

TWO PROPERTIES OF THE POWER SERIES RING

H. AL-EZEH

Department of Mathematics
University of Jordan
Amman, Jordan

(Received July 31, 1986 and in revised form October 29, 1986)

ABSTRACT. For a commutative ring with unity, A , it is proved that the power series ring $A[[X]]$ is a PF-ring if and only if for any two countable subsets S and T of A such that $S \subseteq \text{ann}(T)$, there exists $c \in \text{ann}(T)$ such that $bc = b$ for all $b \in S$. Also it is proved that a power series ring $A[[X]]$ is a PP-ring if and only if A is a PP-ring in which every increasing chain of idempotents in A has a supremum which is an idempotent.

KEY WORDS AND PHRASES. Power series ring, PP-ring, PF-ring, flat, projective, annihilator ideal and idempotent element.

1980 AMS SUBJECT CLASSIFICATION CODE. 13B.

1. INTRODUCTION.

Rings considered in this paper are all commutative with unity. Let $A[[X]]$ be the power series ring over the ring A . Recall that a ring A is called a PF-ring if every principal ideal is a flat A -module. Also a ring A is called a PP-ring if every principal ideal is a projective A -module.

It is proved in Al-Ezeh [1] that a ring A is a PF-ring if and only if the annihilator of each element $a \in A$, $\text{ann}(a)$, is a pure ideal, that is for all $b \in \text{ann}(a)$ there exists $c \in \text{ann}(a)$ such that $bc = b$. A ring A is a PP-ring if and only if for each $a \in A$, $\text{ann}(a)$ is generated by an idempotent, see Evans [2]. In Brewer [3], semihereditary power series rings over von Neumann regular rings are characterized. In this paper we characterize PF- power series rings and PP- power series rings over arbitrary rings.

For any reduced ring A (i.e. a ring with no nonzero nilpotent elements), it was proved in Brewer et al. [4] that

$$\text{ann}_{A[[X]]}(a_0 + a_1X + \dots) = N[[X]]$$

where N is the annihilator of the ideal generated by the coefficients a_0, a_1, \dots

2. MAIN RESULTS.

LEMMA 1. Any PF-ring A is a reduced ring.

PROOF. Assume that there is a nonzero nilpotent element in A . Let n be the least positive integer greater than 1 such that $a^n = 0$. So $a \in \text{ann}(a^{n-1})$. Because A is a PF-ring there exists $b \in \text{ann}(a^{n-1})$ such that $ab = a$. Thus $a^{n-1} \stackrel{A}{=} (ab)^{n-1} = a^{n-1}b^{n-1} = 0$ since $ba^{n-1} = 0$.

Contradiction. So any PP-ring is a reduced ring.

THEOREM 2. The power series ring $A[[X]]$ is a PF-ring if and only if for any two countable sets $S = \{b_0, b_1, b_2, \dots\}$ and $T = \{a_0, a_1, \dots\}$ such that $S \subseteq \text{ann}_A(T)$, there exists $c \in \text{ann}(T)$ such that $b_i c = b_i$ for $i = 0, 1, 2, \dots$

PROOF. First, we prove that $A[[X]]$ is a PF-ring.

Let $g(X) = b_0 + b_1 X + \dots$, and

$$f(X) = a_0 + a_1 X + \dots, \text{ and let}$$

$$g(X) \in \text{ann}_{A[[X]]}(f(X)). \text{ Then } g(X) f(X) = 0.$$

The ring A is in particular a PF-ring because for all $b \in \text{ann}(a)$, there exists

$c \in \text{ann}(a)$ such that $bc = b$. So by Lemma 1, A is a reduced ring. Thus

$$b_i a_j = \text{for all } i = 0, 1, \dots; j = 0, 1, 2, \dots$$

So

$\{b_0, b_1, \dots\} \subseteq \text{ann}(a_0, a_1, \dots)$. So by assumption, there exists $c \in \text{ann}(a_0, a_1, \dots)$ such that $b_i c = b_i$ for all $i = 0, 1, \dots$. Hence $g(X)c = g(X)$ and $c \in \text{ann}_{A[[X]]}(f(X))$. Consequently, the ring $A[[X]]$ is a PF-ring. Conversely, assume $A[[X]]$ is a PF-ring.

Let $\{b_0, b_1, \dots\} \subseteq \text{ann}(a_0, a_1, \dots)$. Let $g(X) = b_0 + b_1 X + \dots$, and $f(X) = a_0 + a_1 + \dots$ Then $g(X) f(X) = 0$. Therefore $g(X) \in \text{ann}_{A[[X]]}(f(X))$. Thus there exists $h(X) = c_0 + c_1 X + \dots$

in $\text{ann}_{A[[X]]}(f(X))$ such that $g(X) h(X) = g(X)$.

Consequently, $h(X) f(X) = 0$ and $g(X) (h(X) - 1) = 0$. Since A is reduced,

$c_i a_j = 0$ for all $i = 0, 1, \dots, j = 0, 1, 2, \dots$ and $b_i (c_0 - 1) = 0$ for all i and $b_i c_j = 0$ for all $j \geq 1$. Hence $\{c_0, c_1, \dots\} \subseteq \text{ann}(a_0, a_1, \dots)$ and $b_i (c_0 - 1) = 0$. So $c_0 \in \text{ann}(a_0, a_1, \dots)$ and $b_i c_0 = b_i$ for all $i = 0, 1, \dots$. Therefore the above condition holds.

Because any PP-ring is a PF-ring, every PP-ring is a reduced ring. On a reduced ring A , a partial order relation can be defined by $a \leq b$ if $ab = a^2$. The following lemma is given in Brewer[3] and Brewer et al.[4].

LEMMA 3. The relation \leq defined above on a reduced ring A is a partial order.

PROOF. Clearly the relation \leq is reflexive. Now assume $a \leq b$ and $b \leq a$. Then $ab = a^2$ and $ba = b^2$. So, $(a-b)^2 = a^2 - 2ab + b^2 = 0$. Because A is reduced $a - b = 0$,

or $a = b$. To prove transitivity of \leq , assume $a \leq b$ and $b \leq c$. So $ab = a^2$ and $bc = b^2$. Consider

$$\begin{aligned} (ac - ab)^2 &= a^2(c^2 - 2cb + b^2) \\ &= a^2(c^2 - b^2) \\ &= ab(c - b)(c + b) \\ &= 0 \end{aligned}$$

because $b(c - b) = 0$. Since A is reduced, $ac - ab = 0$ or $ac = ab = a^2$. Therefore $a \leq b$.

THEOREM 4. The power series ring $A[[X]]$ is a PP-ring if and only if A is a PP-ring in which every increasing chain of idempotents of A with respect to \leq has a supremum which is an idempotent element in A .

PROOF. Assume $A[[X]]$ is a PP-ring. Let $a \in A$. Since $A[[X]]$ is a PP-ring and idempotents in $A[[X]]$ are in A , $\text{ann}_A(a) = eA[[X]]$. We claim $\text{ann}(a) = eA$. Because $\text{ann}_A(a) = eA[[X]]$, $ea = 0$, $rea = 0$ for all $r \in A$. Hence $eA \subseteq \text{ann}(a)$. Now let $b \in \text{ann}(a)$. Hence $b \in \text{ann}_A(a)$. Thus $b = eg(X)$ for some $g(X) = b_0 + b_1X + \dots$. Consequently, $b = eb_0$.

That is $b \in eA$. Whence A is a PP-ring.

To complete the proof of this direction, let $e_0 \leq e_1 \leq e_2 \dots$ be an increasing chain of idempotents in A . Because $A[[X]]$ is a PP-ring and since idempotents of $A[[X]]$ are in A , $\text{ann}_{A[[X]]}(e_0 + e_1X + \dots) = eA[[X]]$. Now we claim $1 - e = \sup\{e_0, e_1, \dots\}$.

Since $ee_i = 0$, $e_i(1 - e) = e_i = e_i^2$, $i = 0, 1, \dots$.

So $e_i \leq 1 - e$ for all $i = 0, 1, \dots$. Let y be an upper bound of $\{e_0, e_1, \dots\}$. So $e_i \leq y$ for $i = 0, 1, \dots$.

Hence $1 - y \in \text{ann}_{A[[X]]}(e_0 + e_1X + \dots)$.

Thus $1 - y = ec$ for some $c \in A$. Consequently,

$$\begin{aligned} y(1 - e) &= (1 - ce)(1 - e) \\ &= 1 - ec - e + ec \\ &= 1 - e \end{aligned}$$

So $1 - e \leq y$. Therefore $1 - e = \sup\{e_0, e_1, \dots\}$.

To prove the other way around, consider $\text{ann}_{A[[X]]}(f(X))$ where $f(X) = a_0 + a_1X + \dots$.

Hence

$$\text{ann}_{A[[X]]}(f(X)) = \text{ann}_A(a_0, a_1, \dots)[[X]]$$

$$\text{ann}_A(a_0, a_1, \dots) = \bigcap_{i=0}^{\infty} \text{ann}(a_i)$$

$$= \bigcap_{i=0}^{\infty} e_i A, \quad e_i^2 = e_i$$

because A is a PP-ring.

Let $d_0 = e_0$, $d_1 = e_0 e_1$, ..., $d_n = d_{n-1} e_n$, ...

One can easily check that

$$\bigcap_{i=0}^{\infty} e_i A = \bigcap_{i=0}^{\infty} d_i A$$

Also it is clear that

$$d_0 \geq d_1 \geq d_2 \dots$$

Therefore

$$1 - d_0 \leq 1 - d_1 \leq 1 - d_2 \dots$$

By assumption, this increasing chain of idempotents has a supremum which is an idempotent. Let

$$\text{Sup}\{1 - d_0, 1 - d_1, 1 - d_2, \dots\} = d. \quad \text{So}$$

$$(1 - d_i) d = 1 - d_i \quad \text{for all } i = 0, 1, \dots .$$

We claim that

$$\bigcap_{i=0}^{\infty} d_i A = (1 - d)A.$$

Now $1 - d \geq d_i$. So $(1 - d)d_i = 1 - d$. Hence

$$(1 - d)A \subseteq d_i A \quad \text{for all } i = 0, 1, \dots .$$

Thus $(1 - d)A \subseteq \bigcap_{i=0}^{\infty} d_i A$.

Let $y \in \bigcap_{i=0}^{\infty} d_i A$. Then $y = d_i y_i$, $i, 0, 1, \dots .$

Consequently

$$\begin{aligned} (1 - d_i)(1 - y) &= 1 + d_i y - d_i - y \\ &= 1 - d_i \end{aligned}$$

Because $y d_i = d_i^2 = d_i^2 y_i = d_i y_i = y$.

Therefore $1 - d_i \leq 1 - y \quad \text{for all } i = 0, 1, \dots .$

Because $d = \sup\{1 - d_0, 1 - d_1, 1 - d_2, \dots\}$,
 $d \leq 1 - y$. So $d = d(1 - y) = d - dy$

Hence $dy = 0$. Thus $y(1 - d) = y - yd = y$

That is $y \in (1 - d)A$. Therefore $\bigcap_{i=0}^{\infty} d_i A = (1 - d)A$.

Consequently,

$$\text{ann}_{A[[X]]}(f(X)) = (1 - d)A[[X]]$$

Therefore $A[[X]]$ is a PP-ring.

REFERENCES

1. AL-EZEH, H. On Some properties of Polynomial rings. I.J.M.M.S To appear
2. EVANS, M. On commutative PP-rings. Pac. J. Math. 41(1972) 687-697.
3. BREWER, J. "Power series over commutative rings". Lecture Notes in pure and applied Mathematics No. 64, Marcel Dekker, New York and Basel (1981).
4. BREWER, J. , RUTTER, E. and WATKINS, J. Coherence and weak global dimension of $R[[X]]$ when R is Von Neumann regular, J. of Algebra 46(1977) 278-289.

Special Issue on Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods:** artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning

- **Application fields:** asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects:** decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site <http://www.hindawi.com/journals/jamds/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/>, according to the following timetable:

| | |
|------------------------|------------------|
| Manuscript Due | December 1, 2008 |
| First Round of Reviews | March 1, 2009 |
| Publication Date | June 1, 2009 |

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk