

## GAPS IN THE SEQUENCE $n^2 \vartheta \pmod{1}$

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**ABSTRACT:** Let  $\vartheta$  be an irrational number and let  $\{t\}$  denote the fractional part of  $t$ . For each  $N$  let  $I_0, I_1, \dots, I_N$  be the intervals resulting from the partition of  $[0,1]$  by the points  $\{k^2 \vartheta\}$ ,  $k = 1, 2, \dots, N$ . Let  $T(N)$  be the number of distinct lengths these intervals can assume. It is shown that  $T(N) \rightarrow \infty$ . This is in contrast to the case of the sequence  $\{n\vartheta\}$ , where  $T(N) \leq 3$ .

**KEY WORDS AND PHRASES.** Uniform distribution mod 1.

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### 1. INTRODUCTION.

Let  $\vartheta$  be an irrational number and let  $\{t\}$  denote the fractional part of  $t$  ( $\{t\} = t \pmod{1} = t - [t]$ , where  $[.]$  is the greatest integer function). For each fixed  $N$  the points  $\{\vartheta\}, \{2\vartheta\}, \{3\vartheta\}, \dots, \{N\vartheta\}$  partition on the interval  $[0,1]$  into  $N+1$  subintervals. It is well known that the lengths of these intervals can assume only 3 values:  $\alpha$ ,  $\beta$  and  $\alpha+\beta$ . The values of  $\alpha$  and  $\beta$  can be actually given explicitly in terms of  $N$  and the continued fraction expansion of  $\vartheta$ . This is known as Steinhaus conjecture and it was first proved by Swierczkowski in [1]. For an excellent exposition of all this, see [2]. In this note we investigate the analogous problem for the sequence  $\{n^2 \vartheta\}$ . It turns out that in this case the number of different lengths these subintervals can assume, is unbounded. More precisely we have the following results.

### 2. MAIN RESULTS.

**Theorem 1** Let  $\vartheta$  be an irrational. For each integer  $N$  let  $I_0, I_1, \dots, I_N$  be the  $N+1$  subintervals resulting from partition of  $[0,1]$  by the points  $\{k^2 \vartheta\}$ ,  $k = 1, 2, \dots, N$ . Let  $T(N)$  be the number of distinct lengths these subintervals assume. Then for each  $\epsilon > 0$ ,

$$T(N) \geq N \exp\left\{-(1+\epsilon) \ln 2^2 \frac{\ln N}{\ln \ln N}\right\} \quad \text{for } N \geq N(\epsilon). \quad (2.1)$$

In particular  $T(N) \geq N^{1-\delta}$  for every  $\delta > 0$  and  $N \geq N(\delta)$ .

In what follows  $\vartheta > 0$  is some fixed irrational. We need the following four simple lemmas.

LEMMA 1. For any integers  $r, s$

$$\{(r+s)\vartheta\} = \{r\vartheta\} + \{s\vartheta\} - E \quad (2.2)$$

where  $E = 0$  or  $1$ .

PROOF. We have

$$\begin{aligned} (r + s)\vartheta &= \{(r+s)\vartheta\} + [(r+s)\vartheta] \\ &= \{r\vartheta\} + \{s\vartheta\} + [r\vartheta] + [s\vartheta] = \{r\vartheta\} + \{s\vartheta\} + \text{integer} \end{aligned}$$

Thus, if  $0 < \{r\vartheta\} + \{s\vartheta\} < 1$  then (2) holds with  $E = 0$ ,

and if  $1 < \{r\vartheta\} + \{s\vartheta\} < 2$  then (2) holds with  $E = 1$ .

LEMMA 2. Suppose  $x, y$  are integers,  $\{x\vartheta\} < \{y\vartheta\}$ . Then

$$\{y\vartheta\} - \{x\vartheta\} = \begin{cases} \{(y-x)\vartheta\} & \text{if } x < y \\ 1 - \{(x-y)\vartheta\} & \text{if } y < x \end{cases} \quad (2.3)$$

PROOF. Suppose  $x < y$  so that  $y = x + k$ . Then by Lemma 1

$$\{y\vartheta\} = \{x\vartheta\} + \{k\vartheta\} - E.$$

If  $E = 1$  then  $\{y\vartheta\} < \{x\vartheta\}$  contrary to hypothesis, so that  $E = 0$  and (2.3) holds.

If  $y < x$ , let  $x = y + k$ ,  $k > 0$ . Again, by Lemma 1

$$\{x\vartheta\} = \{y\vartheta\} + \{k\vartheta\} - E.$$

If  $E = 0$  then  $\{x\vartheta\} > \{y\vartheta\}$  so that  $E = -1$  and (2.3) holds again.

LEMMA 3. For any two non-negative integers  $x, y$ ,  $\{x\vartheta\} \neq 1 - \{y\vartheta\}$ .

PROOF. If  $\{x\vartheta\} + \{y\vartheta\} = 1$  then by Lemma 1

$$\{(x+y)\vartheta\} = \{x\vartheta\} + \{y\vartheta\} - E = 1 - E = 0 \text{ or } 1$$

contradicting the fact that  $\vartheta$  is irrational.

LEMMA 4. Suppose  $x_1, y_1, x_2, y_2$  are non-negative integers and let

$$A = \{y_1\vartheta\} - \{x_1\vartheta\} > 0, \quad B = \{y_2\vartheta\} - \{x_2\vartheta\} > 0.$$

If  $A = B$  then  $y_1 - x_1 = y_2 - x_2$ .

PROOF. We will use Lemmas 2 and 3 and consider 4 cases

I:  $x_1 < y_1, x_2 < y_2$ ;

II:  $x_1 < y_1, x_2 > y_2$ ;

III:  $x_1 > y_1, x_2 < y_2$ ;

IV:  $x_1 > y_1, x_2 > y_2$ .

In case I we get from Lemma 2

$$A = \{(y_1 - x_1)\vartheta\}, \quad B = \{(y_2 - x_2)\vartheta\}$$

so  $A = B$  implies  $y_1 - x_1 = y_2 - x_2$ .

In case II, by Lemma 2 we get

$$A = \{(y_1 - x_1)\vartheta\}, \quad B = 1 - \{(x_2 - y_2)\vartheta\}$$

so  $A = B$  cannot hold by Lemma 3.

Similarly,  $A = B$  cannot hold in case III, and  $A = B$  implies  $y_1 - x_1 = y_2 - x_2$  in case IV.

We are now ready to prove the Theorem 1. Let  $N$  be fixed and consider the partition of  $[0,1]$  by the points  $\{0^{2\vartheta}\} = 0, \{1^{2\vartheta}\}, \{2^{2\vartheta}\}, \{3^{2\vartheta}\}, \dots, \{N^{2\vartheta}\}$ . If we exclude the right most interval (i.e. the interval  $[\{x^{2\vartheta}\}, 1]$  for some  $x$ ), we are left with a collection  $A(N)$  of  $N$  intervals. If two of these intervals  $[\{x_1^{2\vartheta}\}, \{y_1^{2\vartheta}\}]$  and  $[\{x_2^{2\vartheta}\}, \{y_2^{2\vartheta}\}]$  are of equal length then

$$y_1^2 - x_1^2 = y_2^2 - x_2^2 \quad (2.4)$$

by Lemma 4. Let  $T(N)$  be the number of distinct lengths these intervals from  $A(N)$  can assume. The collection  $A(N)$  is then divided into  $T(N)$  subsets, any two intervals from one subset are of equal length. One of these subsets must contain  $N/T(N)$  intervals. Thus, by (2.4), there exists an integer  $k$ ,  $1 \leq k \leq N^2$  such that the equation

$$k = y^2 - x^2 = (y-x)(y+x) \quad (2.5)$$

has  $N/T(N)$  solutions in integers  $x, y$ ,  $1 \leq x < y \leq N^2$ . Each such solution produces 2 distinct divisors of  $k$ . If  $y_1^2 - x_1^2 = y_2^2 - x_2^2$ ,  $1 \leq x_i < y_i \leq N^2$  for  $i = 1, 2$  and  $(x_1, y_1) \neq (x_2, y_2)$ , then  $y_1 - x_1 \neq y_2 - x_2$  and  $y_1 + x_1 \neq y_2 + x_2$ . Thus

$$N/T(N) \leq \frac{1}{2}d(k) \quad (2.6)$$

where  $d(z)$  is the number of divisors of  $z$ . It is well known that for each  $\epsilon > 0$

$$d(z) < \exp\{(1+\epsilon) \ln 2 \frac{\ln z}{\ln \ln z}\} = \varphi(\epsilon, z) \quad \text{for } z \geq z(\epsilon)$$

This was first proved by Wigert in [3], see also [4], Satz 5.2. Since  $k < N^2$  we get from (2.6)

$$\begin{aligned} 2N/T(N) &\leq \varphi(\epsilon, k) < \varphi(\epsilon, N^2) \\ &= \exp\left\{(1+\epsilon) \ln 2 \frac{2 \ln N}{\ln 2 + \ln \ln N}\right\} \\ &\leq \exp\left\{(1+\epsilon) \ln 2^2 \frac{\ln N}{\ln \ln N}\right\} \quad \text{for } N > N_1(\epsilon) \end{aligned} \quad (2.7)$$

Solving this inequality for  $T(N)$  gives (2.1).

The argument carries over almost without any change to the sequence  $\{n^{p\vartheta}\}$  for any integer  $p > 1$ . The corresponding estimate is then as follows.

**THEOREM 2.** Let  $\vartheta$  be an irrational and  $p > 1$  an integer. For each integer  $N$  let  $I_0, I_1, \dots, I_N$  be the  $N+1$  subintervals resulting from partition of  $[0,1]$  by the points  $\{k^{2\vartheta}\}$ ,  $k = 1, 2, \dots, N$ . Let  $T_p(N)$  be the number of distinct lengths these intervals can assume. Then for each  $\epsilon > 0$

$$T_p(N) \geq N \exp\left\{-(1+\epsilon) \ln 2^p \frac{\ln N}{\ln \ln N}\right\} \quad \text{for } N \geq N(\epsilon).$$

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