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Research Article

On Fuzzy ε-Contractive Mappings in Fuzzy Metric Spaces

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We answer into affirmative an open question raised by A. Razani in 2005. An essential role in our proofs is played by the separation axiom in the definition of a fuzzy metric space in the sense of George and Veeramani.

1. Preliminaries

In this section, we recall some definitions and results that will be used in the sequel.

Definition 1.1 (see [1]). A triple $(X,M,*)$, where X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2\times(0,\infty)$, is said to be a *fuzzy metric space* (in the sense of George and Veeramani) if the following conditions are satisfied for all $x,y\in X$ and $s,t>0$:
(GV-1) $M(x,y,t)>0$;
(GV-2) $M(x,y,t)=1$ if and only if $x=y$;
(GV-3) $M(x,y,t)=M(y,x,t)$;
(GV-4) $M(x,y,\cdot)$ is continuous;
(GV-5) $M(x,z,t+s)\geq M(x,y,t)*M(y,z,s)$.
Note (see [2]) that the "separation" condition (GV-2) means that

$$\begin{aligned} &M(x,x,t)=1 \quad \forall x\in X,\ \forall t>0,\\ &x\neq y\Longrightarrow M(x,y,t)<1 \quad \forall t>0. \end{aligned} \tag{1.1}$$

Definition 1.2 (see [1]). Let $(X,M,*)$ be a fuzzy metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be *convergent* if there is $x\in X$ such that $\lim_{n\to\infty}M(x_n,x,t)=1$ for each $t>0$

(the notation $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ will be used). A mapping $f : X \to X$ is said to be *continuous* if $f(x_n) \to f(x)$ whenever (x_n) is a sequence in X convergent to x.

Definition 1.3 (see [3]). Let $(X,M,*)$ be a fuzzy metric space and $0 < \varepsilon < 1$. A mapping $f : X \to X$ is called *fuzzy ε-contractive* if $M(f(x),f(y),t) > M(x,y,t)$ whenever $1-\varepsilon < M(x,y,t) < 1$.

The next continuity lemma can be found in [4] (also see [5, Theorem 12.2.3]).

Lemma 1.4. *Let $(X,M,*)$ be a fuzzy metric space. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} M(x_n,y_n,t) = M(x,y,t)$ for all $t > 0$.*

2. Main results

The following theorem has been proved by Razani in [3].

Theorem 2.1 (see [3, Theorem 3.3]). *Let $(X,M,*)$ be a fuzzy metric space, where the continuous t-norm is defined as $a * b = \min\{a,b\}$. Suppose f is a fuzzy ε-contractive self-mapping of X such that there exists a point $x \in X$ whose sequence of iterates $(f^n(x))$ contains a convergent subsequence $(f^{n_i}(x))$. Then $\xi = \lim_{i\to\infty} f^{n_i}(x)$ is a periodic point, that is, there is a positive integer k such that $f^k(\xi) = \xi$.*

In [3, Question 3.7], it has been asked whether Theorem 2.1 would remain true if $*$ is replaced by an arbitrary t-norm.

With Theorem 2.3, we answer into affirmative this question. In the proofs of our theorems, we need the following.

Lemma 2.2. *Every fuzzy ε-contractive mapping in a fuzzy metric space is continuous.*

Proof. The continuity of the fuzzy ε-contractive mapping f is an immediate consequence of the implication

$$M(x,y,t) > 1-\varepsilon \Longrightarrow M(f(x),f(y),t) \geq M(x,y,t) \tag{2.1}$$

which can be proved as follows: if $M(x,y,t) < 1$, then $M(x,y,t) > 1-\varepsilon$ implies $M(f(x), f(y),t) > M(x,y,t)$, while if $M(x,y,t) = 1$ then, due to (GV-2), we have $x = y$, hence $M(f(x),f(y),t) = M(x,y,t)$. □

Theorem 2.3. *Let $(X,M,*)$ be a fuzzy metric space. Then for every fuzzy ε-contractive mapping f on X with the property that there exists a point $x \in X$ whose sequence of iterates $(f^n(x))_{n\in\mathbb{N}}$ contains a convergent subsequence, the point $\xi = \lim_{i\to\infty} f^{n_i}(x)$ is a periodic point.*

Proof. Since $*$ is continuous, there is $\delta \in (0,\varepsilon)$ such that $(1-\delta)*(1-\delta) > 1-\varepsilon$. Also, there is a positive integer N_1 such that $i \geq N_1$ implies $M(f^{n_i}(x),\xi,t/2) > 1-\delta$, for all $t > 0$. Fix a $k \geq N_1$ and denote $n_{k+1} - n_k$ by s. As f is fuzzy ε-contractive and $M(f^{n_k}(x),\xi,t/2) > 1-\varepsilon$, we have

$$M\left(f^{n_k+1}(x), f(\xi), \frac{t}{2}\right) \geq M\left(f^{n_k}(x), \xi, \frac{t}{2}\right) > 1-\delta > 1-\varepsilon \tag{2.2}$$

and, after $n_{k+1} - n_k$ iterations, $M(f^{n_{k+1}}(x), f^s(\xi), t/2) > 1 - \delta$. Therefore,

$$\begin{aligned} M(\xi, f^s(\xi), t) &\geq M\left(f^{n_{k+1}}(x), \xi, \frac{t}{2}\right) * M\left(f^{n_{k+1}}(x), f^s(\xi), \frac{t}{2}\right) \\ &\geq (1-\delta) * (1-\delta) > 1 - \varepsilon \quad \forall t > 0. \end{aligned} \tag{2.3}$$

Since f is continuous, $\lim_{i\to\infty} f^{n_i}(x) = \xi$ implies $\lim_{i\to\infty} f^{n_i+s}(x) = f^s(\xi)$, therefore, by Lemma 1.4,

$$\lim_{i\to\infty} M(f^{n_i}(x), f^{n_i+s}(x), t) = M(\xi, f^s(\xi), t) \quad \forall t > 0. \tag{2.4}$$

As the sequence of real numbers $(z_n)_{n\geq n_k}$, $z_n := M(f^n(x), f^{n+s}(x), t)_{n\geq n_k}$ is convergent for every $t > 0$ (being nondecreasing and bounded), one has

$$\lim_{n\to\infty} M(f^n(x), f^{n+s}(x), t) = M(\xi, f^s(\xi), t) \quad \forall t > 0. \tag{2.5}$$

On the other hand, from $f^{n_i}(f(x)) = f(f^{n_i}(x)) \to_{i\to\infty} f(\xi)$ and $f^{n_i}(f^{s+1}(x)) = f^{s+1}(f^{n_i}(x)) \to f^{s+1}(\xi)$, it follows that

$$\lim_{i\to\infty} M(f^{n_i+1}(x), f^{n_i+1+s}(x), t) = M(f(\xi), f^{s+1}(\xi), t) \quad \forall t > 0, \tag{2.6}$$

that is,

$$M(\xi, f^s(\xi), t) = M(f(\xi), f^{s+1}(\xi), t) \quad \forall t > 0. \tag{2.7}$$

We claim that $f^s(\xi) = \xi$. Indeed, if $f^s(\xi) \neq \xi$ then, due to (GV-2), $M(\xi, f^s(\xi), t) < 1$, for all $t > 0$ and since $M(\xi, f^s(\xi), t) > 1 - \varepsilon$ for all $t > 0$, we have $1 - \varepsilon < M(\xi, f^s(\xi), t) < 1$ for all $t > 0$. This implies $M(\xi, f^s(\xi), t) < M(f(\xi), f^{s+1}(\xi), t)$ for all $t > 0$, which is a contradiction. Therefore, $f^s(\xi) = \xi$, concluding the proof. □

Example 2.4. Consider fuzzy metric space $(N^*, M, *)$, where $N^* = \{1, 2, \ldots\}$, $a * b = \min\{a, b\}$, and

$$M(x, y, t) = \begin{cases} \dfrac{1}{2}, & x \neq y, \\ 1, & x = y, \end{cases} \tag{2.8}$$

for all $t > 0$. The mapping $f : N^* \to N^*$,

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is even}, \\ 2 & \text{if } x \text{ is odd}, \end{cases} \tag{2.9}$$

is fuzzy 1/2-contractive, in the absence of the condition $1 - \varepsilon < M(x, y, t) < 1$. The sequence of the successive approximations of 1 is $2, 1, 2, 1, 2, 1, \ldots$, and its subsequence $1, 1, \ldots$ converges to 1, which is a periodic point for f.

In the following, we show that the assertion [3, Corollary 3.5] claiming that in the conditions of Theorem 2.1 we cannot have $M(\xi, f(\xi), t) > 1 - \varepsilon$, is not correct. As a matter of fact, we will show in Theorem 2.7 that $M(\xi, f(\xi), t) > 1-\varepsilon$ is a sufficient condition for the existence of a fixed point for a fuzzy ε-contractive mapping.

Example 2.5. Consider the standard fuzzy metric space $(X,M,*)$, where $X = (-\infty,\infty)$, $M(x,y,t) = t/(t+|x-y|)$, $a*b = \min\{a,b\}$, and the mapping $f : X \to X$, $f(x) = x/2$. Since

$$M(f(x), f(y), t) = \frac{2t}{2t+|x-y|} > \frac{t}{t+|x-y|} = M(x,y,t) \tag{2.10}$$

for all $x, y \in X$, $x \neq y$, and $t > 0$, f is fuzzy ε-contractive for every $\varepsilon \in (0,1)$ and it is immediate that the sequence of iterates of any point converges to 0. As 0 is a fixed point of f, we have $M(0, f(0), t) = 1 > 1-\varepsilon$ for every $\varepsilon \in (0,1)$.

The error in the proof of the corollary derives from the fact that the (strict) inequality $M(f^2(\xi), f(\xi), t) > M(f(\xi), \xi, t)$ (see [3]) takes place only if $M(f(\xi), \xi, t) \neq 1$, that is, (due to (GV-2)) only if $f(\xi) \neq \xi$. The next proposition is a correct version of [3, Corollary 3.5].

Proposition 2.6. *Let $(X,M,*)$ be a fuzzy metric space and let $f : X \to X$ be a fuzzy ε-contractive mapping. Suppose that there is $\zeta \in X$ such that $M(\xi, f(\xi), t) > 1-\varepsilon$ for some $t > 0$ and $f^k(\xi) = \xi$ for some integer $k \geq 1$. Then $f(\xi) = \xi$.*

Proof. From $M(\xi, f(\xi), t) > 1-\varepsilon$, it follows that

$$M(f^l(\xi), f^{l+1}(\xi), t) \geq M(f(\xi), f^2(\xi), t) \tag{2.11}$$

for all $l \geq 1$. Thus,

$$M(\xi, f(\xi), t) = M(f^k(\xi), f^{k+1}(\xi), t) \geq M(f(\xi), f^2(\xi), t). \tag{2.12}$$

If we had $f(\xi) \neq \xi$, then due to (GV-2), $M(\xi, f(\xi), t) \neq 1$. As $M(\xi, f(\xi), t) > 1-\varepsilon$, from the definition of a ε-fuzzy contractive mapping, the strict inequality

$$M(f(\xi), f^2(\xi), t) > M(\xi, f(\xi), t) \tag{2.13}$$

would follow, and thus we would obtain

$$M(\xi, f(\xi), t) \geq M(f(\xi), f^2(\xi), t) > M(\xi, f(\xi), t). \tag{2.14}$$

This contradiction completes the proof. □

A sufficient condition for the existence of a fixed point for a fuzzy ε-contraction is given in the next theorem.

Theorem 2.7. *Let $(X,M,*)$ be a fuzzy metric space and let $f : X \to X$ be a fuzzy ε-contractive mapping. Suppose that for some $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ contains a convergent subsequence and let $\zeta \in X$ be its limit. If there exists $t_0 > 0$ such that $M(x, f(x), t_0) > 1-\varepsilon$ and $M(\zeta, f(\zeta), t_0) > 1-\varepsilon$, then ζ is a fixed point of f.*

Proof. Let $x_n = f^n(x)$ and let $(x_{n_k})_{k\in\mathbb{N}}$ be a convergent subsequence of (x_n). As the sequence $(f(x_{n_k}))_{k\in\mathbb{N}}$ converges to $f(\zeta)$ and the sequence $(f(f(x_{n_k})))_{k\in\mathbb{N}}$ converges to

$f(f(\zeta))$ (see Lemma 2.2), we have (see Lemma 1.4)

$$\begin{aligned} M(x_{n_k}, f(x_{n_k}), t) &\longrightarrow M(\zeta, f(\zeta), t) \quad \forall t > 0, \\ M(f(x_{n_k}), f^2(x_{n_k}), t) &\longrightarrow M(f(\zeta), f^2(\zeta), t) \quad \forall t > 0. \end{aligned} \tag{2.15}$$

Since $M(x, f(x), t_0) > 1 - \varepsilon$, the sequence $(z_n)_{n\in\mathbb{N}}$, $z_n := M(x_n, f(x_n), t_0)$ is a nondecreasing sequence of numbers in $[0,1]$, therefore it is convergent. As its subsequence $(M(x_{n_k}, f(x_{n_k}), t_0))$ converges to $M(\zeta, f(\zeta), t_0)$, it follows that z_n converges to $M(\zeta, f(\zeta), t_0)$. Also,

$$\lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} M(f(x_n), f^2(x_n), t_0) = M(f(\zeta), f^2(\zeta), t_0), \tag{2.16}$$

therefore the equality $M(\zeta, f(\zeta), t_0) = M(f(\zeta), f^2(\zeta), t_0)$ holds.

Suppose $\zeta \neq f(\zeta)$. Then, due to (GV-2), $M(\zeta, f(\zeta), t_0)$ is not 1, hence $1 - \varepsilon < M(\zeta, f(\zeta), t_0) < 1$. This implies that $M(f(\zeta), f^2(\zeta), t_0) > M(\zeta, f(\zeta), t_0)$, contradicting the above equality. Therefore, ζ is a fixed point of f. □

Example 2.8. Let $X = (0,\infty)$, $M(x,y,t) = \min\{x,y\}/\max\{x,y\}$ for all $t > 0$ and $a * b = ab$. Then (see [6]), $(X, M, *)$ is a fuzzy metric space. Since $\sqrt{t} > t$ for all $t \in (0,1)$, the mapping $f : X \to X$, $f(x) = \sqrt{x}$, is fuzzy ε-contractive for every $\varepsilon \in (0,1)$ and the sequence $(f^n(1))_{n\in\mathbb{N}}$ is convergent to 1, the fixed point of f. Note that the condition $M(1, f(1), t) > 1 - \varepsilon$ is not satisfied by the mapping in Example 2.4.

Remark 2.9. Theorem 2.3 is Theorem 2.4 in our archived manuscript 35106, submitted in the 4th of August 2005 to FPTA. Recently, Ćirić et al. [7] solved a similar question of Razani for mappings in intuitionistic fuzzy metric spaces.

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