

DIFFEOMORPHISM GROUPS OF CONNECTED SUM OF A PRODUCT OF SPHERES AND CLASSIFICATION OF MANIFOLDS

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ABSTRACT. In [1] and [2] a classification of a manifold M of the type $(n, p, 1)$ was given where $H_p(M) = H_{n-p}(M) = \mathbb{Z}$ is the only non-trivial homology groups. In this paper we give a complete classification of manifolds of the type $(n, p, 2)$ and we extend the result to manifolds of type (n, p, r) where r is any positive integer and $p = 3, 5, 6, 7 \pmod{8}$.

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0. INTRODUCTION.

In [1] Edward C. Turner worked on a classification of a manifold M of the type (n, p, r) where this means that M is simply connected smooth n -manifold and $H_p(M) \approx H_{n-1}(M) \approx \mathbb{Z}^r$ the only non-trivial homology groups except for the top and bottom groups. He gave a classification of such manifolds for the case $r=1$ and $p = 3, 5, 6, 7 \pmod{8}$. So Turner gave a classification of M of type $(n, p, 1)$ and $p = 3, 5, 6, 7 \pmod{8}$. In [2] Hajime Sato independently obtained similar results for M of the type $(n, p, 1)$. The question which naturally follows is: Suppose $r = 2, 3, 4$ and so on, what is the classification of such M ? i.e., what is the classification of M of the type $(n, p, 2)$, $(n, p, 3)$ and so on? In this paper we will study manifolds for the type $(n, p, 2)$ and give its complete classification and then generalize the result to manifolds M of the type (n, p, r) where r is an integer and $p = 3, 5, 6, 7 \pmod{8}$.

In §1 we prove the following

THEOREM 1.1 Let M be an n -dimensional oriented, closed, simply connected manifold of the type $(n, p, 2)$ with $p = 3, 5, 6, 7 \pmod{8}$. Then M is diffeomorphic to $S^p \times D^{q+1} \# S^p \times D^{q+1} \cup S^p \times D^{q+1} \# S^p \times D^{q+1}$ where $n = p+q+1$, $\#$ means connected sum along the boundary as defined by Milnor and Karvaire [3] and $h : S^p \times S^q \# S^p \times S^q \rightarrow S^p \times S^q \# S^p \times S^q$ is a diffeomorphism.

In §2 we compute the group $\tilde{\pi}_0 \text{Diff}(S^p \times S^q \# S^p \times S^q)$ of pseudo-diffeotopy classes of diffeomorphisms of $S^p \times S^q \# S^p \times S^q$ $p < q$.

Let $GL(2, \mathbb{Z})$ denote the set of 2×2 unimodular matrices and H the subgroup of $GL(2, \mathbb{Z})$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - cd \equiv 0 \pmod{2}$ and \mathbb{Z}_4 the subgroup of $GL(2, \mathbb{Z})$ of order 4 generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We will adopt the notation $M_{p,q} = \text{Diff}(S^p \times S^q \# S^p \times S^q)$ and $M_{p,q}^+$ the subgroup of $M_{p,q}$ consisting of diffeomorphisms which induce identity map on all homology groups. We will then prove the following

THEOREM 2.1 (i) If $p+q$ is even, then

$$\frac{\tilde{\pi}_0(M_{p,q})}{\tilde{\pi}_0(M_{p,q}^+)} \approx \begin{cases} \mathbb{Z}_4 \oplus \mathbb{Z}_4 & \text{if } p \text{ is even, } q \text{ is even} \\ GL(2, \mathbb{Z}) \oplus GL(2, \mathbb{Z}) & \text{if } p, q = 1, 3, 7 \\ H \oplus H & \text{if } p, q \text{ odd but } \neq 1, 3, 7 \\ GL(2, \mathbb{Z}) \oplus H & \text{if } p = 1, 3, 7, q \text{ is odd but } \neq 1, 3, 7 \end{cases}$$

(ii) If $p+q$ is odd then

$$\frac{\tilde{\pi}_0(M_{p,q})}{\tilde{\pi}_0(M_{p,q}^+)} \approx \begin{cases} \mathbb{Z}_4 \oplus H & \text{if } p \text{ is even, } q \text{ is odd but } \neq 1, 3, 7 \\ \mathbb{Z}_4 \oplus GL(2, \mathbb{Z}) & \text{if } p \text{ is even and } q = 1, 3, 7 \end{cases}$$

We will further prove the following.

THEOREM 2.15 If $p < q$ and $p \equiv 3, 5, 6, 7 \pmod{8}$ the order of the group $\tilde{\pi}_0(M_{p,q}^+)$ is twice the order of the group $\pi_q(SO(p+1)) \oplus \theta^{p+q+1}$.

In §3 we apply the result in §2 to prove the following

THEOREM 3.7 Let M be an n -dimensional, smooth, closed, oriented manifold such that $n = p+q+1$ and

$$H_i(M) = \begin{cases} \mathbb{Z} & i = 0, n \\ \mathbb{Z} \oplus \mathbb{Z} & i = p, q+1 \\ 0 & \text{elsewhere} \end{cases}$$

then if $p = 3, 5, 6, 7 \pmod{8}$ the number of differentiable manifolds up to diffeomorphism satisfying the above is equal to twice the order of the group $\pi_q(SO(p+1)) \oplus \theta^{p+q+1}$. With induction hypothesis and technique used in §1 and §2, one can prove the following

THEOREM 3.8 If M is a smooth, closed simply connected manifold of type (n, p, r) where $n = p+q+1$ and $p = 3, 5, 6, 7 \pmod{8}$, then the number of differentiable manifolds up to diffeomorphism satisfying the above is equal to

$$r \text{ times the order of } \pi_q(SO(p+1)) \oplus \theta^{p+q+1}.$$

1. MANIFOLDS OF TYPE (n, p, r)

DEFINITION: Let M be a closed, simply connected n -manifold. M is said to be of type (n, p, r) if

$$H_i(M) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ \mathbb{Z}^r & \text{if } i = p, q+1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{where } n = p+q+1$$

We recall from Milnor and Kervaire [3]

DEFINITION: Let M_1 and M_2 be $(p+q+1)$ -manifolds with boundary and H^{p+q+1}

be half-disc, i.e.,

$$H^{p+q+1} = \{x = x_1, x_2, \dots, x_{p+q+1} \mid |x| \leq 1, x_1 \geq 0\}$$

Let D^{p+q} be the subset of H^{p+q+1} for which $x_1 = 0$. We can choose embeddings

$$i_\alpha : (H^{p+q+1}, D^{p+q}) \longrightarrow (M_\alpha, \partial M_\alpha) \quad \alpha = 1, 2$$

so that $i_2 \cdot i_1^{-1}$ reverses orientation. We then form the sum $(M_1 - i_1(0)) + (M_2 - i_2(0))$ by identifying $i_1(tu)$ with $i_2((1-t)u)$ for $0 < t < 1$ $u \in S^{p+q} \cap H^{p+q+1}$. This sum is called the connected sum along the boundary and will be denoted by $M_1 \#_{\partial} M_2$.

REMARK: (1) Notice that the boundary of $M_1 \#_{\partial} M_2$ is $\partial M_1 \# \partial M_2$.

(2) $M_1 \#_{\partial} M_2$ has the homotopy type of $M_1 \vee M_2$: the union with a single point in common.

THEOREM 1.1 If M is a smooth manifold of type $(n, p, 2)$ where $n = p+q+1$ and $p = 3, 5, 6, 7 \pmod{8}$ then there exists a diffeomorphism

$$h : S^p \times S^q \# S^p \times S^q \longrightarrow S^p \times S^q \# S^p \times S^q$$

which induce identity on homology such that M is diffeomorphic to

$$S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \cup S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}.$$

PROOF: Let $\{M, \lambda_1, \lambda_2\}$ be a manifold of type $(n, p, 2)$ and λ_1, λ_2 represent the generators of the first and second summands of $H_p(M) \approx \mathbf{Z} \oplus \mathbf{Z}$. We can choose embeddings $\varphi_i : S^p \longrightarrow M$ so as to represent the homology class λ_i $i = 1, 2$. Since $p < q$, two homotopic embeddings are isotopic. Let $\alpha_i \in \pi_{p-1} SO(q+1)$ be the characteristic class of the embedded sphere S^p , since $p = 3, 5, 6, 7 \pmod{8}$, the normal bundle of the embedded sphere is trivial. It follows that φ_i extends to an embedding $\varphi_i^* : S^p \times D^{q+1} \longrightarrow M$ such that its homology class is λ_i . Then we can form a connected sum along the boundary of the two embedded copies of $S^p \times D^{q+1}$ to get $S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}$. We then have an embedding $i : S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \longrightarrow M$ such that $i_*[S^p] = \lambda_1 + \lambda_2 \in H_p(M)$. Notice that the boundary of $S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}$ is $S^p \times S^q \# S^p \times S^q$ and since $S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}$ has the homotopy type of $S^p \times D^{q+1} \vee S^p \times D^{q+1}$ then it is easy to see that

$$H_i(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}) = \begin{cases} \mathbf{Z} & \text{for } i = 0 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{for } i = p \end{cases}.$$

It is also easy to see that

$$H_i(M - \text{Int}(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})) = \begin{cases} \mathbf{Z} & \text{for } i = 0 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{for } i = p \end{cases}.$$

Now since $S^p \times D^{q+1}$ is a trivial disc bundle over S^p then it has cross sections; hence, there exists orientation reversing diffeomorphism of $S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}$ onto itself. Thus there exists an orientation reversing embedding

$$j : S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \longrightarrow M - \text{Int}(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})$$

such that $j_*[S^p] = \lambda_1 + \lambda_2$ and in fact this embedding is a homotopy equivalence. It follows by [4, Thm. 4.1] that $S^p \times D^{q+1} \# \overset{\partial}{S^p} \times D^{q+1}$ is diffeomorphic to $M - \text{Int}(S^p \times D^{q+1} \# S^p \times D^{q+1})$. Consequently, it follows that M is diffeomorphic to $S^p \times D^{q+1} \# S^p \times D^{q+1} \cup \overset{\partial}{S^p} \times D^{q+1} \# S^p \times D^{q+1}$ for an orientation preserving diffeomorphism $h: S^p \times S^q \# S^p \times S^q \rightarrow S^p \times S^q \# S^p \times S^q$. From the embeddings in the proof, it is clear that h induces identity on homology.

2. THE GROUP $\tilde{\pi}_0 \text{Diff}(S^p \times S^q \# S^p \times S^q)$

For convenience, we adopt the notation $M_{p,q} = \text{Diff}(S^p \times S^q \# S^p \times S^q)$ and $M_{p,q}^+$ the subset of $M_{p,q}$ consisting of diffeomorphisms of $S^p \times S^q \# S^p \times S^q$ which induce identity on all homology groups.

DEFINITION: Let M be an oriented smooth manifold. $\text{Diff}(M)$ is the group of orientation preserving diffeomorphisms of M . Let $f, g \in \text{Diff}(M)$, f and g are said to be pseudo-diffeotopic if there exists a diffeomorphism H of $M \times I$ such that $H(x, 0) = (f(x), 0)$ and $H(x, 1) = (g(x), 1)$ for all $x \in M$. The pseudo-diffeotopy class of diffeomorphisms of M is denoted by $\tilde{\pi}_0(\text{Diff}M)$. We wish to compute $\tilde{\pi}_0(M_{p,q})$ for $p < q$. If $f \in M_{p,q}$ then f induces an automorphism

$$f_*: H_*(S^p \times S^q \# S^p \times S^q) \rightarrow H_*(S^p \times S^q \# S^p \times S^q)$$

of homology groups of $S^p \times S^q \# S^p \times S^q$. Since pseudo-diffeotopic diffeomorphisms induce equal automorphism on homology then we have a well-defined homomorphism

$$\Phi: \tilde{\pi}_0(M_{p,q}) \rightarrow \text{Auto}(H_*(S^p \times S^q \# S^p \times S^q))$$

where $\text{Auto}(H_*(S^p \times S^q \# S^p \times S^q))$ denotes the group of dimension preserving automorphisms of $H_*(S^p \times S^q \# S^p \times S^q)$.

THEOREM 2.1 (i) If $p+q$ is even then

$$\Phi(\tilde{\pi}_0(M_{p,q})) = \begin{cases} \mathbf{Z}_4 \oplus \mathbf{Z}_4 & \text{if } p, q \text{ are even} \\ \text{GL}(2, \mathbf{Z}) \oplus \text{GL}(2, \mathbf{Z}) & \text{if } p, q \text{ are } 1, 3, 7 \\ \mathbf{H} \oplus \mathbf{H} & \text{if } p, q \text{ are odd but } \neq 1, 3, 7 \\ \text{GL}(2, \mathbf{Z}) \oplus \mathbf{H} & \text{if } p = 1, 3, 7 \text{ and } q \text{ is odd but } \neq 1, 3, 7 \end{cases}$$

The following propositions give the proof of Theorem 2.1.

PROPOSITION 2.1 If $p+q$ is even, p is even, then

$$\Phi(\tilde{\pi}_0(M_{p,q})) = \mathbf{Z}_4 \oplus \mathbf{Z}_4.$$

PROOF: Since $p+q$ is even and p is even then q must also be even. We have

$$H_i(S^p \times S^q \# S^p \times S^q) = \begin{cases} \mathbf{Z} & \text{if } i=0, p+q \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } i=p \text{ or } q \\ 0 & \text{elsewhere} \end{cases}.$$

Generators of $H_0(S^p \times S^q \# S^p \times S^q)$ and $H_{p+q}(S^p \times S^q \# S^p \times S^q)$ are mapped to the same generators but $H_p(S^p \times S^q \# S^p \times S^q) = \mathbf{Z} \oplus \mathbf{Z}$. If $f \in M_{p,q}$, we shall denote by $\Phi(f)_p$ the automorphism $f_*: H_p(S^p \times S^q \# S^p \times S^q) \rightarrow H_p(S^p \times S^q \# S^p \times S^q)$ induced by the image f under Φ in dimension p . Then $\Phi(f)_p = f_*: \mathbf{Z} \oplus \mathbf{Z} \rightarrow \mathbf{Z} \oplus \mathbf{Z}$ is the induced

automorphism. If e_1, e_2 are the generators of the first and second summand of $H_p(S^p \times S^q \# S^p \times S^q)$ if \circ denotes the intersection then $e_1 \circ e_1 = 0$, $e_2 \circ e_2 = 0$, $e_1 \circ e_2 = 1$ and $e_2 \circ e_1 = -1$. Let $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in GL(2, \mathbb{Z})$, if $\Phi(f)_p$ takes e_1, e_2 to e'_1, e'_2 respectively then $e'_1 = a_1 e_1 + a_2 e_2$ and $e'_2 = a_3 e_1 + a_4 e_2$ then

$$\begin{aligned} e'_1 \circ e'_1 &= (a_1 e_1 + a_2 e_2) \circ (a_1 e_1 + a_2 e_2) \\ &= a_1 a_1 e_1 \circ e_1 + a_1 a_2 e_1 \circ e_2 + a_2 a_1 e_2 \circ e_1 + a_2 a_2 e_2 \circ e_2 \\ &= a_1 a_2 e_1 \circ e_2 + a_2 a_1 e_2 \circ e_1 = a_1 a_2 - a_1 a_2 = 0. \end{aligned}$$

Similarly $e'_2 \circ e'_2 = 0$

$$\begin{aligned} \text{but } e'_1 \circ e'_2 &= (a_1 e_1 + a_2 e_2) \circ (a_3 e_1 + a_4 e_2) \\ &= a_1 a_3 e_1 \circ e_1 + a_1 a_4 e_1 \circ e_2 + a_2 a_3 e_2 \circ e_1 + a_2 a_4 e_2 \circ e_2 \\ &= a_1 a_4 - a_2 a_3 = 1 \text{ since } GL(2, \mathbb{Z}) \text{ is unimodular.} \end{aligned}$$

$$\begin{aligned} e'_2 \circ e'_1 &= (a_3 e_1 + a_4 e_2) \circ (a_1 e_1 + a_2 e_2) = a_3 a_1 e_1 \circ e_1 + a_3 a_2 e_1 \circ e_2 + a_4 a_1 e_2 \circ e_1 \\ &\quad + a_4 a_2 e_2 \circ e_2 = a_3 a_2 - a_4 a_1 = -1 \end{aligned}$$

hence for p even $\Phi(f)_p$ is an element of a subgroup of $GL(2, \mathbb{Z})$ generated by

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This subgroup has elements $\left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix} \right\} \cong \mathbb{Z}_4$. Hence $\Phi(f)_p \in \mathbb{Z}_4$. Similarly for $i = q$ $\Phi(f)_q \in \mathbb{Z}_4$, it then follows that

$$\Phi(\tilde{\pi}_0(M_{p,q})) \subset \mathbb{Z}_4 \oplus \mathbb{Z}_4.$$

We now show that $\mathbb{Z}_4 \oplus \mathbb{Z}_4 \subset \Phi(\tilde{\pi}_0(M_{p,q}))$. We need to show that the generators of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ can be realized as the image of Φ . We shall adopt the notation $(S^p \times S^q)_1 \# (S^p \times S^q)_2$ where the subscripts 1 and 2 denote the first and second summands of $S^p \times S^q \# S^p \times S^q$ and let R_p and R_q be reflections of S^p and S^q respectively. If $(x_1, y_1) \in (S^p \times S^q)_1$ and $(x_2, y_2) \in (S^p \times S^q)_2$, we define $f \in M_{p,q}$

$$\begin{aligned} f(x_1, y_1) &= (R_p(x_2), R_q(y_2)) \\ f(x_2, y_2) &= (x_1, y_1) \end{aligned}$$

$$\text{In other words } f((x_1, y_1)(x_2, y_2)) = ((R_p(x_2), R_q(y_2)), (x_1, y_1))$$

$$(x_1, y_1) \in (S^p \times S^q)_1 \text{ and } (x_2, y_2) \in (S^p \times S^q)_2.$$

For $\Phi(f)_p \in \text{Auto } H_p(M_{p,q})$, if e_1, e_2 are the generators of the first and second summands of $H_p(S^p \times S^q \# S^p \times S^q) = \mathbb{Z} \oplus \mathbb{Z}$ since f takes x_1 to $R_p(x_2)$ and f takes x_2 to x_1 , then it is easily seen that $\Phi(f)_p(e_1) = -e_2$ and $\Phi(f)_p(e_2) = e_1$. Hence $e'_1 = -e_2$ and $e'_2 = e_1$ and so $e'_1 \circ e'_1 = -e_2 \circ -e_2 = 0$, $e'_2 \circ e'_2 = e_1 \circ e_1 = 0$, $e'_1 \circ e'_2 = -e_2 \circ e_1 = 1$ and $e'_2 \circ e'_1 = e_1 \circ -e_2 = -1$. Hence Φ maps f in dimension p to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which generates \mathbb{Z}_4 . Similar argument shows that Φ maps f in dimension q to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which generates \mathbb{Z}_4 . Then Φ maps onto $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ hence the proof.

PROPOSITION 2.2 If $p+q$ is even but $p, q = 1, 3, 7$ then $\Phi(\tilde{\pi}_0(M_{p,q})) = GL(2, \mathbb{Z}) \oplus GL(2, \mathbb{Z})$.

PROOF: From [5, Appendix B] and [6] one sees that $GL(2, \mathbb{Z})$ is generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since $p, q = 1, 3, 7$ it follows by [7, §1] that there exist maps $f : S^p \rightarrow SO(p+1)$ and $g : S^q \rightarrow SO(q+1)$ such that f and g have index +1.

We then define $h \in M_{p,q}$

$$h(x_1, y_1) = (x_1, y_1) \quad (x_1, y_1) \in (S^p \times S^q)_1$$

$$h(x_2, y_2) = (f(x_1) \cdot x_2, g(y_1) \cdot y_2) \quad (x_2, y_2) \in (S^p \times S^q)_2$$

$$\text{i.e., } h((x_1, y_1), (x_2, y_2)) = ((x_1, y_1), (f(x_1) \cdot x_2, g(y_1) \cdot y_2))$$

Since f has index +1 and h takes x_1 to x_1 and x_2 to $f(x_1) \cdot x_2$ then it follows by an easy application of [7, Prop. 1.2] or [6, Prop. 2.3] that $\Phi(h)_p$ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ also since g has index +1 and h takes y_1 to y_1 and y_2 to $g(y_1) \cdot y_2$ then $\Phi(h)_q$ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Hence Φ maps h to $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$. We now define $\alpha \in M_{p,q}$ by

$$\alpha(x_1, y_1) = (R_p(x_2), R_q(y_2)) \quad (x_1, y_1) \in (S^p \times S^q)_1$$

$$\alpha(x_2, y_2) = (x_1, y_1) \quad (x_2, y_2) \in (S^p \times S^q)_2$$

$$\text{i.e., } \alpha((x_1, y_1), (x_2, y_2)) = ((R_p(x_2), R_q(y_2)), (x_1, y_1))$$

Since α takes x_1 to $R_p(x_2)$ and x_2 to x_1 it follows from Proposition 2.1 that $\Phi(\alpha)_p$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and by similar reasoning $\Phi(\alpha)_q$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This means that Φ maps α to $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$. Since $GL(2, \mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then it follows that for $p, q = 1, 3, 7$

$$\Phi(\tilde{\pi}_0(M_{p,q})) \approx GL(2, \mathbb{Z}) \oplus GL(2, \mathbb{Z}).$$

PROPOSITION 2.3 If $p+q$ is even but p and q are odd but $p, q \neq 1, 3, 7$, then $\Phi(\tilde{\pi}_0(M_{p,q})) \approx H \oplus H$.

PROOF: By using Proposition 2.1 and [8, Lemma 5] it is enough to produce a diffeomorphism in $M_{p,q}$ whose image under Φ is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ in each of the dimensions p and q . This is because $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ generate H . However $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ is trivially the image under Φ of identity map and reflections on each coordinate while $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$ is by Proposition 2.1 the image under Φ of an element of $M_{p,q}$. However, there exists a map $\alpha : S^p \rightarrow SO(p+1)$ of index 2 by [8] so also is a map $\beta : S^q \rightarrow SO(q+1)$ of index 2 and then we can define $f \in M_{p,q}$ thus.

$$f(x_1, y_1) = (x_1, y_1) \quad (x_1, y_1) \in (S^p \times S^q)_1$$

$$f(x_2, y_2) = (\alpha(x_1) \cdot x_2, \beta(y_1) \cdot y_2) \quad (x_2, y_2) \in (S^p \times S^q)_2$$

$$\text{i.e., } f((x_1, y_1), (x_2, y_2)) = ((x_1, y_1), (\alpha(x_1) \cdot x_2, \beta(y_1) \cdot y_2)).$$

It easily follows that since f takes x_1 to x_1 and takes x_2 to $\alpha(x_1) \cdot x_2$ with α having index 2 then it follows by applying [7, Lemma 5] that $\Phi(f)_p$ is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Similar argument shows that $\Phi(f)_q$ is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; hence $\Phi(\tilde{\pi}_0(M_{p,q})) \approx H \oplus H$.

PROPOSITION 2.4 If $p+q$ is even, $p = 1, 3, 7$ but q is odd and $\neq 1, 3, 7$ then $\Phi(\tilde{\pi}_0(M_{p,q})) = GL(2, \mathbb{Z}) \oplus H$.

PROOF: $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ generates $GL(2, \mathbb{Z})$ while $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\}$ generates H , since $q \neq 1, 3, 7$ and by [8] there exists $\alpha : S^q \rightarrow SO(q+1)$ of index 2. If R_p is reflection of S^p then we define $h \in M_{p,q}$

$$h(x_1, y_1) = (R_p(x_2), y_1) \quad (x_1, y_1) \in (S^p \times S^q)_1$$

$$h(x_2, y_2) = (x_1, \alpha(y_1) \cdot y_2) \quad (x_2, y_2) \in (S^p \times S^q)_2$$

Since h takes x_1 to $R_p(x_2)$ and takes x_2 to x_1 it follows by Proposition 2.1

that $\Phi(h)_p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; similarly h takes y_1 to y , and y_2 to $\alpha(y_1) \cdot y_2$ and since α has index 2, it follows that $\Phi(h)_q = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Now if R_q is a reflection on S^q and $\beta: S^p \rightarrow SO_{p+1}$ is of index +1 then we define $f \in M_{p, q}$

$$f(x_1, y_1) = (x_1, R_q(y_2)) \quad (x_1, y_1) \in (S^p \times S^q)_1$$

$$f(x_2, y_2) = (\beta(x_1) \cdot x_2, y_1) \quad (x_2, y_2) \in (S^p \times S^q)_2$$

then it is easy to see that $\Phi(f)_p = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\Phi(f)_q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ so the image of h under Φ is $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\}$ and the image of f under Φ is $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ and since $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\}$ generate H and $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ generate $GL(2, \mathbb{Z})$ then it follows that $\Phi(\tilde{\pi}_0(M_{p, q})) \cong GL(2, \mathbb{Z}) \oplus H$. Hence the proof.

REMARK. For p odd but $\neq 1, 3, 7$ and $q = 1, 3, 7$, we have the same result as above using the same method but since by assumption $p < q$ only one dimension (consequently one manifold) comes in here, viz $p=5$, $q=7$, i.e., $S^5 \times S^7 \# S^5 \times S^7$.

Combination of Propositions 2.1, 2.2, 2.3, and 2.4 proves Theorem 2.1(i).

PROPOSITION 2.5 Suppose $p+q$ is odd and p is even and q odd $\neq 1, 3, 7$ then $\tilde{\pi}_0(M_{p, q}) \cong \mathbb{Z}_4 \oplus H$.

PROOF: Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generate \mathbb{Z}_4 and $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\}$ generate H , then we only need to find the diffeomorphism in $M_{p, q}$ that Φ maps to these generators. Similar to Proposition 2.4, we define $f \in M_{p, q}$ by

$$f(x_1, y_1) = (R_p(x_2), y_1) \quad (x_1, y_1) \in (S^p \times S^q)_1$$

$$f(x_2, y_2) = (x_1, \alpha(y_1) \cdot y_2) \quad (x_2, y_2) \in (S^p \times S^q)_2$$

where R_p is the reflection on S^p and $\alpha: S^q \rightarrow SO_{q+1}$ is of index 2 which exists by [3] since $q \neq 1, 3, 7$. It then follows that $\Phi(f)_p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\Phi(f)_q = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Also we define $g \in M_{p, q}$ thus

$$g(x_1, y_1) = (x_1, R_q(y_2)) \quad (x_1, y_1) \in (S^p \times S^q)_1, \quad (x_2, y_2) \in (S^p \times S^q)_2$$

$$g(x_2, y_2) = (x_2, y_1)$$

i.e., $g((x_1, y_1), (x_2, y_2)) = ((x_1, R_q(y_2)), (x_2, y_1))$

where R_q is the reflection on S^q . Since g takes x_1 to x_1 and x_2 to x_2 then $\Phi(g)_p = \text{identity} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and since g takes y_1 to $R_q(y_2)$ and y_2 to y_1 it follows that by applying Proposition 2.1, $\Phi(g)_q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence f is mapped by Φ to $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\}$ while g is mapped by Φ to $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ and since these matrices generate H and \mathbb{Z}_4 respectively then it follows that

$$\Phi(\tilde{\pi}_0(M_{p, q})) \cong \mathbb{Z}_4 \oplus H.$$

PROPOSITION 2.6 Suppose $p+q$ is odd and p is even q is odd and $= 1, 3, 7$. Then $\tilde{\pi}_0(M_{p, q}) \cong \mathbb{Z}_4 \oplus GL(2, \mathbb{Z})$.

PROOF: Again since $q = 1, 3, 7$ by [6, Prop. 2.4] there exists a map $\alpha: S^q \rightarrow SO_{q+1}$ of index 1. Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generates Z_4 and $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ generate $GL(2, \mathbb{Z})$ we define elements of $M_{p,q}$ that are mapped onto these generators. Let $h \in M_{p,q}$ be defined thus

$$h(x_1, y_1) = (R_p(x_2), y_1) \quad \text{where } (x_1, y_1) \in (S^p \times S^q)_1$$

$$h(x_2, y_2) = (x_1, \alpha(y_1) \cdot y_2) \quad (x_2, y_2) \in (S^p \times S^q)_2$$

$$\text{i.e., } h(x_1, y_1), (x_2, y_2) = ((R_p(x_2), y_1), (x_1, \alpha(y_1) \cdot y_2))$$

where R_p is the reflection of S^p . Then it is easy to see that $\Phi(h)_p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ while $\Phi(h)_q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Also one can define $f \in M_{p,q}$ as

$$f(x_1, y_1) = (x_1, R_q(x_2)) \quad \text{where } (x_1, y_1) \in (S^p \times S^q)_1, (x_2, y_2) \in (S^p \times S^q)_2$$

$$f(x_2, y_2) = (x_2, y_1)$$

i.e., $f((x_1, y_1), (x_2, y_2)) = ((x_1, R_q(x_2)), (x_2, y_1))$ where R_q is a reflection of S^q and so it is easily seen that $\Phi(f)_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ while $\Phi(f)_q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ so h is mapped by Φ to $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ while f is mapped by Φ to $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ and since these sets of matrices generate $GL(2, \mathbb{Z})$ and Z_4 respectively then $\tilde{\Phi}(\pi_0(M_{p,q})) \approx Z_4 \oplus GL(2, \mathbb{Z})$. Combining Propositions 2.5 and 2.6, we obtain Theorem 2.1 (ii).

REMARK. If p is odd but $\neq 1, 3, 7$ and q is even, we get the same result as in Proposition 2.5 using equivalent method. Also if $p = 1, 3, 7$ and q is even, we obtain the same result as that of Proposition 2.6.

Since $M_{p,q}^+$ denotes the subgroup of $M_{p,q}$ consisting of diffeomorphisms of $S^p \times S^q \# S^p \times S^q$ which induce identity map on all homology groups, it follows that $M_{p,q}^+$ is the kernel of the homomorphism Φ . We now compute $M_{p,q}^+$. We define a homomorphism

$$G: \tilde{\pi}_0(M_{p,q}^+) \longrightarrow \pi_p SO(q+1)$$

Given an element $\{f\} \in \tilde{\pi}_0(M_{p,q}^+)$, since $\Phi(f)$ is identity, it means that if $i(S^p \times \{p_0\})$ is the usual identity embedding of $S^p \times \{p_0\}$ into $S^p \times S^q \# S^p \times S^q$ where p_0 is a fixed point in S^q far away from the connected sum, then the sphere $S^p \times \{p_0\}$ in $S^p \times S^q \# S^p \times S^q$ represents a generator of the homology $H_p(S^p \times S^q \# S^p \times S^q) \approx \mathbb{Z} \oplus \mathbb{Z}$. Since $\Phi(f)$ is identity, it follows that $f(S^p \times p_0)$ is homologous to $i(S^p \times p_0)$ and since $p < q$ and by Hurewicz theorem, f and i are homotopic and in fact with the dimension restriction, they are diffeotopic. By tubular neighborhood theorem, f is diffeotopic to a map say f'' such that $f''(S^p \times D^q) = S^p \times D^q$ where $f''(x, y) = (x, \alpha(f'')(x) \cdot y)$ and $\alpha(f''): S^p \longrightarrow SO(q)$. Let $i: SO(q) \longrightarrow SO(q+1)$ be the inclusion map and $i_*: \pi_p SO(q) \longrightarrow \pi_p SO(q+1)$ the induced map on the homotopy groups. Then we define

$$G\{f\} = i_*(\alpha(f'')).$$

LEMMA 2.7 G is well-defined.

PROOF: Let $f, h \in M_{p,q}^+$ such that f and h are pseudo-diffeotopic then $f \cdot h^{-1} \in M_{p,q}^+$ is pseudo-diffeotopic to the identity. If $G\{f\} = i_*(\alpha(f''))$ and

$G(h) = i_*(\alpha(h''))$ where $f(x, y) = (x, \alpha(f'')(x) \cdot y)$ and $h(x, y) = (x, \alpha(h'')(x) \cdot y)$ for $(x, y) \in S^p \times D^q$ then it follows that

$$f \cdot h^{-1}(x, y) = (x, \alpha(f'')\alpha(h'')^{-1}(x) \cdot y) \quad (x, y) \in S^p \times D^q .$$

We wish to show that $i_*(\alpha(f'')) = i_*(\alpha(h''))$. Since $G(f) = i_*(\alpha(f'')) \in \pi_p SO(q+1)$ and $G(h) = i_*(\alpha(h'')) \in \pi_p SO(q+1)$ then we can define maps $f_1 \cdot h_1 \in \text{Diff}(S^p \times S^q)$ thus $f_1(x, y) = (x, i_*(\alpha(f'')(x) \cdot y)$ and $h_1(x, y) = (x, i_*(\alpha(h'')(x) \cdot y)$ then consider $f_1 \cdot h_1^{-1} \in \text{Diff}(S^p \times S^q)$ defined by $f_1 \cdot h_1^{-1}(x, y) = (x, i_*(\alpha(f'')) \cdot i_*(\alpha(h''))^{-1}(x) \cdot y)$ $(x, y) \in S^p \times S^q$. Since $f \cdot h^{-1}$ is pseudo-diffeotopic to identity so is $f_1 \cdot h_1^{-1}$ by its definition. Hence $f_1 \cdot h_1^{-1} \in \text{Diff}(S^p \times S^q)$ is diffeotopic to the identity hence it extends to a diffeomorphism g of $D^{p+1} \times S^q$, i.e., there exists $g \in \text{Diff}(D^{p+1} \times S^q)$ such that $g|_{\text{Diff}(S^p \times S^q)} = f_1 \cdot h_1^{-1}$. Let S_β denote the q -sphere bundle over $p+1$ -sphere with characteristic map $\beta : S^p \rightarrow SO(q+1)$. Then we have

$$S_{i_*(\alpha(f'')) \cdot i_*(\alpha(h''))^{-1}} = D^{p+1} \times S^q \bigcup_{f_1 \cdot h_1^{-1}} D^{p+1} \times S^q$$

so this gives a q -sphere bundle over a $p+1$ -sphere with the characteristic class of the equivalent plane bundle being $i_*(\alpha(f'')) \cdot i_*(\alpha(h''))^{-1}$. However, $f_1 \cdot h_1^{-1}$ extends to $g \in \text{Diff}(D^{p+1} \times S^q)$ then we have

$$\begin{aligned} S^{p+1} \times S^q &= D_1^{p+1} \times S_1^q \cup_{\text{Id}} D_2^{p+1} \times S_2^q \\ &\downarrow \qquad \qquad \qquad \downarrow \\ S_{i_*(\alpha(f'')) \cdot i_*(\alpha(h''))^{-1}} &= D_1^{p+1} \times S_1^q \cup_{f_1 \cdot h_1^{-1}} D_2^{p+1} \times S_2^q \end{aligned}$$

Hence we define a map $H : S^{p+1} \times S^q \rightarrow S_{i_*(\alpha(f'')) \cdot i_*(\alpha(h''))^{-1}}$

$$H(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in D_2^{p+1} \times S_2^q \\ g(x, y) & \text{if } (x, y) \in D_1^{p+1} \times S_1^q \end{cases} .$$

H is well-defined and is a diffeomorphism. This means that $S_{i_*(\alpha(f'')) \cdot i_*(\alpha(h''))^{-1}}$ is a trivial q -sphere bundle over S^{p+1} with characteristic class $i_*(\alpha(f'')) \cdot i_*(\alpha(h''))^{-1}$. It then follows from [1, Lemma 3.6(b)] that $i_*(\alpha(f'')) = i_*(\alpha(h''))$. Hence G is well-defined. It is easy to see that G is a homomorphism.

LEMMA 2.8 $G(\tilde{\pi}_0(M_{p,q}^+)) = i_*(\pi_p(SO(q)))$.

PROOF: By the definition of G , $G(\tilde{\pi}_0(M_{p,q}^+)) \subset i_*(\pi_p SO(q))$ we then show that $i_*(\pi_p SO(q)) \subset G(\tilde{\pi}_0(M_{p,q}^+))$. If $\alpha \in i_*(\pi_p SO(q))$ and $\{a\} = \alpha$ where $a : S^p \rightarrow SO(q+1)$ then we can define $f \in M_{p,q}$ by

$$f(x, y) = \begin{cases} (x, a(x) \cdot y) & \text{if } (x, y) \in (S^p \times S^q)_1 \\ (x, y) & \text{if } (x, y) \in (S^p \times S^q)_2 \end{cases}$$

since $a \in i_*(\pi_p SO(q))$ then $f \in \tilde{\pi}_0(M_{p,q}^+)$ and so $G(f) = \alpha \in i_*(\pi_p SO(q))$.

In fact since $p < q$, then $\pi_p(S^q) = 0$ hence it follows from the exact sequence $\pi_{p+1} S^q \rightarrow \pi_p SO_q \xrightarrow{i_*} \pi_p SO_{q+1} \rightarrow \pi_p S^q \rightarrow \dots$ that i_* is an epimorphism and so it is easily seen that G is surjective. Hence the proof.

The next lemma is similar to [6, Lemma 3.3].

LEMMA 2.9 Let $u \in \ker G$, then there exists a representative $f \in M_{p,q}^+$ of u such that f is identity on $S^p \times D^q$.

PROOF: If $p < q-1$, then $\pi_{p+1}(S^q) = 0$ and also $\pi_p(S^q) = 0$ and so it follows from the exact sequence

$$\cdots \rightarrow \pi_{p+1}(S^q) \rightarrow \pi_p(S^{q-1}) \xrightarrow{i_*} \pi_p(S^{q-1}) \rightarrow \pi_p(S^q) \rightarrow \cdots$$

that i_* is an isomorphism hence if $u = \{f\} \in \ker G$ then $G(u) = i_*\alpha(f') = 0$ implies $\alpha(f') = 0$. Since $f(x, y) = (x, \alpha(f')(x) \cdot y)$ for $(x, y) \in S^p \times D^q$ then it means $f(x, y) = (x, y)$ hence f is identity on $S^p \times D^q$. However, in general let $g \in M_{p,q}$ be defined thus, if $S^p \times D_+^q$, $S^p \times D_-^q$ are subsets of $(S^p \times S^q)_1$, away from the connected sum in $M_{p,q}$, we then define

$$g(x, y) = \begin{cases} (x, \alpha(f')^{-1}(x) \cdot y) & \text{for } (x, y) \in S^p \times D_+^q \text{ and } S^p \times D_-^q \subset (S^p \times S^q)_1 \\ (x, y) & (S^p \times S^q)_2 \end{cases}$$

since $i_*\alpha(f') \in \pi_p(S^{q-1})$ we define $g' \in M_{p,q}$ by

$$g'(x, y) = \begin{cases} (x, i_*\alpha(f')^{-1}(x) \cdot y) & \text{if } (x, y) \in (S^p \times S^q)_1 \\ (x, y) & \text{if } (x, y) \in (S^p \times S^q)_2 \end{cases}$$

then g and g' are diffeotopic and since $u \in \ker G$, $G(u) = 0 = i_*\alpha(f')$ then g' is pseudo-diffeotopic to the identity and so follows that g is also pseudo-diffeotopic to the identity in $M_{p,q}$. Then the composition $g \circ f$ is pseudo-diffeotopic to f and clearly by the definition of g , $g \circ f$ keeps $S^p \times D_+^q$ fixed and represents u because it is pseudo-diffeotopic to f . Hence the proof.

We now wish to compute $\ker G$. To do this, we define a homomorphism

$$N : \ker G \rightarrow \tilde{\pi}_0(\text{Diff}^+(S^p \times S^q)) \quad \text{and}$$

show that N is surjective. Here we adopt the notation $\text{Diff}^+(S^p \times S^q)$ to mean the set of all diffeomorphisms of $S^p \times S^q$ to itself which induce identity on all homology groups. Given $u \in \ker G$, let $f \in M_{p,q}^+$ be its representative then it follows from Lemma 2.9 that we can take f to be identity on $S^p \times D^q$. So we have a map

$$f : (S^p \times S^q)_1 \# (S^p \times S^q)_2 \rightarrow (S^p \times S^q)_3 \# (S^p \times S^q)_4 \quad \text{such that}$$

f is identity on $S^p \times D^q \subset (S^p \times S^q)_1$.

Using the technique introduced by Milnor [9] and [3], we perform the spherical modification on the domain $(S^p \times S^q)_1 \# (S^p \times S^q)_2$ that removes $S^p \times D^q \subset (S^p \times S^q)_1$ and replaces it with $D^{p+1} \times S^{q-1}$. Clearly we obtain $(S^p \times S^q)_2$ since $S^p \times D^q \cup D^{p+1} \times S^{q-1}$ is diffeomorphic to S^{p+q} . Since f is the identity on $S^p \times D^q$, we can assume that $f(S^p \times D^q) = S^p \times D^q \subset (S^p \times S^q)_3$ and then perform the corresponding spherical modification on the range $(S^p \times S^q)_3 \# (S^p \times S^q)_4$ to obtain $(S^p \times S^q)_4$.

After this modification we are then left with a diffeomorphism say f' of $(S^p \times S^q)_1$ onto $(S^p \times S^q)_4$, i.e., $f' \in \text{Diff}(S^p \times S^q)$ since $f \in M_{p,q}^+$ then $f' \in \text{Diff}^+(S^p \times S^q)$. So we define $N[f] = \{f'\}$.

LEMMA 2.10 N is well-defined.

PROOF: Let $f, g \in \text{Ker } G$ such that f is pseudo-diffeotopic to g , then f is identity on $S^p \times D^q$ and g is also identity on $S^p \times D^q$. Since f is pseudo-diffeotopic to g then there exists a diffeomorphism

$F \in \text{Diff}((S^p \times S^q \# S^p \times S^q) \times I)$ such that F is identity on $S^p \times D^q \times I$ and $F|_{(S^p \times S^q \# S^p \times S^q) \times 0} = f$ while $F|_{(S^p \times S^q \# S^p \times S^q) \times 1} = g$. If we now perform the spherical modification on the domain $(S^p \times S^q)_1 \# (S^p \times S^q)_2 \times I$ of F by removing $S^p \times D^q \times I \subset (S^p \times S^q)_1 \times I$ and replacing it with $D^{p+1} \times S^{q-1} \times I$, then we obtain the manifold $(S^p \times S^q)_2 \times I$ and since F is identity on $S^p \times D^q \times I$, we then perform the corresponding modification on the range $(S^p \times S^q)_3 \# (S^p \times S^q)_4 \times I$ by removing $S^p \times D^q \times I \subset (S^p \times S^q)_3 \times I$ and replacing it with $D^{p+1} \times S^{q-1} \times I$ to obtain $(S^p \times S^q)_4 \times I$. We then obtain a diffeomorphism

$$F' : (S^p \times S^q)_2 \times I \longrightarrow (S^p \times S^q)_4 \times I$$

i.e., $F' \in \text{Diff}^+(S^p \times S^q \times I)$ hence $N(F) = F'$ and $F'|_{(S^p \times S^q) \times 0} = f'$ and $F'|_{S^p \times S^q \times 1} = g'$ hence f' is pseudo-diffeotopic to g' and so N is well-defined. It is easy to see that N is a homomorphism.

LEMMA 2.11 N is surjective.

PROOF: Let $h' \in \text{Diff}^+(S^p \times S^q)$, we need to find a diffeomorphism $h \in M_{p,q}^+$ such that $N(h) = h'$. If D^{p+q} is a disc in $S^p \times S^q$ then we can assume h' is identity on D^{p+q} then we have $h' \in \text{Diff}^+(S^p \times S^q - D^{p+q})$. We then define $h \in M_{p,q}^+$ thus

$$h(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in (S^p \times S^q)_1 - D^{p+q} \\ h'(x, y) & \text{if } (x, y) \in (S^p \times S^q)_2 - D^{p+q} \end{cases}$$

where $M_{p,q}^+ = \text{Diff}^+(S^p \times S^q)_1 \# (S^p \times S^q)_2$ as earlier stated. h is well-defined and $h \in M_{p,q}^+$. Since h is identity on $(S^p \times S^q)_1$ then it is identity on $S^p \times D^q \subset (S^p \times S^q)_1$ hence $h \in \text{Ker } G$ and clearly $N(h) = h'$ and so N is surjective.

We recall from [6, §3] the homomorphism

$$B : \tilde{\pi}_0 \text{Diff}^+(S^p \times S^q) \longrightarrow \pi_p \text{SO}(q+1) \text{ which is similarly}$$

defined as homomorphism G and where Sato gave a computation of $\text{Ker } B$. We will apply this result of $\text{Ker } B$ to the next lemma.

LEMMA 2.12 $\text{Ker } N$ is in one-to-one correspondence with $\text{Ker } B$.

PROOF: Let $f \in \text{Ker } B$, we will produce a diffeomorphism $f' \in M_{p,q}^+$ such that $f' \in \text{Ker } N$. Since $f \in \text{Ker } B$ then $f \in \text{Diff}^+(S^p \times S^q)$ and $f|_{S^p \times D^q} = \text{identity}$. We define a diffeomorphism $f' : (S^p \times S^q)_1 \# (S^p \times S^q)_2 \longrightarrow (S^p \times S^q)_3 \# (S^p \times S^q)_4$ by

$$f'(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in (S^p \times S^q)_1 - D^{p+q} \\ (x, y) & \text{if } (x, y) \in (S^p \times S^q)_2 - D^{p+q} \end{cases}$$

f' is well-defined and $f' \in M_{p,q}^+$. Since $f' = f$ on $(S^p \times S^q)$, and since $f|_{S^p \times D^q \subset (S^p \times S^q)_1}$ is identity then it follows that $f'|_{S^p \times D^q} = \text{identity}$ and so $f' \in \text{Ker } G$. However, using $S^p \times D^q \subset (S^p \times S^q)_1$ to perform spherical modification on both sides of the domain and range of f' and the fact that f' is the identity on $(S^p \times S^q)_2$ we clearly see that $N(f') = \text{identity} \in \text{Diff}(S^p \times S^q)_2$ hence $f' \in \text{Ker } N$.

Conversely let $f \in \text{Ker } N$, then $N(f) = f' \in \tilde{\pi}_0 \text{Diff}^+(S^p \times S^q)$. We want to show that $f' \in \text{Ker } B$. Since $f \in \text{Ker } N$ then it means the image of f under N is trivial hence $N(f) = f'$ is pseudo-diffeotopic to the identity. We now consider $B(f')$ where $B: \tilde{\pi}_0 \text{Diff}^+(S^p \times S^q) \rightarrow \pi_p(SO(q+1))$ is defined in [6] similar to our homomorphism G . Since $f' \in \text{Diff}^+(S^p \times S^q)$ and $p < q$ then $f'|_{S^p \times D^q} = S^p \times D^q$ where $f'(x, y) = (x, b(f')(x) \cdot y)$ for $(x, y) \in S^p \times D^q$ and $b(f'): S^p \rightarrow SO(q)$. If $i: SO(q) \rightarrow SO(q+1)$ is the inclusion map and $i_*: \pi_p(SO(q)) \rightarrow \pi_p(SO(q+1))$ is the induced homomorphism then $B(f') = i_* b(f') \in \pi_p(SO(q+1))$.

However since f' is pseudo-diffeotopic to the identity then let $H: S^p \times S^q \times I \rightarrow S^p \times S^q \times I$ be the pseudo-diffeotopy between f' and identity id . Then

$$\begin{array}{ccccc} D^{p+1} \times S^q & \cup & D^{p+1} \times S^q = D^{p+1} \times S^q & \cup & S^p \times S^q \times I \cup D^{p+1} \times S^q \\ f \downarrow \approx & & \downarrow \text{id} & & \downarrow H & \downarrow \text{id}_1 & \downarrow \text{id} \\ D^{p+1} \times S^q & \cup & D^{p+1} \times S^q = D^{p+1} \times S^q & \cup & S^p \times S^q \times I \cup D^{p+1} \times S^q \\ \text{id} & & \text{id}'_1 & & \text{id}_2 & & \text{id}_2 \end{array}$$

is the required diffeomorphism between $D^{p+1} \times S^q \cup D^{p+1} \times S^q$ and $D^{p+1} \times S^q \cup D^{p+1} \times S^q = S^{p+1} \times S^q$ where $\text{id}_1(x, y) = (x, y, 1)$, $\text{id}_2(x, y, 0) = (x, y)$, $f'_1(x, y, 0) = f'(x, y)$ and $\text{id}_2(x, y) = \text{id}(x, y, 1) = (x, y)$. However, consider $S_{i_* b(f')}$ the q -sphere bundle over a $(p+1)$ -sphere whose characteristic class of the equivalent normal bundle is $i_* b(f') \in \pi_p(SO(q+1))$ hence $S_{i_* b(f')} = D^{p+1} \times S^q \cup D^{p+1} \times S^q \approx S^{p+1} \times S^q$ by the above diffeomorphism and since $p < q$ it follows by [f, Prop. 3.6] that $i_* b(f') = 0$. Hence $f' \in \text{Ker } B$ and so $\text{Ker } N$ is in one-to-one correspondence with $\text{Ker } B$. Since N is surjective by Lemma 2.11 then we have

LEMMA 2.13 The order of the group $\text{Ker } G$ equals the order of the direct sum group

$$\text{Ker } B \oplus \tilde{\pi}_0 \text{Diff}^+(S^p \times S^q)$$

Also since G is surjective by Lemma 2.8 then it is easily seen that

LEMMA 2.14 The order of $\tilde{\pi}_0(M_{p,q}^+)$ is equal to the order of the direct sum group

$$\pi_p(SO(q+1)) \oplus \text{Ker } B \oplus \tilde{\pi}_0 \text{Diff}^+(S^p \times S^q)$$

However one can easily deduce from [6, §4]

$$\text{LEMMA 2.15 } \text{ker } B \approx \pi_q(SO(p+1)) \oplus \theta^{p+q+1}$$

Also from [6, Thm. II] and [1, Thm. 3.10] we have

$$\text{LEMMA 2.16 } \tilde{\pi}_0 \text{Diff}^+(S^p \times S^q) = \pi_p(SO(q+1)) \oplus \pi_q(SO(p+1)) \oplus \theta^{p+q+1}$$

Combining Lemmas 2.12, 2.13, 2.14, 2.15, and 2.16, we obtain

THEOREM 2.17 For $p < q$, the order of the group $\tilde{\pi}_0(M_{p,q}^+)$ equals twice the order of the group $\pi_p(SO(q+1)) \oplus \pi_q(SO(p+1)) \oplus \theta^{p+q+1}$.

3. CLASSIFICATION OF MANIFOLDS

Consider the class of manifolds $\{M, \lambda_1, \lambda_2\}$ where M is a manifold of type

$(n, p, 2)$ where $n = p+q+1$ and $p = 3, 5, 6, 7 \pmod{8}$ and λ_1, λ_2 are the generators of $H_p(M) = \mathbb{Z} \oplus \mathbb{Z}$. By the proof of Theorem 1.1 we have an embedding $\varphi_i : S^p \times D^{q+1} \rightarrow M$ which represents the homology class λ_i $i = 1, 2$. If we then take the connected sum along the boundary of the two embedded copies of $S^p \times D^{q+1}$ we have an embedding

$$i : S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \rightarrow M \quad \text{such that } i_*(S^p) = \lambda_1 + \lambda_2$$

Two of such manifolds $[M, \lambda_1, \lambda_2]$ and $[M', \lambda'_1, \lambda'_2]$ will be said to be equivalent if there is an orientation preserving diffeomorphism of M onto M' which takes λ_1 to λ'_1 $i = 1, 2$. Let \mathcal{M}_n be the equivalent class of manifolds satisfying these conditions. This equivalent class which is also the diffeomorphism class has a group structure. The operation is connected sum along the boundary $S^p \times S^q \# S^p \times S^q$ of $S^p \times D^{q+1} \# S^p \times D^{q+1}$. For if $[M, \lambda_1, \lambda_2], [M', \lambda'_1, \lambda'_2] \in \mathcal{M}_n$, then let

$i_1 : S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \rightarrow M$ be an orientation preserving embedding such that

$i_{1*}[S^p] = \lambda_1 + \lambda_2$ and since there is an orientation reversing diffeomorphism of $S^p \times D^{q+1} \# S^p \times D^{q+1}$ to itself (because $S^p \times D^{q+1}$ is a trivial $q+1$ -disc bundle over S^p) then we have an orientation reversing embedding $i_2 : S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \rightarrow M'$

such that $i_{2*}[S^p] = \lambda'_1 + \lambda'_2$. We now obtain $M \# M'$ from the disjoint sum

$(M - \text{Int } i_1(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})) \cup (M' - \text{Int } i_2(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}))$ by identifying

$i_1(x)$ with $i_2(x)$ for $x \in S^p \times S^q \# S^p \times S^q$. We will call this operation the connected sum along double p -cycle. Where the $2p$ in $M \# M'$ means that we are

identifying along the boundary of embedded copies of connected sum along the boundary of two copies of $S^p \times D^{q+1}$. It is easy to see that $H_p(M \# M') \cong \mathbb{Z} \oplus \mathbb{Z}$. Since we

have identified $i_1(S^p \times S^q \# S^p \times S^q)$ with $i_2(S^p \times S^q \# S^p \times S^q)$ we can define

$i_{1*}[S^p] = \lambda_1 \# \lambda'_1 + \lambda_2 \# \lambda'_2$ the generators of $H_p(M \# M')$ then we see that $M \# M' \in \mathcal{M}_n$.

LEMMA 3.1 The connected sum along the double p -cycle is well-defined and associative.

PROOF: We need to show that the operation does not depend on the choice of the embeddings. Suppose there is another embedding $\varphi'_i : S^p \times D^{q+1} \rightarrow M$ which represents the homology class λ_i $i = 1, 2$ and gives a corresponding embedding $i'_1 : S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \rightarrow M$. By the tubular neighborhood theorem $\varphi_i(S^p \times D^{q+1})$ and $\varphi'_i(S^p \times D^{q+1})$ differ only by rotation of their fiber, i.e., by an element of $\pi_p SO(q+1) = 0$ since $p = 3, 5, 6, 7 \pmod{8}$ hence the two embeddings are isotopic and so the corresponding embeddings

$$i_1 : S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \rightarrow M \quad \text{and}$$

$$i'_1 : S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \rightarrow M \quad \text{are isotopic.}$$

The definition does not therefore depend on the choice of i_1 . With similar argument it does not depend on i_2 . The connected sum is therefore well-defined. Associativity is easy to check.

LEMMA 3.2 If $\{M, \lambda_1, \lambda_2\}, \{M_1, \lambda_{1_1}, \lambda_{1_2}\} \in \mathcal{M}_n$, such that they are equivalent.

If $\{M', \lambda'_1, \lambda'_2\} \in \mathcal{M}_n$ then $(M \# M', \lambda_1 \# \lambda'_1, \lambda_2 \# \lambda'_2)$ is equivalent to $(M_1 \# M', \lambda_{1_1} \# \lambda'_1, \lambda_{1_2} \# \lambda'_2)$.

PROOF: Since M, M_1 are equivalent in \mathcal{M}_n then there exists an orientation preserving diffeomorphism $f: M \rightarrow M_1$ which carries λ_1 to λ_{1_1} and λ_2 to λ_{1_2} hence it carries the embedding $\varphi_i(S^p \times D^{q+1})$ to the corresponding embedding $\varphi_{1_i}(S^p \times D^{q+1})$ $i=1, 2$ and so f carries the embedding $i(S^p \times D^{q+1} \# S^p \times D^{q+1}) \subset M$ to the embedding $i_1(S^p \times D^{q+1} \# S^p \times D^{q+1}) \subset M_1$ hence f induces a diffeomorphism

$$f': M - \text{Int } i(S^p \times D^{q+1} \# S^p \times D^{q+1}) \xrightarrow{\partial} M_1 - \text{Int } i_1(S^p \times D^{q+1} \# S^p \times D^{q+1})$$

which carries λ_1 to λ_{1_1} and λ_2 to λ_{1_2} .

Trivially we have the identity map

$$\text{id}: M' - \text{Int } i'(S^p \times D^{q+1} \# S^p \times D^{q+1}) \xrightarrow{\partial} M' - \text{Int } i(S^p \times D^{q+1} \# S^p \times D^{q+1})$$

which carries λ'_1 to λ'_1 and λ'_2 to λ'_2 . We then take the connected sum along their boundary $S^p \times S^q \# S^p \times S^q$ to have $M \# M'$ which is disjoint sum of

$M - \text{Int } i(S^p \times D^{q+1} \# S^p \times D^{q+1}) \cup M' - \text{Int } i'(S^p \times D^{q+1} \# S^p \times D^{q+1})$ by identifying $i(x)$ and $i'(x)$ for $x \in S^p \times S^q \# S^p \times S^q$. Similarly $M_1 \# M'$ is the disjoint sum of

$M - \text{Int } i_1(S^p \times D^{q+1} \# S^p \times D^{q+1}) \cup M' - \text{Int } i'(S^p \times D^{q+1} \# S^p \times D^{q+1})$ by identifying $i_1(x)$ and $i'(x)$ for $x \in S^p \times S^q \# S^p \times S^q$. Clearly we have a diffeomorphism

$g: M \# M' \rightarrow M_1 \# M'$ which is f' on M and identity of M' and g carries $\lambda_1 \# \lambda'_1$ to $\lambda_{1_1} \# \lambda'_1$ and $\lambda_2 \# \lambda'_2$ to $\lambda_{1_2} \# \lambda'_2$. Hence $\{M \# M', \lambda_1 \# \lambda'_1, \lambda_2 \# \lambda'_2\}$ is equivalent to $\{M_1 \# M', \lambda_{1_1} \# \lambda'_1, \lambda_{1_2} \# \lambda'_2\}$ in \mathcal{M}_n . That proves the lemma.

If we now take two copies of $S^p \times D^{q+1} \# S^p \times D^{q+1}$ and identify the two copies on

their common boundaries by the identity map, we will obtain the manifold $S^p \times S^{q+1} \# S^p \times S^{q+1}$, i.e., $S^p \times S^{q+1} \# S^p \times S^{q+1} = (S^p \times D^{q+1} \# S^p \times D^{q+1}) \cup (S^p \times D^{q+1} \# S^p \times D^{q+1})$

where $\text{id} = \text{identity}: S^p \times S^q \# S^p \times S^q \rightarrow S^p \times S^q \# S^p \times S^q$. If $\lambda_{0_1}, \lambda_{0_2}$ are the generators of $H_p(S^p \times S^{q+1} \# S^p \times S^{q+1}) = \mathbf{Z} \oplus \mathbf{Z}$ and $-\lambda_1 + (-\lambda_2) \in H_p(-M) = \mathbf{Z} \oplus \mathbf{Z}$ where $i_*(S^p) = -\lambda_1 + -\lambda_2$ and $i: M \rightarrow -M$ is the orientation reversing diffeomorphism then we have the following.

LEMMA 3.3 \mathcal{M}_n is a group with identity element $(S^p \times S^{q+1} \# S^p \times S^{q+1}, \lambda_{0_1}, \lambda_{0_2})$

and for $(M, \lambda_1, \lambda_2) \in \mathcal{M}_n$ $(-M, -\lambda_1, -\lambda_2)$ is the inverse element.

To be able to prove our main theorem later, we need to investigate

$\widetilde{\pi}_0^+ \text{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1})$. As in the case of $\widetilde{\pi}_0(M_{p,q})$, we define a homomorphism $\Phi: \widetilde{\pi}_0^+ \text{Diff}(S^p \times D^{q+1} \# S^p \times D^{q+1}) \rightarrow \text{Auto } H_*(S^p \times D^{q+1} \# S^p \times D^{q+1})$ by induced automorphism of homology groups. Since $S^p \times D^{q+1} \# S^p \times D^{q+1}$ has the homotopy type of $S^p \times D^{q+1} \vee S^p \times D^{q+1}$ then

$$H_i(S^p \times D^{q+1} \# S^p \times D^{q+1}) = \begin{cases} \mathbf{Z} & \text{if } i = 0 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } i = p \end{cases} .$$

Using similar ideas in §2, it is easy to prove the following.

LEMMA 3.4

$$\Phi'(\widetilde{\pi}_0(\text{Diff}(S^p \times D^{q+1} \# S^p \times D^{q+1}))) = \begin{cases} \mathbb{Z}_4 & \text{if } p \text{ is even} \\ \text{GL}(2, \mathbb{Z}) & \text{if } p = 1, 3, 7 \\ \mathbb{H} & \text{if } p \text{ is odd but } \neq 1, 3, 7 \end{cases}$$

Let $\text{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1}) \subset \text{Diff}(S^p \times D^{q+1} \# S^p \times D^{q+1})$ be the set of all diffeomorphisms of $S^p \times D^{q+1} \# S^p \times D^{q+1}$ which induce identity automorphisms on its homology. Then it follows that $\widetilde{\pi}_0(\text{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1}))$ is the kernel of Φ' . We define a homomorphism

$$G' : \pi_p SO(q+1) \longrightarrow \widetilde{\pi}_0(\text{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1}))$$

If $a \in \pi_p SO(q+1)$ and $a = \{a\}$ then we define a map

$$g_a : S^p \times D^{q+1} \# S^p \times D^{q+1} \xrightarrow{\partial} S^p \times D^{q+1} \# S^p \times D^{q+1}$$

by

$$g_a(x, y) = \begin{cases} (x, a(x) \cdot y) & \text{for } (x, y) \in (S^p \times D^{q+1})_1 \\ (x, a(x) \cdot y) & \text{for } (x, y) \in (S^p \times D^{q+1})_2 \end{cases}$$

g_a is clearly well-defined and it is a diffeomorphism and since g_a keeps S^p fixed, it induces identity on all homology groups hence $g_a \in \text{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1})$.

We will define $G'[a] = \{g_a\}$

LEMMA 3.5 G' is well defined.

PROOF: If $a' \in \pi_p SO(q+1)$ such that a is homotopic to a' and let $H : S^p \times I \longrightarrow SO(q+1)$ be the homotopy such that $H(S^p \times 0) = a$ and $H(S^p \times 1) = a'$ then we construct a diffeomorphism F of $(S^p \times D^{q+1} \# S^p \times D^{q+1}) \times I$ by

$$F(x, y, t) = \begin{cases} (x, H(x, t) \cdot y) & (x, y) \in (S^p \times D^{q+1})_1 \\ (x, H(x, t) \cdot y) & (x, y) \in (S^p \times D^{q+1})_2 \end{cases}$$

This is the diffeotopy which connects g_a and $g_{a'}$.

LEMMA 3.6 G' is surjective.

PROOF: Let $\{f\} \in \widetilde{\pi}_0(\text{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1}))$ then f induces identity on all homology groups. However $H_p(S^p \times D^{q+1} \# S^p \times D^{q+1}) \approx \mathbb{Z} \oplus \mathbb{Z}$ and so if λ_1 and λ_2 represents the generators of the first and second summand and the embeddings $i_1 : S^p \times \{p_0\} \longrightarrow S^p \times D^{q+1} \# S^p \times D^{q+1}$ and $i_2 : S^p \times \{p_0\} \longrightarrow S^p \times D^{q+1} \# S^p \times D^{q+1}$ represents the homology class λ_1 and λ_2 respectively, since f induces identity on homology then $f(S^p \times \{p_0\})$ and $i_1(S^p \times \{p_0\})$ are homologous. Since $p < q$ and by Hurewicz theorem i_1 and $f \circ i_1$ are homotopic, by Haefliger [10] and by the diffeotopy extension theorem and tubular neighborhood theorem, there exists f' in the diffeotopy class of f such that $f'(x, y) = (x, a(x) \cdot y)$ for $(x, y) \in (S^p \times D^{q+1})_1$ where $S^p \times D^{q+1}$ is the tubular neighborhood of $S^p \times \{p_0\}$ and $a : S^p \longrightarrow SO(q+1)$. Similar argument applies to the embedding $i_2 : S^p \times \{p_0\} \longrightarrow S^p \times D^{q+1} \# S^p \times D^{q+1}$ and

so we have a map f' in the diffeotopy class of f hence in the diffeotopy class of f' and so f' must be of the form $f'(x, y) = (x, a(x) \cdot y)$ where $(x, y) \in (S^p \times D^{q+1})_2$. It follows that

$$f(x, y) = \begin{cases} (x, a(x) \cdot y) & (x, y) \in (S^p \times D^{q+1})_1 \\ (x, a(x) \cdot y) & (x, y) \in (S^p \times D^{q+1})_2 \end{cases}$$

Hence G' is surjective.

One can easily deduce from Lemma 3.6 that $\tilde{\pi}_0^{\text{Diff}^+}(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})$ is a factor group of $\pi_p(SO_{q+1})$.

THEOREM 3.7 Let M be an n -dimensional closed simply connected manifold of type $(n, p, 2)$ where $n = p+q+1$ with $p = 3, 5, 6, 7 \pmod{8}$ then the number of differentiable manifolds satisfying the above conditions up to diffeomorphism is twice the order of the direct sum group $\pi_p SO(p+1) \oplus \theta^n$.

PROOF: We define a map $C : \tilde{\pi}_0(M_{p,q}^+) \longrightarrow \mathcal{M}_n$ and show that C is an isomorphism. Let $\{f\} \in \tilde{\pi}_0(M_{p,q}^+)$ then f is a diffeomorphism of $S^p \times S^q \# S^p \times S^q$ which induces identity on homology. We then take two copies $(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})_1$ and $(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})_2$ of $S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}$ and attach them on the boundary by f to have $(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})_1 \cup (S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})_2$. An orientation is chosen to be compatible with $(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})_1$ and the manifold obtained belongs to the group \mathcal{M}_n .

The generators of the p -dimensional homology group is fixed to be the one represented by the usual embedding $S^p \times \{p_0\} \longrightarrow (S^p \times D^{q+1})_1 \subset (S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})_1$ and $S^p \times \{p_0\} \longrightarrow (S^p \times D^{q+1})_2 \subset (S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})_2$. We then define $C[f] = (S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}) \cup (S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})$. We now show that C is well-defined. Let $f_0, f_1 \in M_{p,q}^+$ such that f_0 is pseudo-diffeotopic to f_1 then there exists $H : (S^p \times S^q \# S^p \times S^q) \times I \longrightarrow (S^p \times S^q \# S^p \times S^q) \times I$ such that $H(x, y, 0) = f_0$ and $H(x, y, 1) = f_1$ then we wish to show that $(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}) \cup (S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})$ is diffeomorphic to $(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}) \cup (S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})$. We then define a map

$$\begin{array}{ccccccc} (S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}) \cup S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} & = & S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} & \cup (S^p \times S^q \# S^p \times S^q) \times I & \cup S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \\ \downarrow f_0 & & \downarrow id & & \downarrow H & & \downarrow id \\ (S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}) \cup (S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}) & = & S^p \times D^{q+1} \#_{\partial} D^{q+1} & \cup (S^p \times S^q \# S^p \times S^q) \times I & \cup S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \\ \downarrow f_1 & & \downarrow id_1 & & \downarrow f_1 & & \downarrow id \end{array}$$

where $id_0(x, y) = (x, y, 1)$, $id_1(x, y, 0) = (x, y)$, $f_0'(x, y, 0) = f_0(x, y)$ and $f_1'(x, y) = f_1(x, y, 1)$. This is a well-defined map and is the required diffeomorphism from $(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}) \cup (S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})$ to $(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}) \cup (S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})$. Hence C is well-defined and it is easy to see that C is a homomorphism. By Theorem 1.1 it follows that C is surjective. We now need to show that C is injective. Suppose $\{f\} \in \tilde{\pi}_0(M_{p,q}^+)$ and $C(f) = (M, \lambda_1, \lambda_2)$ is trivial, then it follows that

$M = (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \cup_{\partial} (S^p \times D^{q+1} \# S^p \times D^{q+1})_2$ is diffeomorphic to $(S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \cup_{\text{id}} (S^p \times D^{q+1} \# S^p \times D^{q+1})_2 = S^p \times S^{q+1} \# S^p \times S^{q+1}$ with p -dimensional homology generators $\lambda_0, \lambda_1, \lambda_2$, by a diffeomorphism d which carries λ_1 to λ_0 and λ_2 to λ_0 , i.e.,

$$\begin{array}{c} (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \cup_{\partial} (S^p \times D^{q+1} \# S^p \times D^{q+1})_2 \\ \downarrow \\ (S^p \times D^{q+1} \# S^p \times D^{q+1})_1 \cup_{\text{id}} (S^p \times D^{q+1} \# S^p \times D^{q+1})_2 = S^p \times S^{q+1} \# S^p \times S^{q+1} \end{array}$$

It is easy to see that since d carries λ_1 to λ_0 and λ_2 to λ_0 and because $p = 3, 5, 6, 7 \pmod{8}$ then d is the identity on $(S^p \times D^{q+1} \# S^p \times D^{q+1})_1$. On the boundary $S^p \times S^q \# S^p \times S^q$, d is just f . Since d is a diffeomorphism it follows that f extends to a diffeomorphism of $(S^p \times D^{q+1} \# S^p \times D^{q+1})_2$, which means $f \in \text{Diff}^+(S^p \times S^q \# S^p \times S^q)$ is extendable to $\text{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1})$, but by Lemma 3.5, $\pi_0 \text{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1})$ is a factor group of $\pi_p(SO_{q+1})$ but since $p = 3, 5, 6, 7 \pmod{8}$ then $\pi_p(SO_{q+1}) = 0$. Hence f is pseudo-diffeotopic to the identity and so C is injective. It then follows that C is an isomorphism. By Theorem 2.17 and since $p = 3, 5, 6, 7 \pmod{8}$ it follows that the order of the group $\pi_0(M_{p,q}^+)$ is twice the order of the group $\pi_q SO(p+1) \oplus \theta^n$ and since C is an isomorphism the theorem is proved. The methods used here if carefully applied can be used to obtain a general result.

THEOREM 3.8 If M is a smooth, closed simply connected manifold of type (n, p, r) where $n = p+q+1$ and $p = 3, 5, 6, 7 \pmod{8}$ then the number of differentiable manifolds up to diffeomorphism satisfying the above is equal to r times the order of $\pi_q SO(p+1) \oplus \theta^n$.

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