Internat. J. Math. & Math. Sci.
Vol. 10, No.3 (1987) 531-534

SOME GENERATING FUNCTIONS OF LAGUERRE POLYNOMIALS

BIDYUT KUMAR GUHA THAKURTA

Department of Mathematics
R.K.M.V.C. College, Rahara,
24-Parganas, West Bengal
INDIA

(Received November 12, 1985)

ABSTRACT. In this note a class of interesting generating relation, which is stated in the form of theorem, involving Laguerre polynomials is derived. Some applications of the theorem are also given here.

KEY WORDS AND PHRASES. Laguerre polynomials, generating functions.
1980 AMS SUBJECT CLASSIFICATION CODE. 33A65

1. INDRODUCTION.

The Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by,

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \; {}_1F_1\,(-n;\; 1+\alpha;\; x) \tag{1.1}$$

where n is a non-negative integer.

From [1] we have

$$\sum_{n=0}^{\infty} \binom{n+m}{n} L_{n+m}^{(\alpha)}(x)\, w^n = (1-w)^{-1-\alpha-m} \exp\left(\frac{-xw}{1-w}\right) L_m^{(\alpha)}\left(\frac{x}{1-w}\right), \tag{1.2}$$

Observing the existence of the above generating relation (1.2) the present author is interested to investigate the existence of more general generating relation by the group-theoretic method. In fact, the following theorem is obtained as the main result of our investigation.

THEOREM 1. If there exists a generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n\, w^n\, L_{n+m}^{(\alpha)}(x) \tag{1.3}$$

then

$$(1-w)^{-1-\alpha-m} \exp\left(\frac{-wx}{1-w}\right) G\left(\frac{x}{1-w}, \frac{wz}{1-w}\right) = \sum_{n=0}^{\infty} w^n\, f_n(z)\, L_{n+m}^{(\alpha)}(x) \tag{1.4}$$

where

$$f_n(z) = \sum_{k=0}^{n} \binom{n+m}{k+m} a_k\, z^k .$$

The importance of the above theorem lies in the fact that one can get a good number of generating relations from (1.4) by attributing different suitable values to a_n in the relation (1.3).

2. DERIVATION OF THE THEOREM.

THEOREM 1. Using the differential recurrence relation [2]

$$x \frac{d}{dx} (L_{n+m}^{(\alpha)} (x)) = (n+m+1) L_{n+m+1}^{(\alpha)} (x) - (n + m + \alpha + 1 - x) L_{n+m}^{(\alpha)} (x) . \tag{2.1}$$

We find the following partial differential operator,

$$\mathbb{R} = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (-x+m+1)y$$

such that

$$\mathbb{R} (y^{\alpha+n} L_{n+m}^{(\alpha)} (x)) = (n+m+1) y^{\alpha+n+1} L_{n+m+1}^{(\alpha)} (x) . \tag{2.2}$$

The extended form of the group generated by $\mathbb{R}$ is given by,

$$e^{w\mathbb{R}} f(x,y) = (1-wy)^{-m-1} \exp(\frac{-wxy}{1-wy}) f(\frac{x}{1-wy} , \frac{y}{1-wy}) .$$

Let us consider the generating relation of the form:

$$G(x,w) = \sum_{n=o}^{\infty} a_n L_{n+m}^{(\alpha)} (x) w^n . \tag{2.3}$$

Replacing w by wyz and then multiplying both sides by y^α, we get

$$y^\alpha G(x,wyz) = \sum_{n=o}^{\infty} a_n (wyz)^n y^\alpha L_{n+m}^{(\alpha)} (x) = \sum_{n=o}^{\infty} a_n (wz)^n y^{\alpha+n} L_{n+m}^{(\alpha)} (x) .$$

Operating both sides of the above expression by (exp $w\mathbb{R}$), we get

$$(\exp w\mathbb{R}) (y^\alpha G(x,wyz)) = (\exp w\mathbb{R}) (\sum_{n=o}^{\infty} a_n (wz)^n y^{\alpha+n} L_{n+m}^{(\alpha)} (x)) \tag{2.4}$$

The left member of (2.4) becomes

$$(1-wy)^{-1-m} \exp(\frac{-wxy}{1-wy}) (\frac{y}{1-wy})^\alpha G(\frac{x}{1-wy} , \frac{wyz}{1-wy}) . \tag{2.5}$$

The right member of (2.4) is equal to

$$\begin{aligned}
&\sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n (wz)^n \frac{w^k}{k!} \mathbb{R}^k (y^{n+\alpha} L_{n+m}^{(\alpha)} (x)) \\
&= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n \frac{w^{n+k}}{k!} z^n (n+m+1)_k y^{n+\alpha+k} L_{n+m+k}^{(\alpha)} (x) \\
&= y^\alpha \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n z^n (wy)^{n+k} \frac{(n+m+k)!}{k!(n+m)!} L_{n+m+k}^{(\alpha)} (x) \\
&= y^\alpha \sum_{n=o}^{\infty} \sum_{k=o}^{n} (wy)^n (a_{n-k} z^{n-k} \frac{(n+m)!}{k! (n-k+m)!}) L_{n+m}^{(\alpha)} (x) \\
&= y^\alpha \sum_{n=o}^{\infty} (wy)^n f_n (z) L_{n+m}^{(\alpha)} (x)
\end{aligned} \tag{2.6}$$

where

$$f_n (z) = \sum_{k=o}^{n} \binom{n+m}{k+m} a_k z^k .$$

Equating (2.5) and (2.6) and then putting y=1, we get

$$(1-w)^{-1-\alpha-m} \exp(\frac{-wx}{1-w}) G(\frac{x}{1-w} , \frac{wz}{1-w}) = \sum_{n=o}^{\infty} w^n f_n(z) L_{n+m}^{(\alpha)} (x) \tag{2.7}$$

where,

$f_n(z) = \sum_{k=o}^{n} \binom{n+m}{k+m} a_k z^n$, this completes the proof of the theorem.

On the other hand, if we consider the continuous transformations group defined by the infinitesimal operator

$$\mathbb{R}_1 = e^t \left(\frac{\partial}{\partial t} - \frac{1}{2} x \frac{\partial}{\partial z} - x \frac{\partial}{\partial x}\right)$$

then the equations of finite transformations of the group are

$$x' = (\exp w \mathbb{R}_1)x, \ y' = (\exp w\mathbb{R}_1)y, \ z' = (\exp w\mathbb{R}_1)z \tag{2.8}$$

where w is the parameter of the group under consideration.

Also we know that

$$\begin{aligned}(\exp w \mathbb{R}_1)\, f(x,y,z) &= f((\exp w\mathbb{R}_1)x,\ (\exp w\mathbb{R}_1)y,\ (\exp w\mathbb{R}_1)z) \\ &= f\,(x',\ y',\ z')\end{aligned} \tag{2.9}$$

From [3] we see that the effect of the operator $(\exp w \mathbb{R}_1)$ on the variables are as follows:

$$\begin{aligned} x' &= x/\ (1-we^t) \\ y' &= t - \log\,(1-we^t) \\ z' &= z - x\, we^t/\ 2(1-we^t). \end{aligned} \tag{2.10}$$

and

$$\mathbb{R}_1\, F_{n+m}\,(x,t,z) = (n+m+1)\, F_{n+m+1}\,(x,t,z)\ , \tag{2.11}$$

where

$$F_{n+m}\,(x,t,z) = \exp\,[(n+m)t + \frac{\alpha+1}{2}\, t + z - \frac{x}{2}]\ x^{(\alpha+1)/2}\ L_{n+m}^{(\alpha)}\,(x).$$

Now replacing w by wye^t in (2.3) and then multiplying both members by

$$\exp\,[\ mt + \frac{\alpha+1}{2}\, t + z - \frac{x}{2}\ ]\ x^{(\alpha+1)/2}$$

we get

$$\begin{aligned} G(x,\ wye^t)\ \exp\{mt + \frac{\alpha+1}{2}\, t + z - \frac{x}{2}\}\ x^{(\alpha+1)/2} \\ = \sum_{n=o}^{\infty} a_n\ (wy)^n\ F_{n+m}(x,t,z)\ . \end{aligned} \tag{2.12}$$

Operating both members of the above expression by $(\exp w\ \mathbb{R}_1)$ and using (2.8), (2.9) and (2.11), we get

$$\begin{aligned} G(x',\ wye^{t'})\ \exp\{mt' + \frac{\alpha+1}{2}\, t' + z' - \frac{x'}{2}\}\ \ (x')^{(\alpha+1)/2} \\ = \sum_{n=o}^{\infty} w^n\ f_n(y)\ F_{n+m}\,(x,t,z)\ , \end{aligned} \tag{2.13}$$

where

$$f_n(y) = \sum_{k=o}^{n} \binom{n+m}{k+m}\, a_k\ y^k\ .$$

Putting the values of x', y', z' from (2.10) and then substituting $t = z = o$ we finally obtain

if

$$G(x,w) = \sum_{n=o}^{\infty} a_n\ w^n\ L_{n+m}^{(\alpha)}\,(x)$$

then

$$\begin{aligned} (1-w)^{-1-\alpha-m}\ \exp\left(\frac{-xw}{1-w}\right)\ G\left(\frac{x}{1-w}\ ,\ \frac{wy}{1-w}\right) \\ = \sum_{n=o}^{\infty} w^n\ f_n(y)\ L_{n+m}^{(\alpha)}\,(x) \end{aligned} \tag{2.14}$$

where

$$f_n(y) = \sum_{k=o}^{n} \binom{n+m}{k+m}\, a_k\, y^k\ ,$$

which is same as (2.7).

From above we see that if $\mathbb{R}_1$ be used the calculation becomes much harder than when $\mathbb{R}$ is used.

COROLLARY 1. If we put m=o in the above theorem, we get the following well-known theorem derived by W.A. Al-Salam [4], and the second author [5].

"If there exists a generating relation of the form

$$G(x,w) = \sum_{n=o}^{\infty} a_n w^n L_n^{(\alpha)}(x)$$

then

$$(1-w)^{-\alpha-1} \exp\left(\frac{-wx}{1-w}\right) G\left(\frac{x}{1-w}, \frac{wz}{1-w}\right) = \sum_{n=o}^{\infty} w^n f_n(z) L_n^{(\alpha)}(x)$$

where

$$f_n(z) = \sum_{k=o}^{n} \binom{n}{k} a_k z^k .$$

APPLICATION. As a nice application of our theorem, we consider the generating relation given in (1.2), i.e.,

$$\sum_{n=o}^{\infty} \binom{n+m}{n} L_{n+m}^{(\alpha)}(x) w^n = (1-w)^{-1-\alpha-m} \exp\left(\frac{-xw}{1-w}\right) L_m^{(\alpha)}\left(\frac{x}{1-w}\right) .$$

If we put $a_n = \binom{m+n}{n}$ in our theorem, we get

$$(1-w-wz)^{-1-\alpha-m} \exp\left(\frac{-wx(1+z)}{1-w-wz}\right) L_m^{(\alpha)}\left(\frac{x}{1-w-wz}\right) = \sum_{n=o}^{\infty} w^n f_n(z) L_{n+m}^{(\alpha)}(x)$$

where

$$f_n(z) = \sum_{k=o}^{m} \binom{n+m}{k+m} \binom{m+n}{n} z^k .$$

ACKNOWLEDGEMENT. I am grateful to Dr. A.K. Chongdar for his help in the preparation of this work.

REFERENCES

1. RAINVILLE, E.D. Special Functions, Chelsea Publishing Company, Bronz, New York, (1960), 211.

2. ERDELYI, A. et al. Higher Transcendental Functions, McGraw-Hill Book Company, Inc., New York (1953), 189.

3. DAS, M.K. Sur les Polynomes de Laguerre, du Point de vue de L'Algebra de Lie, C.R. Acad. Sc. Paris 270A (1970), 380-383.

4. AL-SALAM, W.A. Operational Representation for the Laguerre and Other Polynomials, Duke Math. Jour. 31 (1964), 127-142.

5. CHONGDAR, A.K. On a Class of Trilateral Generating Relations with Tchebychev Polynomials from the View Point of One Parameter Group of Continuous Transformations, Bull Cal. Math. Soc. 73 (1981), 127-140.

Journal of Applied Mathematics and Decision Sciences

Special Issue on Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk