Internat. J. Math. & Math. Sci.
VOL. 11 NO. 1 (1988) 201-204

RESEARCH NOTES

A NOTE ON THE UNIQUE SOLVABILITY OF A CLASS OF NONLINEAR EQUATIONS

RABINDRANATH SEN

Department of Applied Mathematics
University College of Science
92, Acharya Prafulla Chandra Road
Calcutta - 700009

and

SULEKHA MUKHERJEE

Department of Mathematics
University of Kalyani
Kalyani, Dt.Nadia
West Bengal, India

(Received April 28, 1986 and in revised form September 17, 1986)

ABSTRACT. The aim of the present note is to devise a simple criterion for the existence of the unique solution of a class of nonlinear equations whose solvability is taken for granted.

KEY WORDS AND PHRASES. Hammerstein equation, Monotonically Decomposable operators.
1980 AMS SUBJECT CLASSIFICATION CODES. PRIMARY. 66J15, 47H17.

1. INTRODUCTION.

In equations describing physical problems presence of more than one solution may sometimes create complications. One is often led to a solution that may differ from the desired solution and hence a lack of agreement of the solution with the experimental result occurs. We are therefore motivated in devising a simple criterion by which the uniqueness of the solution of a class of equations is guaranteed.

In what follows we take X to be a complete supermatic space [1] and f to be an element of X.

A is a nonlinear mapping of X into X and we are interested in solving the equation

$$u = Au + f, \quad f \in X \tag{1.1}$$

by the iterates of the form

$$u_{n+1} = A u_n + f \ (u_o \text{ prechosen}) \tag{1.2}$$

Section 2 contains the convergence theorem and an example is appended in section 3. Earlier Sen [2], Sen and Mukherjee [3] proved the unique solvability of the nonlinear equation $Au = Pu$ in the setting of a metric space.

2. CONVERGENCE.

THEOREM 2.1. Let the following conditions be fulfilled:

i) There exists a bounded linear operator L mapping X into X s.t.

a) $\rho\ (Au, Av) \leqq \rho(Lu, Lv),\ \forall\ u, v \in X$

b) $\rho\ (L^p Au, L^p Av) \leqq \rho(L^{p+1}u, L^{p+1}v),\ p = 0,1, \ldots (m-1)$

c) $\rho\ (L^m u, L^m v) \leqq q\, \rho(u,v),\ \forall\ u, v \in X,\ 0 < q < 1$ and for fixed m

ii) f belongs to the range of $(I-A)$.

Then the sequence u_n defined by (1.2) converges to the unique solution of (1.1).

PROOF. By condition (ii) there exists a

$$u^* \in X \quad \text{s.t} \qquad u^* = Au^* + f \tag{2.1}$$

The space being supermetric and the use of the conditions a), b) and c) yields

$$\begin{aligned} \rho(u_{m+1}, u^*) &= \rho\ (Au_m, Au^*) \\ &\leqq \rho(L^{(m-1)}u_o, L^{(m-1)}u^*) \\ &\leqq q\, \rho(Lu_o, Lu^*) \end{aligned} \tag{2.2}$$

Hence $\rho(u_{nm}, u^*) \leqq q^n \rho(u_o, u^*)$ --- 0 as $n \to \infty$ $(0 < q < 1)$ (2.3)

If v^* is another solution of the equation

$$\begin{aligned} \rho\ (u^*, v^*) &= \rho\ (Au^* + f, Av^* + f) \\ &\leqq q\, \rho(u^*, v^*)\ (0 < q < 1) \end{aligned} \tag{2.4}$$

Hence, $u^* = v^*$

Therefore $\{u_n\}$ converges uniquely to the solution of (1.1) where f belongs to the range of (I.A).

3. EXAMPLE.

In this section we consider the following Hammerstein equation

$$u(x) = 1 + \int_0^1 |x-t|\ [\ u(t) - \tfrac{1}{2} u^2(t)\ ]\ dt \tag{3.1}$$

in the setting of C(0,1).

By using the theory of Monotonically Decomposable Operator (MDO) [1] Collatz proved [4] the existence of a solution $u(x)$ with

$$2(x - x^2) \leqq u(x) \leqq 2(1 - x - x^2) \tag{3.2}$$

Let $X = \{u(x)\ /\ 2(x - x^2) \leqq u(x) \leqq 2(1 - x - x^2)\}$

We would show that in X the equation (3.1) admits of a unique solution. Here

$$Au = \int_0^1 |x-t|\ [\ u(t) - \tfrac{1}{2} u^2(t)\ ]\ dt \tag{3.3}$$

Let us choose $Lu = \int_0^1 |x-t|\ u(t)\ dt$ (3.4)

We take $\rho\ (u,v) = ||\ u-v\ || = \max\limits_{0 \leq x \leq 1} |u(x) - v(x)|\quad 0 \leqq x \leqq 1$ (3.5)

$\forall\ u(x),\ v\ (x)\quad C(0,1).$

Let us consider the metric in X induced by the metric in C(0,1) and Complete X w.r.t. the induced metric so that X is a complete supermetric space.

$$Au - Av = \int_0^1 |x-t| \ (u(t)-v(t))dt - \frac{1}{2} (u(\xi) + v(\xi)) \int_0^1 |x-t| \ u(t) \ dt - \frac{1}{2} (u(\eta) + v(\eta)) \int_0^1 |x-t| \ v(t) \ dt \qquad (3.6)$$

$$0 < \xi < 1$$

$$0 < \eta < 1$$

$$\max_{o\le x\le 1} \left| 1 - \frac{u(x) + v(x)}{2} \right| \le 1, \quad \forall \ u(x), v(x) \in X \qquad (3.7)$$

$$\rho \ (Lu,Lv) = \max_{0\le x\le 1} |u(\bar{\xi})-v(\bar{\eta})| \int_0^1 |x-t| \ dt \ (0<\bar{\xi}<1, \ o<\bar{\eta}<1) \qquad (3.8)$$

$$\int_0^1 |x-t| \ v(t) \ dt \le \frac{v(\bar{\eta})}{|u(\bar{\xi}) - v(\bar{\eta})|} \ \rho(Lu,Lv) \qquad (3.9)$$

Neglecting quantities of second order in ξ,η

$$|Au-Av| \le [\ 1 - \frac{1}{2} \ (u(\xi) + v(\xi) \ ) + v(\bar{\eta})] \ \rho(Lu, \ Lv)$$

Now $1 - \frac{1}{2} \ (u(\xi) + v(\xi) + v(\bar{\eta}))$

$$1 - 2 \ (\xi - \xi^2) + 2 \ (1 - \bar{\eta} + \bar{\eta}^2) \simeq 1 \qquad (3.11)$$

Therefore $\rho(Au, Av) \le \rho(Lu, Lv), \ \forall u, v \in X$ (3.12)

Simple manipulation shows that

$$\rho(LAu, \ LAv) \le \rho(L^2u, \ L^2v) \qquad (3.13)$$

Using Schwartz inequality we have

$$\rho(L^2u, \ L^2v) \le \frac{1}{3} \ \rho(u,v) \qquad (3.14)$$

It may however be noted that both A and L map X into X where X is a complete metric space.

Hence by Theorem 2.1 $\{u_n\}$ defined by

$$u_{n+1} = 1 - \int_0^1 |x-t| \ [u_n(t) - \frac{1}{2} u_n^2(t))] \ dt \quad n = 0,1,2,...$$

Converges to the unique solution of equation (3.1) in X.

3.1. In many nonlinear equations it is possible to generate L's which could be prototypes of the linearized versions of A.

ACKNOWLEDGEMENT. The authors are grateful to the referee for some comments.

REFERENCES

1. COLLATZ, L. Functional Analysis and Numerical Mathematics, Academic Press, N.Y., 1966.

2. SEN, R. Approximate Iterative Process in a Supermetric Space, Bull. Cal. Math. Soc. 63(1971), 121-123.

3. SEN, R. and MUKHERJEE, S. On Iterative Solutions of Nonlinear Functional Equations in a Metric Space, International J. Math. and Math. Sci. 6(1983), 161-170.

4. COLLATZ, L. Nonlinear Functional Analysis and Applications, (Ed.L.B.Rall), Academic Press, 1971.

Mathematical Problems in Engineering

Special Issue on Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil ; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru