

## M-QUASI-HYPONORMAL COMPOSITION OPERATORS

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**ABSTRACT.** A necessary and sufficient condition is obtained for M-quasi-hyponormal composition operators. It has also been proved that the class of M-quasi-hyponormal composition operators coincides with the class of M-paranormal composition operators. Existence of M-hyponormal composition operators which are not hyponormal; and M-quasi-hyponormal composition operators which are not M-hyponormal and quasi-hyponormal are also shown.

**KEY WORDS AND PHRASES.** M-hyponormal, M-quasi-hyponormal, M-paranormal, normal composition operators.

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### 1. PRELIMINARIES.

Let  $(X, S, m)$  be a sigma-finite measure space and  $T$  a measurable transformation from  $X$  into itself (that is one  $mT^{-1}(E) = 0$  whenever  $m(E) = 0$  for  $E \in S$ ). Then the equation  $C_T f = f \circ T$  for every  $f$  in  $L^2(m)$  defines a linear transformation. If  $C_T$  is bounded with range in  $L^2(m)$ , then it is called composition operator. If  $X = N$  the set of all non-zero positive integers and  $m$  is counting measure on the family of all subsets of  $N$ , then  $L^2(m) = \ell^2$  (the Hilbert space of all square summable sequences).

Let  $f_o = \frac{dmT^{-1}}{dm}$  be the Radon-Nikodym derivative of the measure  $mT^{-1}$  with respect to the measure  $m$ ,

$$\frac{dm(T \circ T)}{dmT^{-1}} = g_o, \quad \frac{dm(T \circ T)}{dm} = h_o$$

Then  $h_o = f_o \cdot g_o$ .

Let  $B(H)$  denote the Banach algebra of all bounded linear operators on the Hilbert space  $H$ . An operator  $T \in B(H)$  is called M-quasi-hyponormal if there exists  $M > 0$  such that

$$M^2 T^* T^2 - (T^* T)^2 \geq 0$$

or equivalently  $\|T^*Tx\| \leq M \|T^2x\|$  for all  $x$  in  $H$  [1].  $T$  is said to be  $M$ -paranormal [2] if for all unit vectors  $x$  in  $H$

$$\|Tx\|^2 \leq M \|T^2x\|.$$

$T$  is said to be  $M$ -hyponormal [2] if

$$\|Tx\|^2 \leq M \|Tx\| \text{ for all } x \text{ in } H.$$

The purpose of this paper is to generalize the results on quasi-hyponormal composition operators in [3] for  $M$ -quasi-hyponormal composition operators.

## 2. $M$ -QUASI-HYPONORMAL COMPOSITION OPERATORS.

In this section we obtain a necessary and sufficient condition for  $M$ -quasi-hyponormal composition operators and then show that the class of  $M$ -quasi-hyponormal composition operators on  $\ell^2$  coincides with the class of  $M$ -paranormal composition operators. We also show the existence of  $M$ -hyponormal composition operators which are not hyponormal, and  $M$ -quasi-hyponormal composition operators which are not  $M$ -hyponormal and quasi-hyponormal.

**THEOREM 2.1.** Let  $C_T \in B(\ell^2)$ . Then  $C_T$  is  $M$ -quasi-hyponormal if and only if  $f_o^2 \leq M^2 h_o$ .

**PROOF.** Since for any  $f$  in  $\ell^2$ ,

$$(C_T^* C_T^2 f, f) = (C_T^2 f, C_T^2 f) = \int h_o |f|^2 dm,$$

$$= (M_{h_o} f, f),$$

where  $M_{h_o}$  is the multiplication operator induced by  $h_o$ , therefore  $C_T^* C_T^2 = M_{h_o}^2$ .

Similarly it can be seen that  $C_T^* C_T = M_{f_o}$ .  $C_T$  is  $M$ -quasi-hyponormal if and only if

$$M^2 C_T^* C_T^2 - (C_T^* C_T)^2 \geq 0.$$

This implies that

$$M^2 M_{h_o} - M_{f_o}^2 \geq 0,$$

that is  $f_o^2 \leq M^2 h_o$ .

Hence the result.

**COROLLARY.** Let  $C_T \in B(\ell^2)$ . Then  $C_T$  is  $M$ -quasi-hyponormal if and only if  $f_o^2 \leq M^2 g_o$ .

**PROOF.** Since  $h_o = f_o \cdot g_o$  and  $f_o$  is positive, therefore, by above theorem we get the result.

**THEOREM 2.2.** Let  $C_T \in B(\ell^2)$ . Then  $C_T$  is  $M$ -quasi-hyponormal if and only if  $C_T$  is  $M$ -paranormal.

**PROOF.** Necessity is true for any bounded operator  $A$ . For the sufficiency, let  $C_T$  be  $M$ -paranormal, then

$$||C_T x_{\{n\}}||^2 \leq M ||C_T^2 x_{\{n\}}|| \quad \text{for all } n \in N$$

$$\text{or } \int |x_{\{n\}} \circ T|^2 dm \leq M (\int |x_{\{n\}} \circ T|^2 dm)^{1/2}$$

$$\text{or } \int |x_{\{n\}}|^2 dm T^{-1} \leq M (\int |x_{\{n\}}|^2 dm (T \circ T)^{-1})^{1/2}$$

$$\text{or } \int_{\{n\}} f_o dm \leq M (\int_{\{n\}} h_o dm)^{1/2}$$

$$\text{or } f_o^2(n) \leq M^2 h_o(n) \quad \text{for all } n \in N.$$

Hence  $f_o^2 \leq M^2 h_o$ ;  $C_T$  is M-quasi-hyponormal.

**THEOREM 2.3.** Let  $C_T \in B(\ell^2)$  and  $T: N \rightarrow N$  be one-to-one. Then the following are equivalent.

- (i) Normal
- (ii) M-hyponormal
- (iii) M-quasi-hyponormal.

**PROOF.** (i) implies (ii), (ii) implies (iii) are always true for any bounded operator A. We show that (iii) implies (i). Let  $C_T$  be M-quasi-hyponormal. Then  $||C_T^* C_T f|| \leq M ||C_T^2 f||$  for all  $f$  in  $\ell^2$ . Now T is onto because if T is not onto then  $N|T(N)$  is non-empty and for  $n \in N|T(N)$

$$||C_T^* C_T x_{\{n\}}|| = 1 \text{ and } ||C_T^* C_T x_{\{n\}}|| = 0.$$

There exists no  $M > 0$  such that  $C_T$  is M-quasi-hyponormal which is a contradiction.

Since T is one-to-one, therefore, T is invertible, by Theorem 2.2 [4]  $C_T$  is invertible and  $C_T$  is normal by Theorem 2.1 [3].

Here we give an example of a composition operator on  $\ell^2$  which is M-hyponormal but not hyponormal.

**EXAMPLE 1.** Let  $T: N \rightarrow N$  be the mapping such that

$$T(1) = 2, \quad T(2) = 1, \quad T(3) = 2 \text{ and}$$

$$T(3n+m) = n+2, \quad m = 1, 2, 3 \quad \text{and } n \in N.$$

Then  $C_T$  is not hyponormal as  $f_o \circ T \neq f_o$  for  $n = 1$ .  $C_T$  is M-hyponormal for  $M \geq \sqrt{2}$ .

**EXAMPLE 2.** Let  $T: N \rightarrow N$  be defined by  $T(1) = 2, T(2) = 1, T(3n+m) = n+1, m = 0, 1, 2$  and  $n \in N$ . Then  $C_T$  is  $\sqrt{2}$  - quasi-hyponormal but  $C_T$  is not  $\sqrt{2}$  - hyponormal.  $C_T$  is not quasi-hyponormal also.

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