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TWO NEW FINITE DIFFERENCE METHODS FOR COMPUTING EIGENVALUES OF A FOURTH ORDER LINEAR BOUNDARY VALUE PROBLEM

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ABSTRACT. This paper describes some new finite difference methods of order 2 and 4 for computing eigenvalues of a two-point boundary value problem associated with a fourth order differential equation of the form $(py'')'' + (q - \lambda r)y = 0$. Numerical results for two typical eigenvalue problems are tabulated to demonstrate practical usefulness of our methods.

KEY WORDS AND PHRASES. Band-matrices, finite-difference methods, generalized eigenvalue problem, positive definite matrices, two-point boundary value problems.
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1. INTRODUCTION.

We shall consider the fourth order linear differential equation

$$\frac{d^2}{dx^2}\left[p(x)\,\frac{d^2y}{dx^2}\right] + [q(x) - \lambda r(x)]y = 0, \quad -\infty \le a \le x < b < \infty, \tag{1.1}$$

associated with the following pairs of homogeneous boundary conditions

$$y(a) = y(b) = y''(a) = y''(b) = 0. \tag{1.2}$$

Such boundary value problems occur in applied mathematics, engineering and modern physics, (see ref. [1-4]. In the differential equation (1.1) the functions $p(x)$, $q(x)$, $r(x) \in C[a,b]$ and satisfy the conditions

$$p(x) > 0,\ q(x) \ge 0 \ \text{ and } r(x) > 0,\ x \in [a,b]. \tag{1.3}$$

We cannot compute the exact values of the eigenvalues λ for which the boundary value problem (1.1) - (1.2) has a nontrivial eigensolution $y(x)$ for arbitrary chocies of the functions $p(x)$, $q(x)$ and $r(x)$. We resort to numerical methods for computing approximate values of λ. The most commonly used technique for approximating λ for which the system (1.1) - (1.2) has a nontrivial eigenfunction $y(x)$ is by finite difference methods.

Recently, the author [2] has analysed some new finite different methods of order 2 and 4 for computing eigenvalues of a two point boundary value problem involving the differential equation (1.1) with $p(x) \equiv 1$ associated with one of the following pairs of homogeneous boundary conditions:

$$\begin{array}{ll} \text{(a)} & y(a) = y(b) = y'(a) = y'(b) = 0 \\ \text{(b)} & \text{the same boundary conditions as (1.2)} \\ \text{(c)} & y(a) = y'(a) = y''(b) = y'''(b) = 0. \end{array} \quad (1.4)$$

Chawla and Katti [3] have developed a numerical finite difference method of order 2 for approximating the lowest eigenvalue λ of the system (1.1) - (1.4(a)) with $p(x) \equiv 1$. A fourth order method was later developed by Chawla [4] for the numerical treatment of the same problem. This latter method leads to a generalized seven-band symmetric matrix eigenvalue problem.

Let λ be any eigenvalue of the system (1.1) - (1.2) and let $y(x) \not\equiv 0$ be the corresponding eigenfunction. Then on multiplying (1.1) by $y(x)$ and integrating the resulting equation from a to b, we find after integration by parts and on using (1.2), that

$$\lambda = \frac{\int_a^b p(y'')^2\, dx + \int_a^b qy^2 dx}{\int_a^b ry^2 dx} > 0 \quad (1.5)$$

in view of (1.3).

The purpose of this brief report is to present two new finite difference methods for computing approximate values of λ for the system (1.1) - (1.2). These methods lead to generalized five-band and nine-band symmetrix matrix eigenvalue problems and provide $0(h^2)$ and $0(h^4)$ -convergent approximations for the eigenvalues.

2. A SECOND ORDER METHOD

For a positive integer $N \geq 5$, let $h = (b - a)/(N + 1)$ and $x_i = a + ih$, $i = 0(1)N + 1$. We shall designate $y_i = y(x_i)$, $p_i = p(x_i)$, $q_i = q(x_i)$ and $r_i = r(x_i)$. Note that the differential system (1.1) - (1.2) is equivalent to

$$\begin{array}{ll} \text{(a)} & y''(x) = v(x)/p(x)\ , \ y(a) = y(b) = 0\ , \\ \text{(b)} & v''(x) + [q(x) - \lambda r(x)]\, y(x) = 0\ , \\ & v(a) = v(b) = 0. \end{array} \quad (2.1)$$

Now the central difference approximation to 2.1(a) is

$$-y_{i-1} + 2y_i - y_{i+1} + h^2(v_i/p_i) + \frac{h^4}{12}\, y^{(4)}(\Theta_i) = 0\ , \quad (2.2)$$

$$\Theta_i \in (x_{i-1}, x_{i+1})\ , \ i = 1(1)N\ .$$

The preceding system can be conveniently written in matrix form

$$JY + h^2P^{-1}V + \frac{h^4}{12} T_1 = 0 \quad (2.3)$$

where $Y = (y_i)$, $V = (v_i)$, $T_1 = (\rho_i)$ are N-dimensional column vectors with $\rho_i = y^{(4)}(\Theta_i)$, $P = \text{diag}\,(p_i)$, and $J = (j_{mn})$ is a tridiagonal matrix so that

$$j_{mn} = \begin{cases} 2, & m = n \\ -1, & |m - n| = 1 \\ 0, & |m - n| > 1 . \end{cases} \tag{2.4}$$

In an analogous manner, on discretizing 2.1(b), we get

$$JV - h^2QY + \lambda h^2RY + \frac{h^4}{12} T_2 = 0 \tag{2.5}$$

where $Q = \text{diag}\,(q_i)$, $R = \text{diag}\,(r_i)$ and $T_2 = (\sigma_i)$ with $\sigma_i = v^{(4)}(\phi_i)$, $\phi_i \in (x_{i-1}, x_{i+1})$. Next, we eliminate v between (2.3) and (2.5) to obtain

$$AY \equiv (JPJ + h^4Q)Y = \lambda h^4RY + \Gamma , \tag{2.6}$$

where

$$\Gamma = \frac{1}{12}[h^6T_2 - h^4JPT_1] . \tag{2.7}$$

It can be verified that the matrix $A = JPJ + h^4Q$ is a five-band symmetric matrix. Now, in (2.6), neglect truncation error Γ, replace Y by $\tilde{Y}$, then our method for computing approximations Λ for λ of the system (1.1) - (1.2) can be expressed as a generalized seven-band symmetric matrix eigenvalue problem

$$A\tilde{Y} = \Lambda h^4 R\tilde{Y} . \tag{2.8}$$

In fact the matrix JPJ is a positive definite matrix and hence for any step-size $h > 0$, the approximations Λ for λ by (2.8) are real and positive for all $p(x) > 0$ and $r(x) > 0$. That our method provides $O(h^2)$ convergent approximations Λ for λ can be established following Grigorieff [5]. We omit the proof of convergence for brevity.

3. A FOURTH ORDER METHOD

Following Shoosmith [6] the boundary value problems 2.1(a) and 2.1(b) are discretized by the finite difference scheme

$$\begin{aligned} &\text{(a)} \quad 14y_0 - 29y_1 + 16y_2 - y_3 = h^2[y_0'' + 12y_1''] , \\ &\text{(b)} \quad (1 - \tfrac{\delta^2}{12})\delta^2 y_i = h^2 y_i'' , \quad i = 2(1)N-1, \\ &\text{(c)} \quad -y_{n-2} + 16y_{N-1} - 29y_N + 14y_{N+1} = h^2[12y_N'' + y_{N+1}''] . \end{aligned} \tag{3.1}$$

It turns out the boundary value problem 3.1(a) gives rise to the linear equations

$$M\tilde{Y} + 12h^2p^{-1}\tilde{V} = 0 . \tag{3.2}$$

Similarly, for the system 2.1(b), we obtain the linear equations

$$M\tilde{V} = 12h^2Q\tilde{Y} - 12\Lambda h^2R\tilde{Y} , \tag{3.3}$$

where the five-band N x N matrix M is given by

$$M = \begin{bmatrix} 29 & -16 & 1 & & & & \\ -16 & 30 & -16 & 1 & & & \\ 1 & -16 & 30 & -16 & 1 & & \\ - & - & - & - & - & - & - \\ & & 1 & -16 & 30 & -16 & 1 \\ & & & 1 & -16 & 30 & -16 \\ & & & & 1 & -16 & 29 \end{bmatrix} . \qquad (3.4)$$

The elimination of $\tilde{V}$ from (3.2) and (3.3) gives our method for computing Λ for λ of (1.1) - (1.2) in the form

$$(MPM + 144h^4Q)\tilde{Y} = 144\Lambda h^4R\tilde{Y} , \qquad (3.5)$$

where the matrix MPM is a nine-band positive definite matrix and hence for any step-size $h > 0$, the approximations Λ for λ by (3.5) are real and positive for all $p(x)$, $r(x) > 0$. As before, it can be proved from the results of Grigorieff [5] that our present method provided $0(h^4)$ convergent approximations Λ for λ.

4. NUMERICAL RESULTS

In order to illustrate our methods of order 2 and 4 for the approximation of λ satisfying (1.1) - (1.2), we consider the eigenvalue problems:

$$[(1 + x^2)y'']'' + [\frac{1}{(1 + x^2)} - \lambda(1 + x)^4]y = 0 , \qquad (4.1)$$

$$y(0) = y(1) = y''(0) = y''(1) = 0 .$$

The smallest eigenvalue λ_1 = 22.754, 058, 480, . . .

$$[e^xy'']'' + [\sin x - \lambda \cos x]y = 0 , \qquad (4.2)$$

$$y(0) = y(1) = y''(0) = y''(1) = 0 .$$

The smallest eigenvalue of the system (4.2) is λ_1 = 181.345, 488, 233, . . . We list the approximations Λ_1 for λ_1 and the relative errors $| 1 - \frac{\lambda_1}{\Lambda_1} |$ for various values of the step-size h. It is readily verified that the relative errors (Table I) based on generalized eigenvalue problem (2.8) provide $0(h^2)$ - convergent approximations for the smallest eigenvalue of the system (4.1) and (4.2). Similarly, the relative errors (Table II) based on the generalized eigenvalue problem (3.5) do indeed provide $0(h^4)$ -convergent approximations for the smallest eigenvalue of the systems (4.1) and (4.2).

TABLE I

Results based on (2.8), second order approximations

Problem	N	Λ_1	$\left\|1 - \frac{\lambda_1}{\Lambda_1}\right\|$
(4.1)	7	22.187	2.557-2*
	15	22.610	6.352-3
	31	22.718	1.586-3
	63	22.745	3.962-4
	127	22.752	9.907-5
	255	22.753	2.480-5
(4.2)	7	176.641	2.664-2
	15	180.159	6.588-3
	31	181.048	1.642-3
	63	181.271	4.103-4
	127	181.327	1.025-4
	255	181.341	2.560-5

*We write 2.557-2 for 2.557×10^{-2}.

TABLE II

Results based on (3.5), 4th order approximations

Problem	N	Λ_1	$\left\|1 - \frac{\lambda^1}{\Lambda_1}\right\|$
(4.1)	7	22.746, 419	3.358-4
	15	22.753, 574	2.129-5
	31	22.754, 027	1.358-6
	63	22.754, 056	1.078-7
(4.2)	7	181.244, 637	5.564-4
	15	181.339, 089	3.529-5
	31	181.345, 093	2.175-6
	63	181.345, 470	9.728-8

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