4.5)$$

where  $r=N-3$  and  $T_n(x)$  is the  $n$ th Chebyshev polynomial of the first kind. The reason for this choice of basis is the need to handle the derivative terms in the matrix  $A$  without introducing artificial singularities. To calculate the elements in (4.4.b), we expand the functions  $B(x)$  and  $G(x)$  by Chebyshev series and use Fast Fourier Techniques to approximate the expansion coefficients. Thence we relate the elements  $A_{ij}$ ,  $B_{ij}$  and  $G_i$  of (4.4.b) to the coefficients of these expansions. This together with a numerical example will be considered in a subsequent paper.

While we do not attempt an error analysis here, the rapidity of convergence in calculating the matrix equation (4.4) has been considered formally by Delves and Mead [3], Freeman et al [4] and Delves and Biau [5]. In these papers it is shown that a complete characterisation of the convergence of the calculation can be given in terms of an assumed structure of the matrix  $L$  in (4.4) and the convergence of the Fourier coefficients of the right hand function  $G(x)$  in (4.1.a). Both a priori and a posteriori truncation error estimates are provided by Delves [6] where a very similar treatment to the one given in this section is used for Fredholm integral equations and from which we take (ignoring the a priori estimate since it contains an unknown constant):

$$\text{A posteriori estimate: } \sim \frac{C}{s-1} N^{-(s-1)} \sim N a_N \quad (4.6)$$

which is a standard bound;  $s = \min(p, q)$  where  $p$  and  $q$  depend on the differentiability of  $B(x)$  and  $G(x)$ . The procedure given in this section can easily be extended to two dimensions in a straightforward manner and details are omitted.

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