

Research Article

Strong Convergence Theorems for a Finite Family of Nonexpansive Mappings

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We modified the classic Mann iterative process to have strong convergence theorem for a finite family of nonexpansive mappings in the framework of Hilbert spaces. Our results improve and extend the results announced by many others.

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1. Introduction and preliminaries

Let H be a real Hilbert space, C a nonempty closed convex subset of H , and $T : C \rightarrow C$ a mapping. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is called a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T , that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a self-mapping $f : C \rightarrow C$ is a contraction on C , if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in C$. We use Π_C to denote the collection of all contractions on C , that is, $\Pi_C = \{f \mid f : C \rightarrow C \text{ a contraction}\}$. An operator A is strongly positive if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2 \quad \forall x \in H. \quad (1.1)$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems (see, e.g., [1, 2] and the references therein). A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H :

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \quad (1.2)$$

2 Fixed Point Theory and Applications

where C is the fixed point set of a nonexpansive mapping S , and b is a given point in H . In [2], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n b, \quad n \geq 0, \quad (1.3)$$

converges strongly to the unique solution of the minimization problem (1.2) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Recently, Marino and Xu [1] introduced a new iterative scheme by the viscosity approximation method

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \quad (1.4)$$

They proved that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the variational inequality $\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0$, $x \in C$, which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x), \quad (1.5)$$

where C is the fixed point set of a nonexpansive mapping S , and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$.)

Mann's iteration process [3] is often used to approximate a fixed point of a nonexpansive mapping, which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Tx_n, \quad n \geq 0, \quad (1.6)$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^\infty$ is in the interval $[0, 1]$. But Mann's iteration process has only weak convergence, in general. For example, Reich [4] shows that if E is a uniformly convex Banach space and has a Frechet differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\sum \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by process (1.6) converges weakly to a point in $F(T)$. Therefore, many authors try to modify Mann's iteration process to have strong convergence.

Kim and Xu [5] introduced the following iteration process:

$$x_0 = x \in C \text{ arbitrarily chosen,}$$

$$y_n = \beta_n x_n + (1 - \beta_n) Tx_n, \quad (1.7)$$

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n.$$

They proved that the sequence $\{x_n\}$ defined by (1.7) converges strongly to a fixed point of T provided the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Recently, Yao et al. [6] also modified Mann's iterative scheme (1.7) and got a strong convergence theorem. They improved the results of Kim and Xu [5] to some extent.

In this paper, we study the mapping W_n defined by

$$\begin{aligned}
 U_{n0} &= I, \\
 U_{n1} &= \gamma_{n1} T_1 U_{n0} + (1 - \gamma_{n1}) I, \\
 U_{n2} &= \gamma_{n2} T_2 U_{n1} + (1 - \gamma_{n2}) I, \\
 &\vdots \\
 U_{n,N-1} &= \gamma_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \gamma_{n,N-1}) I, \\
 W_n &= U_{nN} = \gamma_{nN} T_N U_{n,N-1} + (1 - \gamma_{nN}) I,
 \end{aligned} \tag{1.8}$$

where $\{\gamma_{n1}\}, \{\gamma_{n2}\}, \dots, \{\gamma_{nN}\} \in (0, 1]$. Such a mapping W_n is called the W_n -mapping generated by T_1, T_2, \dots, T_N and $\{\gamma_{n1}\}, \{\gamma_{n2}\}, \dots, \{\gamma_{nN}\}$. Nonexpansivity of each T_i ensures the nonexpansivity of W_n . It follows from [7, Lemma 3.1] that $F(W_n) = \cap_{i=1}^N F(T_i)$.

Very recently, Xu [2] studied the following iterative scheme:

$$x_{n+1} = \alpha_n u + (I - \alpha_n A) T_{n+1} x_n, \quad n \geq 0, \tag{1.9}$$

where A is a linear bounded operator, $T_n = T_{n \bmod N}$ and the mod function takes values in $\{1, 2, \dots, N\}$. He proved that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the minimization problem (1.2) provided T_n satisfy

$$F(T_N \cdots T_2 T_1) = F(T_1 T_N \cdots T_3 T_2) = \cdots = F(T_{N-1} \cdots T_1 T_n), \tag{1.10}$$

and $\{\alpha_n\} \in (0, 1)$ satisfying the following control conditions:

- (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+N}| < \infty$ or $\lim_{n \rightarrow \infty} \alpha_n / \alpha_{n+N} = 0$.

Remark 1.1. There are many nonexpansive mappings, which do not satisfy (1.10). For example, if $X = [0, 1]$ and T_1, T_2 are defined by $T_1 x = x/2 + 1/2$ and $T_2 x = x/4$, then $F(T_1 T_2) = \{4/7\}$, whereas $F(T_2 T_1) = \{1/7\}$.

In this paper, motivated by Kim and Xu [5], Marino and Xu [1], Xu [2], and Yao et al. [6], we introduce a composite iteration scheme as follows:

$$\begin{aligned}
 x_0 &= x \in C \text{ arbitrarily chosen,} \\
 y_n &= \beta_n x_n + (1 - \beta_n) W_n x_n, \\
 x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n,
 \end{aligned} \tag{1.11}$$

where $f \in \Pi_C$ is a contraction, and A is a linear bounded operator. We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that $\{x_n\}$ defined by (1.11) converges to a common fixed point of the finite family of nonexpansive mappings, which solves some variation inequality and is also the optimality condition for the minimization problem (1.5).

4 Fixed Point Theory and Applications

Now, we consider some special cases of iterative scheme (1.11). When $A = I$ and $\gamma = 1$ in (1.11), we have that (1.11) collapses to

$$\begin{aligned} x_0 &= x \in C \text{ arbitrarily chosen,} \\ y_n &= \beta_n x_n + (1 - \beta_n) W_n x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) y_n. \end{aligned} \tag{1.12}$$

When $A = I$ and $\gamma = 1$ in (1.11), $N = 1$ and $\{\gamma_{n1}\} = 1$ in (1.8), we have that (1.11) collapses to the iterative scheme which was considered by Yao et al. [6]. When $A = I$ and $\gamma = 1$ in (1.11), $N = 1$ and $\{\gamma_{n1}\} = 1$ in (1.8), and $f(y) = u \in C$ for all $y \in C$ in (1.11), we have that (1.11) reduces to (1.7), which was considered by Kim and Xu [5].

In order to prove our main results, we need the following lemmas.

LEMMA 1.2. *In a Hilbert space H , there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, (x + y) \rangle, \quad x, y \in H. \tag{1.13}$$

LEMMA 1.3 (Suzuki [8]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let β_n be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n \rightarrow \infty} (|y_{n+1} - y_n| - \|x_{n+1} - x_n\|) \leq 0. \tag{1.14}$$

Then $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

LEMMA 1.4 (Xu [2]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \tag{1.15}$$

where γ_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} \alpha_n = 0$.

LEMMA 1.5 (Marino and Xu [1]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$.*

LEMMA 1.6 (Marino and Xu [1]). *Let H be a Hilbert space. Let A be a strongly positive linear bounded selfadjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $T : C \rightarrow C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto t\gamma f(x) + (1 - tA)x$. Then $\{x_t\}$ converges strongly as $t \rightarrow 0$ to a fixed point \bar{x} of T , which solves the variational inequality*

$$\langle (A - \gamma f)\bar{x}, z - \bar{x} \rangle \leq 0, \quad z \in F(T). \tag{1.16}$$

2. Main results

THEOREM 2.1. *Let C be a closed convex subset of a Hilbert space H . Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ and W_n is defined by (1.8). Assume that*

$0 < \gamma < \bar{\gamma}/\alpha$ and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Given a map $f \in \Pi_C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ in $(0, 1)$, the following conditions are satisfied:

- (C1) $\sum_{n=0}^\infty \alpha_n = \infty$;
- (C2) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C3) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (C4) $\lim_{n \rightarrow \infty} |\gamma_{n,i} - \gamma_{n-1,i}| = 0$, for all $i = 1, 2, \dots, N$.

Let $\{x_n\}_{n=1}^\infty$ be the composite process defined by (1.11). Then $\{x_n\}_{n=1}^\infty$ converges strongly to $q \in F$, which also solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad p \in F. \quad (2.1)$$

Proof. First, we observe that $\{x_n\}_{n=0}^\infty$ is bounded. Indeed, take a point $p \in F$ and notice that

$$\|y_n - p\| \leq \beta_n \|x_n - p\| + (1 - \beta_n) \|W_n x_n - p\| \leq \|x_n - p\|. \quad (2.2)$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - Ap) + (I - \alpha_n A)(y_n - p)\| \\ &\leq [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)]\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\|. \end{aligned} \quad (2.3)$$

By simple inductions, we have $\|x_n - p\| \leq \max\{\|x_0 - p\|, \|Ap - \gamma f(p)\|/(\bar{\gamma} - \gamma\alpha)\}$, which gives that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$.

Next, we claim that

$$\|x_{n+1} - x_n\| \longrightarrow 0. \quad (2.4)$$

Put $l_n = (x_{n+1} - \beta_n x_n)/(1 - \beta_n)$. Now, we compute $l_{n+1} - l_n$, that is,

$$x_{n+1} = (1 - \beta_n)l_n + \beta_n x_n, \quad n \geq 0. \quad (2.5)$$

Observing that

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}A)y_{n+1} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} \\ &\quad - \frac{\alpha_n\gamma f(x_n) + (I - \alpha_nA)y_n - \beta_nx_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - Ay_{n+1})}{1 - \beta_{n+1}} - \frac{\alpha_n(\gamma f(x_n) - Ay_n)}{1 - \beta_n} \\ &\quad + W_{n+1}x_{n+1} - W_n x_n, \end{aligned} \quad (2.6)$$

6 Fixed Point Theory and Applications

we have

$$\begin{aligned} \|l_{n+1} - l_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|Ay_n - \gamma f(x_n)\| \\ &\quad + \|x_{n+1} - x_n\| + \|W_{n+1}x_n - W_nx_n\|. \end{aligned} \quad (2.7)$$

Next, we will use M to denote the possible different constants appearing in the following reasoning. It follows from the definition of W_n that

$$\begin{aligned} &\|W_{n+1}x_n - W_nx_n\| \\ &= \|\gamma_{n+1,N}T_NU_{n+1,N-1}x_n + (1 - \gamma_{n+1,N})x_n - \gamma_{n,N}T_NU_{n,N-1}x_n - (1 - \gamma_{n,N})x_n\| \\ &\leq |\gamma_{n+1,N} - \gamma_{n,N}| \|x_n\| + \|\gamma_{n+1,N}T_NU_{n+1,N-1}x_n - \gamma_{n,N}T_NU_{n,N-1}x_n\| \\ &\leq |\gamma_{n+1,N} - \gamma_{n,N}| \|x_n\| + \|\gamma_{n+1,N}(T_NU_{n+1,N-1}x_n - T_NU_{n,N-1}x_n)\| \\ &\quad + |\gamma_{n+1,N} - \gamma_{n,N}| \|T_NU_{n,N-1}x_n\| \\ &\leq 2M |\gamma_{n+1,N} - \gamma_{n,N}| + \gamma_{n+1,N} \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\|. \end{aligned} \quad (2.8)$$

Next, we consider

$$\begin{aligned} &\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\ &= \|\gamma_{n+1,N-1}T_{N-1}U_{n+1,N-2}x_n + (1 - \gamma_{n+1,N-1})x_n \\ &\quad - \gamma_{n,N-1}T_{N-1}U_{n,N-2}x_n - (1 - \gamma_{n,N-1})x_n\| \\ &\leq |\gamma_{n+1,N-1} - \gamma_{n,N-1}| \|x_n\| + \gamma_{n+1,N-1} \|T_{N-1}U_{n+1,N-2}y_n - T_{N-1}U_{n,N-2}x_n\| \\ &\quad + |\gamma_{n+1,N-1} - \gamma_{n,N-1}| \|T_{N-1}U_{n,N-2}x_n\| \\ &\leq 2M |\gamma_{n+1,N-1} - \gamma_{n,N-1}| + \|U_{n+1,N-2}x_n - U_{n,N-2}x_n\|. \end{aligned} \quad (2.9)$$

It follows that

$$\begin{aligned} &\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\ &\leq 2M |\gamma_{n+1,N-1} - \gamma_{n,N-1}| + 2M |\gamma_{n+1,N-2} - \gamma_{n,N-2}| + \|U_{n+1,N-3}x_n - U_{n,N-3}x_n\| \\ &\leq 2M \sum_{i=2}^{N-1} |\gamma_{n+1,i} - \gamma_{n,i}| + \|U_{n+1,1}x_n - U_{n,1}x_n\| \\ &\leq 2M \sum_{i=1}^{N-1} |\gamma_{n+1,i} - \gamma_{n,i}|. \end{aligned} \quad (2.10)$$

Substituting (2.10) into (2.8) yields that

$$\begin{aligned} \|W_{n+1}x_n - W_nx_n\| &\leq 2M |\gamma_{n+1,N} - \gamma_{n,N}| + 2\gamma_{n+1,N}M \sum_{i=1}^{N-1} |\gamma_{n+1,i} - \gamma_{n,i}| \\ &\leq 2M \sum_{i=1}^N |\gamma_{n+1,i} - \gamma_{n,i}|. \end{aligned} \quad (2.11)$$

It follows that

$$\begin{aligned} &\|l_{n+1} - l_n\| - \|x_n - x_{n-1}\| \\ &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|Ay_n - \gamma f(x_n)\| + 2M \sum_{i=1}^N |\gamma_{n+1,i} - \gamma_{n,i}|. \end{aligned} \quad (2.12)$$

Observing conditions (C1), (C4) and takeing the limits as $n \rightarrow \infty$, we get

$$\limsup_{n \rightarrow \infty} (\|l_{n+1} - l_n\| - \|x_n - x_{n-1}\|) \leq 0. \quad (2.13)$$

We can obtain $\lim_{n \rightarrow \infty} \|l_n - x_n\| = 0$ easily by Lemma 1.3. Since $x_{n+1} - x_n = (1 - \beta_n)(l_n - x_n)$, we have that (2.4) holds. Observing that $x_{n+1} - y_n = \alpha_n(\gamma f(x_n) - Ay_n)$, we can easily get $\lim_{n \rightarrow \infty} \|y_n - x_{n+1}\| = 0$, which implies that

$$\|y_n - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|, \quad (2.14)$$

that is,

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \quad (2.15)$$

On the other hand, we have

$$\|W_nx_n - x_n\| \leq \|x_n - y_n\| + \|y_n - W_nx_n\| \leq \|x_n - y_n\| + \beta_n \|x_n - W_nx_n\|, \quad (2.16)$$

which implies $(1 - \beta_n) \|W_nx_n - x_n\| \leq \|x_n - y_n\|$. From condition (C3) and (2.15), we obtain

$$\|W_nx_n - x_n\| \longrightarrow 0. \quad (2.17)$$

Next, we claim that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \leq 0, \quad (2.18)$$

where $q = \lim_{t \rightarrow 0} x_t$ with x_t being the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)W_nx$. Then, x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I - tA)W_nx_t$. Thus, we

8 Fixed Point Theory and Applications

have $\|x_t - x_n\| = \|(I - tA)(W_n x_t - x_n) + t(\gamma f(x_t) - Ax_n)\|$. It follows from Lemma 1.2 that

$$\begin{aligned}
\|x_t - x_n\|^2 &= \|(I - tA)(W_n x_t - x_n) + t(\gamma f(x_t) - Ax_n)\|^2 \\
&\leq (1 - \bar{\gamma}t)^2 \|W_n x_t - x_n\|^2 + 2t \langle \gamma f(x_t) - Ax_n, x_t - x_n \rangle \\
&\leq (1 - 2\bar{\gamma}t + (\bar{\gamma}t)^2) \|x_t - x_n\|^2 + f_n(t) \\
&\quad + 2t \langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle,
\end{aligned} \tag{2.19}$$

where

$$f_n(t) = (2\|x_t - x_n\| + \|x_n - W_n x_n\|) \|x_n - W_n x_n\| \rightarrow 0, \quad \text{as } n \rightarrow 0. \tag{2.20}$$

It follows that

$$\langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq \frac{\bar{\gamma}t}{2} \langle Ax_t - Ax_n, x_t - x_n \rangle + \frac{1}{2} f_n(t). \tag{2.21}$$

Let $n \rightarrow \infty$ in (2.21) and note that (2.20) yields

$$\limsup_{n \rightarrow \infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq \frac{t}{2} M, \tag{2.22}$$

where $M > 0$ is a constant such that $M \geq \bar{\gamma} \langle Ax_t - Ax_n, x_t - x_n \rangle$ for all $t \in (0, 1)$ and $n \geq 1$. Taking $t \rightarrow 0$ from (2.22), we have $\limsup_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq 0$. Since H is a Hilbert space, the order of $\limsup_{t \rightarrow 0}$ and $\limsup_{n \rightarrow \infty}$ is exchangeable, and hence (2.18) holds. Now from Lemma 1.2, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(I - \alpha_n A)(y_n - q) + \alpha_n(\gamma f(x_n) - Aq)\|^2 \\
&\leq \|(I - \alpha_n A)(y_n - q)\|^2 + 2\alpha_n \langle \gamma f(x_n) - Aq, x_{n+1} - q \rangle \\
&\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + \alpha_n \gamma \alpha (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) \\
&\quad + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle,
\end{aligned} \tag{2.23}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \frac{(1 - \alpha_n \bar{\gamma})^2 + \alpha_n \gamma \alpha}{1 - \alpha_n \gamma \alpha} \|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n \gamma \alpha} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
&\leq \left[1 - \frac{2\alpha_n(\bar{\gamma} - \alpha\gamma)}{1 - \alpha_n \gamma \alpha} \right] \|x_n - q\|^2 \\
&\quad + \frac{2\alpha_n(\bar{\gamma} - \alpha\gamma)}{1 - \alpha_n \gamma \alpha} \left[\frac{1}{\bar{\gamma} - \alpha\gamma} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle + \frac{\alpha_n \bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)} M \right].
\end{aligned} \tag{2.24}$$

Put $l_n = 2\alpha_n(\bar{y} - \alpha_n\gamma)/(1 - \alpha_n\alpha\gamma)$ and $t_n = 1/(\bar{y} - \alpha\gamma)\langle\gamma f(q) - Aq, x_{n+1} - q\rangle + \alpha_n\bar{y}^2/(2(\bar{y} - \alpha\gamma))M$, that is,

$$\|x_{n+1} - q\|^2 \leq (1 - l_n)\|x_n - q\|^2 + l_n t_n. \quad (2.25)$$

It follows from conditions (C1), (C2), and (2.22) that $\lim_{n \rightarrow \infty} l_n = 0$, $\sum_{n=1}^{\infty} l_n = \infty$, and $\limsup_{n \rightarrow \infty} t_n \leq 0$. Apply Lemma 1.4 to (2.25) to conclude that $x_n \rightarrow q$. This completes the proof. \square

Remark 2.2. Our results relax the condition of Kim and Xu [1] imposed on control sequences and also improve the results of Yao et al. [6] from one single mapping to a finite family of nonexpansive mappings, respectively.

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