

GAPS IN THE SEQUENCE $n^2 \vartheta \pmod{1}$

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(Received September 16, 1985)

ABSTRACT: Let ϑ be an irrational number and let $\{t\}$ denote the fractional part of t . For each N let I_0, I_1, \dots, I_N be the intervals resulting from the partition of $[0,1]$ by the points $\{k^2\vartheta\}$, $k = 1, 2, \dots, N$. Let $T(N)$ be the number of distinct lengths these intervals can assume. It is shown that $T(N) \rightarrow \infty$. This is in contrast to the case of the sequence $\{n\vartheta\}$, where $T(N) \leq 3$.

KEY WORDS AND PHRASES. Uniform distribution mod 1.

1980 AMS SUBJECT CLASSIFICATION CODE. Primary 10K31

1. INTRODUCTION.

Let ϑ be an irrational number and let $\{t\}$ denote the fractional part of t ($\{t\} = t \pmod{1} = t - [t]$, where $[.]$ is the greatest integer function). For each fixed N the points $\{\vartheta\}, \{2\vartheta\}, \{3\vartheta\}, \dots, \{N\vartheta\}$ partition on the interval $[0,1]$ into $N+1$ subintervals. It is well known that the lengths of these intervals can assume only 3 values: α , β and $\alpha+\beta$. The values of α and β can be actually given explicitly in terms of N and the continued fraction expansion of ϑ . This is known as Steinhaus conjecture and it was first proved by Swierczkowski in [1]. For an excellent exposition of all this, see [2]. In this note we investigate the analogous problem for the sequence $\{n^2\vartheta\}$. It turns out that in this case the number of different lengths these subintervals can assume, is unbounded. More precisely we have the following results.

2. MAIN RESULTS.

Theorem 1 Let ϑ be an irrational. For each integer N let I_0, I_1, \dots, I_N be the $N+1$ subintervals resulting from partition of $[0,1]$ by the points $\{k^2\vartheta\}$, $k = 1, 2, \dots, N$. Let $T(N)$ be the number of distinct lengths these subintervals assume. Then for each $\varepsilon > 0$,

$$T(N) \geq N \exp\left\{-\left(1+\varepsilon\right) \ln 2^2 \frac{\ln N}{\ln \ln N}\right\} \quad \text{for } N \geq N(\varepsilon). \quad (2.1)$$

In particular $T(N) \geq N^{1-\delta}$ for every $\delta > 0$ and $N \geq N(\delta)$.

In what follows $\delta > 0$ is some fixed irrational. We need the following four simple lemmas.

LEMMA 1. For any integers r, s

$$\{(r+s)\delta\} = \{r\delta\} + \{s\delta\} - E \quad (2.2)$$

where $E = 0$ or 1.

PROOF. We have

$$\begin{aligned} (r+s)\delta &= \{(r+s)\delta\} + [(r+s)\delta] \\ &= \{r\delta\} + \{s\delta\} + [r\delta] + [s\delta] = \{r\delta\} + \{s\delta\} + \text{integer} \end{aligned}$$

Thus, if $0 < \{r\delta\} + \{s\delta\} < 1$ then (2) holds with $E = 0$,

and if $1 < \{r\delta\} + \{s\delta\} < 2$ then (2) holds with $E = 1$.

LEMMA 2. Suppose x, y are integers, $\{x\delta\} < \{y\delta\}$. Then

$$\{y\delta\} - \{x\delta\} = \begin{cases} \{(y-x)\delta\} & \text{if } x < y \\ 1 - \{(x-y)\delta\} & \text{if } y < x \end{cases} \quad (2.3)$$

PROOF. Suppose $x < y$ so that $y = x + k$. Then by Lemma 1

$$\{y\delta\} = \{x\delta\} + \{k\delta\} - E.$$

If $E = 1$ then $\{y\delta\} < \{x\delta\}$ contrary to hypothesis, so that $E = 0$ and (2.3) holds.

If $y < x$, let $x = y + k$, $k > 0$. Again, by Lemma 1

$$\{x\delta\} = \{y\delta\} + \{k\delta\} - E.$$

If $E = 0$ then $\{x\delta\} > \{y\delta\}$ so that $E = -1$ and (2.3) holds again.

LEMMA 3. For any two non-negative integers x, y , $\{x\delta\} \neq 1 - \{y\delta\}$.

PROOF. If $\{x\delta\} + \{y\delta\} = 1$ then by Lemma 1

$$\{(x+y)\delta\} = \{x\delta\} + \{y\delta\} - E = 1 - E = 0 \text{ or } 1$$

contradicting the fact that δ is irrational.

LEMMA 4. Suppose x_1, y_1, x_2, y_2 are non-negative integers and let

$$A = \{y_1\delta\} - \{x_1\delta\} > 0, \quad B = \{y_2\delta\} - \{x_2\delta\} > 0.$$

If $A = B$ then $y_1 - x_1 = y_2 - x_2$.

PROOF. We will use Lemmas 2 and 3 and consider 4 cases

I: $x_1 < y_1, x_2 < y_2$;

II: $x_1 < y_1, x_2 > y_2$;

III: $x_1 > y_1, x_2 < y_2$;

IV: $x_1 > y_1, x_2 > y_2$.

In case I we get from Lemma 2

$$A = \{(y_1 - x_1)\delta\}, \quad B = \{(y_2 - x_2)\delta\}$$

so $A = B$ implies $y_1 - x_1 = y_2 - x_2$.

In case II, by Lemma 2 we get

$$A = \{(y_1 - x_1)\delta\}, \quad B = 1 - \{(x_2 - y_2)\delta\}$$

so $A = B$ cannot hold by Lemma 3.

Similarly, $A = B$ cannot hold in case III, and $A = B$ implies $y_1 - x_1 = y_2 - x_2$ in case IV.

We are now ready to prove the Theorem 1. Let N be fixed and consider the partition of $[0,1]$ by the points $\{0^2\vartheta\} = 0, \{1^2\vartheta\}, \{2^2\vartheta\}, \{3^2\vartheta\}, \dots, \{N^2\vartheta\}$. If we exclude the right most interval (i.e. the interval $[\{N^2\vartheta\}, 1]$ for some x), we are left with a collection $A(N)$ of N intervals. If two of these intervals $[\{x_1^2\vartheta\}, \{y_1^2\vartheta\}]$ and $[\{x_2^2\vartheta\}, \{y_2^2\vartheta\}]$ are of equal length then

$$y_1^2 - x_1^2 = y_2^2 - x_2^2 \quad (2.4)$$

by Lemma 4. Let $T(N)$ be the number of distinct lengths these intervals from $A(N)$ can assume. The collection $A(N)$ is then divided into $T(N)$ subsets, any two intervals from one subset are of equal length. One of these subsets must contain $N/T(N)$ intervals. Thus, by (2.4), there exists an integer k , $1 \leq k \leq N^2$ such that the equation

$$k = y^2 - x^2 = (y-x)(y+x) \quad (2.5)$$

has $N/T(N)$ solutions in integers x, y , $1 \leq x < y \leq N^2$. Each such solution produces 2 distinct divisors of k . If $y_1^2 - x_1^2 = y_2^2 - x_2^2$, $1 \leq x_i < y_i \leq N^2$ for $i = 1, 2$ and $(x_1, y_1) \neq (x_2, y_2)$, then $y_1 - x_1 \neq y_2 - x_2$ and $y_1 + x_1 \neq y_2 + x_2$. Thus

$$N/T(N) \leq \frac{1}{2}d(k) \quad (2.6)$$

where $d(z)$ is the number of divisors of z . It is well known that for each $\epsilon > 0$

$$d(z) < \exp\left\{(\ln z + \epsilon) \ln 2 \frac{\ln z}{\ln \ln z}\right\} = \varphi(\epsilon, z) \quad \text{for } z \geq z(\epsilon)$$

This was first proved by Wigert in [3], see also [4], Satz 5.2. Since $k < N^2$ we get from (2.6)

$$\begin{aligned} 2N/T(N) &\leq \varphi(\epsilon, k) < \varphi(\epsilon, N^2) \\ &= \exp\left\{(\ln N + \epsilon) \ln 2 \frac{2 \ln N}{\ln 2 + \ln \ln N}\right\} \\ &\leq \exp\left\{(\ln N + \epsilon) \ln 2^2 \frac{\ln N}{\ln \ln N}\right\} \quad \text{for } N > N_1(\epsilon) \end{aligned} \quad (2.7)$$

Solving this inequality for $T(N)$ gives (2.1).

The argument carries over almost without any change to the sequence $\{n^p\vartheta\}$ for any integer $p > 1$. The corresponding estimate is then as follows.

THEOREM 2. Let ϑ be an irrational and $p > 1$ an integer. For each integer N let I_0, I_1, \dots, I_N be the $N+1$ subintervals resulting from partition of $[0,1]$ by the points $\{k^p\vartheta\}$, $k = 1, 2, \dots, N$. Let $T_p(N)$ be the number of distinct lengths these intervals can assume. Then for each $\epsilon > 0$

$$T_p(N) \geq N \exp\left\{-(\ln N + \epsilon) \ln 2^p \frac{\ln N}{\ln \ln N}\right\} \quad \text{for } N \geq N(\epsilon).$$

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