OPTIMALITY AND EXISTENCE FOR
LIPSCHITZ EQUATIONS

JOHNNY HENDERSON

Department of Algebra, Combinatorics, & Analysis
Auburn University
Auburn, Alabama 36849

ABSTRACT: Solutions of certain boundary value problems are shown to exist for the nth order differential equation $y^{(n)} = f(t,y,y',\ldots,y^{(n-1)})$, where f is continuous on a slab $(a,b) \times R^n$ and f satisfies a Lipschitz condition on the slab. Optimal length subintervals of (a,b) are determined, in terms of the Lipschitz coefficients, on which there exist unique solutions.

1. INTRODUCTION.

We will be concerned with the existence of solutions of boundary value problems for the nth order differential equation

$$y^{(n)} = f(t,y,y',\ldots,y^{(n-1)}), \qquad (1.1)$$

where f is continuous on a slab $(a,b) \times R^n$ and satisfies a Lipschitz condition,

$$\left| f(t,y_1,\ldots,y_n) - f(t,z_1,\ldots,z_n) \right| \leq \sum_{i=1}^{n} k_i \left| y_i - z_i \right|, \qquad (1.2)$$

on the slab.

A number of papers have appeared in which optimal length subintervals of (a,b) are determined, in terms of the Lipschitz coefficients k_i, $1 \leq i \leq n$, on which solutions of certain boundary value problems for (1.1) are unique; see, for example [1-15]. Of motivational importance in this work are the papers by Jackson [10-11] in which he applied methods from control theory in establishing optimal length subintervals, in terms of the Lipschitz coefficients, on which solutions of conjugate boundary value problems and right focal point boundary value problems for (1.1) are unique. It then follows from uniqueness implies existence results due to Hartman [16-17] and Klaasen [18] in the conjugate case and Henderson [19] in the right focal point case, that unique solutions exist on the optimal intervals given in [10-11].

In [7-8], we adapted Jackson's control theory arguments, in conjunction with
uniqueness implies existence results, and determined optimal length subin-
tervals of (a,b) on which there exist unique solutions of several classes of
boundary value problems for third and fourth order ordinary differential equations
satisfying Lipschitz conditions. In a recent work [9], we followed the pattern of
[7-8, 10-11], by applying the Pontryagin Maximum Principle to a linearization of
(1.1), and determined optimal length subintervals of (a,b), in terms of k_i, on which
solutions are unique for boundary value problems for (1.1) satisfying

$$y^{(i)}(t_1') = y_{i+1}, \qquad 0 \leq i \leq n-h+k-1,$$

$$y^{(i)}(t_i) = y_{n-h+(i+1)}, \qquad k \leq i \leq h-1,$$

(1.3)

where $a < t_1' < t_k \leq \cdots \leq t_{h-1} < b$, $0 \leq k < h \leq n$, and $y_i \epsilon$ **R**, $1 \leq i \leq n$.

In this work, we now address the problem of existence of solutions of (1.1),
(1.3) on the optimal intervals for uniqueness given in [9]. We state in Section 2
some of the results concerning optimality and uniqueness obtained in [9] which are
pertinent to the arguments here. Then in Section 3, we are able to prove that on
subintervals of length less than the optimal length given in Section 2 and for
certain values of k and h, solutions of (1.1), (1.3) exist. For this restricted
set of k and h, the existence result is in some sense analogous to the
uniqueness implies existence results in [16-19].

2. OPTIMALITY AND UNIQUENESS.

In this section, we state a Theorem and a Corollary from [9, Thm. 3 & the
Cor.], in which optimal length subintervals of (a,b) in terms of the Lipschitz co-
efficients k_i, $1 \leq i \leq n$, are determined on which solutions of (1.1), (1.3) are
unique.

THEOREM 1. Let $0 \leq k < h \leq n$ be given and let $\gamma = \min\{\gamma_\ell | k \leq \ell < h\}$, where
γ_ℓ is the smallest positive number such that there exists a solution x(t) of the
boundary value problem

$$x^{(n)} = (-1)^{h-\ell}[k_1 x + \sum_{i=1}^{n} k_i |x^{(i-1)}|],$$

$$x^{(i)}(0) = 0, \ 0 \leq i \leq n-h+\ell-1,$$

$$x^{(i)}(\gamma_\ell) = 0, \ \ell \leq i \leq h-1,$$

with x(t) > 0 on $(0,\gamma_\ell)$, or $\gamma_\ell = +\infty$ if no such solution exists. For any
$k \leq \ell < h$, if y(t) and z(t) are distinct solutions of (1.1) such that

$$y^{(i)}(t_1') = z^{(i)}(t_1'), \ 0 \leq i \leq n - h + \ell - 1,$$

$$y^{(i)}(t_i) = z^{(i)}(t_i), \ \ell \leq i \leq h - 1,$$

$a < t_1' < t_\ell \leq \cdots \leq t_{k-1} < b$, and if $t_{h-1} - t_1' < \gamma$, it follows that $y(t) \equiv z(t)$ on
(a,b), and this is best possible for the class of all differential equations
satisfying the Lipschitz condition (1.2).

REMARK. Jackson [11] has proved Theorem 1 for the case when h = n and k = 0. Now, on subintervals of length less than the constant γ in Theorem 1, it follows from Rolle's Theorem that solutions of a number of other boundary value problems for (1.1) are unique. For example, we can state the following.

COROLLARY 2. Let γ be as in Theorem 1. For any $k \leq \ell < h$ and $h - \ell \leq j \leq h$, if $y(t)$ and $z(t)$ are solutions of (1.1) such that

$$y^{(i)}(t_i') = z^{(i)}(t_i'), \quad 0 \leq i \leq n-h + \ell-1,$$

$$y^{(i+j-h)}(t_i) = y^{(i+j-h)}(t_i), \quad \ell \leq i \leq h-1,$$

$a < t_1' < t_\ell \leq \dots \leq t_{h-1} < b$, and if $t_{h-1} - t_1' < \gamma$, it follows that $y(t) \equiv z(t)$ on (a,b), and this is best possible.

3. EXISTENCE OF SOLUTIONS.

Analogous to uniqueness implies existence results proved by Hartman [16-17], Klaasen [18], and Henderson [19], we give a proof in this section that, on the optimal subintervals for uniqueness determined in Section 2 and for a restricted set of values of h and k, the boundary value problems (1.1), (1.3) have unique solutions. For the proof, we use somewhat standard shooting methods. We will give the proof only for two-point problems, with the proof for multipoint problems being similar.

THEOREM 3. Let $[n/2] \leq h \leq n$ be given, ([·] denotes the greatest integer function). Let $k = 0$ and let $\gamma = \min\{\gamma_\ell \mid 0 \leq \ell < h\}$ be as defined in Theorem 1. Then the boundary value problem

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}),$$

$$y^{(i)}(t_1) = y_{i+1}, \quad 0 \leq i \leq n-h+\ell-1,$$

$$y^{(i)}(t_2) = y_{n-h+(i+1)}, \quad \ell \leq i \leq h-1,$$

where $a < t_1 < t_2 < b$, $0 \leq \ell < h$, has a unique solution for any assignment of $y_i \in R$, $1 \leq i \leq n$, provided $t_2 - t_1 < \gamma$. Furthermore, this result is best possible for the class of all differential equations which satisfy the Lipschitz condition (1.2).

PROOF. Let $a < t_1 < t_2 < b$, with $t_2 - t_1 < \gamma$, and $y_i \in R$, $1 \leq i \leq n$, be given. We prove the existence of solutions for a much larger family of boundary value problems than those in the statement of the theorem. In fact, we prove the existence of solutions of the two-point problems which belong to the class of problems in Corollary 2. For induction purposes, we arrange these problems in a lower triangular array,

$$(1,1)$$
$$(2,1) \quad (2,2)$$
$$(h,1) \quad (h,2) \cdots (h,h),$$

where the boundary value problem for (1) associated with the (μ,ν)-position,

$1 \leq \nu \leq \mu \leq h$, satisfies

$$y^{(i)}(t_1) = y_{i+1}, \quad 0 \leq i \leq n-\nu-1,$$

$$y^{(i)}(t_2) = y_{n-\mu+(i+1)}, \quad \mu-\nu \leq i \leq \mu-1.$$

Under this arrangement, the boundary value problems for (1.1) along the principal diagonal (μ,μ), $1 \leq \mu \leq h$, are conjugate type problems, whereas the boundary value problems in the statement of this theorem are associated with the entries along the bottom row (h,ν), $1 \leq \nu \leq h$.

By Corollary 2, solutions of all the problems in this array are unique on subintervals of length less than γ. Moreover, by the constraints on h and k, it follows that solutions of all conjugate type boundary value problems for (1.1) are unique. Then, it follows from the uniqueness implies existence result of Hartman [16-17] and Klaasen [18] that the conjugate boundary value problems, and in particular those associated with the entries on the main diagonal, have unique solutions. (This is the reason for the constraints on h and k.) For existence of solutions of the remaining problems associated with the array, we will use the shooting method coupled with an induction along the subdiagonals on the array.

In that direction, choose any boundary value problem for (1) associated with the first subdiagonal $(\mu,\mu-1)$, where $2 \leq \mu \leq h$; that is, we are concerned with solutions of (1) satisfying

$$y^{(i)}(t_1) = y_{i+1}, \quad 0 \leq i \leq n-\mu,$$

$$y^{(i)}(t_2) = y_{n-\mu+(i+1)}, \quad 1 \leq i \leq \mu-1.$$

In applying the shooting method, let $z(t)$ be the solution of (1) satisfying conditions associated with the (μ,μ)-position,

$$z^{(i)}(t_1) = y_{i+1}, \quad 0 \leq i \leq n-\mu-1,$$

$$z(t_2) = 0,$$

$$z^{(i)}(t_2) = y_{n-\mu+(i+1)}, \quad 1 \leq i \leq \mu-1,$$

and define $S = \{y^{(n-\mu)}(t_1) \mid y(t)$ is a solution of (1) satisfying $y^{(i)}(t_1) = z^{(i)}(t_1)$, $0 \leq i \leq n-\mu-1$, and $y^{(i)}(t_2) = z^{(i)}(t_2)$, $1 \leq i \leq \mu-1\}$. $S \neq \emptyset$ since $z^{(n-\mu)}(t_1) \in S$, and since solutions of the problems corresponding to $(\mu,\mu-1)$-position are unique, it follows from a standard application of the Brouwer Invariance of Domain Theorem that S is open, (see [20-21] for a typical argument).

We claim that S is also a closed subset of R. Assuming the claim to be false, it follows that there is a limit point $r_0 \in \overline{S} \setminus S$. Hence, there exists a strictly monotone sequence $\{r_j\} \subset S$ of numbers converging to r_0. We may assume without loss of generality that $r_j \uparrow r_0$. For each $j \geq 1$, let $y_j(t)$ denote the solution of (1) given by the definition of S satisfying,

$$y_j^{(i)}(t_1) = z^{(i)}(t_1), \ 0 \leq i \leq n-\mu-1,$$

$$y_j^{(n-\mu)}(t_1) = r_j,$$

$$y_j^{(i)}(t_2) = z^{(i)}(t_2), \ 1 \leq i \leq \mu-1.$$

From Corollary 2, it follows that, for each $j \geq 1$, $y_j(t) < y_{j+1}(t)$ on $(t_1, t_2]$.
Furthermore, since f satisfies the Lipschitz condition (2), it follows that a
compactness condition on sequences of solutions of (1) is satisfied, (see [10]); from
this compactness condition and the fact that $r_0 \neq 3$, we have that $\{y_j(t)\}$ is not
uniformly bounded on each compact subinterval of (a,b), and in particular, is not
uniformly bounded above on each compact subinterval of $[t_1, t_2]$.

Now let $u(t)$ be the solution of the problem for (1) associated with the
$(\mu-1, \mu-1)$-position,

$$u^{(i)}(t_1) = y_{i+1}, \ 0 \leq i \leq n-\mu-1,$$

$$u^{(n-\mu)}(t_1) = r_0,$$

$$u(t_2) = 0,$$

$$u^{(i)}(t_2) = y_{n-\mu+(i+1)}, \ 1 \leq i \leq \mu-2.$$

It follows that, for some $\delta > 0$, $y_1^{(i)}(t) < u^{(i)}(t)$ on $(t_1, t_1 + \delta)$, $0 \leq i \leq n-\mu$,
and either (i) $(-1)^{i+1} y_1^{(i)}(t) < (-1)^{i+1} u^{(i)}(t)$ on $(t_1-\delta, t_1)$, $0 \leq i \leq n-\mu$, when $n-\mu$
is odd, or (ii) $(-1)^i y_1^{(i)}(t) < (-1)^i u^{(i)}(t)$ on $(t_1-\delta, t_1)$, $0 \leq i \leq n-\mu$, when $n-\mu$
is even. We will assume that $n-\mu$ is odd and also that $t_2 - (t_1 - \delta) < \gamma$.

It follows that there exists a subsequence $\{y_{j_k}(t)\}$ such that, for each $k \geq 1$,
$y_{j_k}^{(n-\mu)}(t)$ intersects $u^{(n-\mu)}(t)$ at a point $\rho_k \in (t_1, t_1 + \delta)$ and $y_{j_k}^{(n-\mu)}(t)$
intersects $u^{(n-\mu)}(t)$ or $y_1^{(n-\mu)}(t)$ at a point $\sigma_k \in (t_1 - \delta, t_1)$ and $\sigma_k \neq t_1$ and
$\rho_k \neq t_1$. By choosing successive subsequences and relabeling, we may assume that
$t_1-\delta < \sigma_k < t_1 < \rho_k < t_1 + \delta$ are the first points where these intersections occur.
Now, if there is an infinite subsequence, which we relabel as $\{y_{j_k}(t)\}$, such that
$y_{j_k}^{(n-\mu)}(\sigma_k) = y_1^{(n-\mu)}(\sigma_k)$, we have that, for each k, $(-1)^{i+1} y_1^{(i)}(t)$
$< (-1)^{i+1} y_{j_k}^{(i)}(t) < (-1)^{i+1} u^{(i)}(t)$ on (σ_k, t_1), $0 \leq i \leq n-\mu$. In this
case $\lim\limits_{k \to \infty} y_{j_k}^{(i)}(\sigma_k) = y_1^{(i)}(t_1)$, $0 < i < n-\mu$. But, it is also the case that
$y_{j_k}^{(i)}(t_2) = y_1^{(i)}(t_2)$, $1 < i < \mu-1$, and so from the continuous dependence of solutions

on boundary conditions of problems associated with the $(\mu,\mu-1)$-position, it follows

that $\{y_{j_k}^{(i)}(t)\}$ converges uniformly to $y_1^{(i)}(t)$ on compact subintervals of (a,b),

$0 \le i \le n-1$. This is impossible, since $y_{j_k}^{(n-\mu)}(t_1) = r_{j_k} + r_0 > r_1 = y_1^{(n-\mu)}(t_1)$.

In the case that there is an infinite subsequence, which we relabel again as

$\{y_{j_k}(t)\}$, such that $y_{j_k}^{(n-\mu)}(\sigma_k) = u^{(n-\mu)}(\sigma_k)$ and $y_{j_k}^{(n-\mu)}(\rho_k) = u^{(n-\mu)}(\rho_k)$, it

follows that, for each k, $(-1)^{i+1}y_1^{(i)}(t) < (-1)^{i+1}y_{j_k}^{(i)}(t) < (-1)^{i+1}u^{(i)}(t)$ on

(σ_k, t_1) and $y_1^{(i)}(t) < y_{j_k}^{(i)}(t) < u^{(i)}(t)$ on (t_1, ρ_k), $0 \le i \le n-\mu$, and

$y_{j_k}^{(n-\mu+1)}(\tau_k) = u^{(n-\mu+1)}(\tau_k)$, some $\tau_k \in (\sigma_k, \rho_k)$. It follows that $\lim_{k \to \infty} y_{j_k}^{(i)}(\sigma_k)$

$= u^{(i)}(t_1)$, $0 \le i \le n-\mu$, and that $\lim_{k \to \infty} y_{j_k}^{(n-\mu+1)}(\tau_k) = u^{(n-\mu+1)}(t_1)$; it is also the

case that $y_{j_k}^{(i)}(t_2) = u^{(i)}(t_2)$, $1 \le i \le \mu-2$. From uniqueness of solutions of

boundary value problems for (1) corresponding to the $(\mu-1, \mu-2)$-position coupled with

an argument similar to the one used in the proof of the first theorem of [13, Thm. 1]

and the fact that $t_2 - (t_1 - \delta) < \gamma$, solutions of this latter type of problem for

(1) are unique and thus depend continuously upon boundary conditions; it follows that

$\{y_{j_k}^{(i)}(t)\}$ converges uniformly to $u^{(i)}(t)$ on compact subintervals of (a,b),

$0 \le i \le n-1$. In particular, $u^{(\mu-1)}(t_2) = y_{j_k}^{(\mu-1)}(t_2)$, for all k, and it then

follows that $u^{(n-\mu)}(t_1) = r_0 \in S$; again, a contradiction.

Thus, S is also closed and hence $S \equiv R$. Choosing $y_{n-\mu} \in S$, the corresponding

solution of (1) satisfies the boundary value problem corresponding to the

$(\mu,\mu-1)$-position. Hence, boundary value problems for (1) associated with the first

subdiagonal, $(\mu,\mu-1)$, $2 \le \mu \le h$, have unique solutions.

For the induction, assume now that $2 < m+1 \le h$ and that, for each $1 \le s \le m$,

the boundary value problems for (1) associated with the subdiagonals

$(\mu,\mu-(s-1))$, $s \le \mu \le h$, have unique solutions.

For $s = m+1$, we now argue that boundary value problems for (1) corresponding to

the subdiagonal $(\mu,\mu-m)$, where $m+1 \le \mu \le h$, have unique solutions. Choosing any

such $(\mu,\mu-m)$, we are concerned with solutions of (1) satisfying

$$y^{(i)}(t_1) = y_{i+1}, \quad 0 \le i \le n-\mu+m-1,$$

$$y^{(i)}(t_2) = y_{n-\mu+(i+1)}, \quad m \le i \le \mu-1.$$

For the shooting scheme here, let $z(t)$ be the solution of (1) corresponding to the

$(\mu,\mu-(m-1))$-position satisfying

$$z^{(1)}(t_1) = y_{i+1}, \quad 0 < i < n - \mu+m-2,$$

$$z^{(m-1)}(t_2) = 0,$$

$$z^{(i)}(t_2) = y_{n-\mu+(i+1)}, \quad m < i < \mu-1.$$

In this case, define $S_1 = \{y^{(n-\mu+m-1)}(t_1) \mid y(t)$ is a solution of (1) satisfying $y^{(i)}(t_1) = z^{(i)}(t_1)$, $0 < i < n-\mu+m-2$, and $y^{(i)}(t_2) = z^{(i)}(t_2)$, $m < i < \mu-1\}$. In a manner analogous to above, it can be argued that S_1 is a nonempty subset of R which is both open and closed, so that $S_1 \equiv R$. Choosing $y_{n-\mu+m} \in S_1$, the corresponding solution of (1) is the desired solution. Hence, boundary value problems for (1) associated with the diagonal $(\mu,\mu-m)$, $m+1 \leq \mu \leq h$, have unique solutions.

Therefore, by induction, boundary value problems for (1) associated with each entry in the triangular array have unique solutions, and the conclusion of the theorem holds from the cases corresponding to the bottom row (h,ν), $1 \leq \nu \leq h$.

Using shooting methods and an induction similar to above, one can prove existence of solutions of multipoint boundary value problems (1), (3) with the same constraints on h and k.

THEOREM 4. Let $[n/2] \leq h \leq n$ be given. Let $k = 0$ and let $\gamma = \min \{\gamma_\ell \mid 0 \leq \ell < h\}$ be as defined in Theorem 1. Then the boundary value problem

$$y^{(n)} = f(t,y,y', \ldots, y^{(n-1)}),$$

$$y^{(i)}(t_1') = y_{i+1}, \quad 0 \leq i \leq n-h+\ell-1,$$

$$y^{(i)}(t_1) = y_{n-h+(i+1)}, \quad \ell \leq i \leq h-1,$$

where $a < t_1' < t_\ell \leq \ldots \leq t_{h-1} < b$, $0 \leq \ell < h$, has a unique solution for any assignment of $y_i \in R$, $1 \leq i \leq n$, provided $t_{h-1} - t_1' < \gamma$. This result is best possible for the class of all differential equations which satisfy the Lipschitz condition (1.2).

REFERENCES

1. AGARWAL, R., Best possible length estimates for nonlinear boundary value problems, Bull. Inst. Math. Acad. Sinica 9 (1931), 169-177.

2. AGARWAL, R. and CHOW, Y., Iterative methods for a fourth order boundary value problem, J. Comp. Appl. Math. 10 (1984), 203-217.

3. AGARWAL, R. and WILSON, S., On a fourth order boundary value problem, Utilitas Math. 26 (1984), 297-310.

4. GINGOLD, H., Uniqueness of solutions of boundary value problems of systems of ordinary differential equations, Pac. J. Math. 75 (1978), 107-136.

5. GINGOLD, H., Integ-ation criteria for second order nonlinear boundary value
    problems, J. Math. Anal. Appl. 73 (1980), 392-410.

6. GINGOLD, H. and GUSTAFSON, G., Uniqueness for nth order de la Vallee Poussin
    boundary value problems, Applicable Anal. 20 (1985), 201-220.

7. HENDERSON, J., Best interval lengths for boundary value problems for third order
    Lipschitz equations, SIAM J. Math. Anal., in press.

8. HENDERSON, J. and MCGWIER, R., Uniqueness, existence, and optimality for fourth
    order Lipschitz equations, J. Differential Equations, in press.

9. HENDERSON, J., Boundary value problems for nth order Lipschitz equations, J.
    Math. Anal. Appl., in press.

10. JACKSON, L., Existence and uniqeness of solutions of boundary value problems for
    Lipschitz equations, J. Differential Equations 32 (1979), 76-90.

11. JACKSON, L., Boundary value problems for Lipschitz equations, "Differential
    Equations" (S. Ahmed, M. Keener, A. Lazer, eds.), Academic Press, New York,
    1980, 31-50.

12. MELENTSOVA, YU., A best possible estimate of the nonoscillation interval for a
    linear differential equation with coefficients bounded in L_r, Diff. Urav. 13
    (1977), 1776-1786.

13. MELENTSOVA, YU, and MIL'SHTEIN, G., An optimal estimate of the length on which a
    multipoint boundary value problem possesses a solution, Diff. Urav. 10 (1974),
    1630-1641.

14. MELENTSOVA, YU. and MIL'SHTEIN, G., Optimal estimation of the nonoscillation
    interval for linear differential equations with bounded coefficients, Diff.
    Urav. 17 (1981), 2160-2175.

15. TROCH, I., On the interval of disconjugacy of linear autonomous differential
    equations, SIAM J. Math. Anal. 12 (1981), 78-89.

16. HARTMAN, P., Unrestricted n-parameter families, Rend. Circ. Mat. Palermo (2) 7
    (1958), 123-142.

17. HARTMAN, P., On n-parameter families and interpolation problems for nonlinear
    ordinary differential equations, Trans. Amer. Math. Soc. 154 (1971), 201-226.

18. KLAASEN, G., Existence theorems for boundary value problems for nth order
    ordinary differential equations, Rocky Mtn. J. Math. 3 (1973), 457-472.

19. HENDERSON, J. Existence of solutions of right focal point boundary value
    problems for ordinary differential equations, Nonlinear Anal. 5 (1981),
    989-1002.

20. HENDERSON, J., Uniqueness of solutions of right focal point boundary value prob-
    lems for ordinary differential equations, J. Differential Equation 41 (1981),
    218-227.

21. JACKSON, L., Uniqueness of solutions of boundary value problems for ordinary
    differential equations, SIAM J. Math. Appl. 24 (1973), 535-538.

Special Issue on
Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www .hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http:// mts.hindawi.com/ according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São Josè dos Campos, 12227-010 São Paulo, Brazil ; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk