

STRONG CONVERGENCE TO COMMON FIXED POINTS OF NONEXPANSIVE MAPPINGS WITHOUT COMMUTATIVITY ASSUMPTION

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We introduce an iteration scheme for nonexpansive mappings in a Hilbert space and prove that the iteration converges strongly to common fixed points of the mappings without commutativity assumption.

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1. Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H . A mapping T of C into itself is said to be nonexpansive

$$\|Tx - Ty\| \leq \|x - y\|, \quad (1.1)$$

for each $x, y \in C$. For a mapping T of C into itself, we denote by $F(T)$ the set of fixed points of T . We also denote by \mathbb{N} and \mathbb{R}^+ the set of positive integers and nonnegative real numbers, respectively.

Baillon [1] proved the first nonlinear ergodic theorem. Let C be a nonempty bounded convex closed subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, for an arbitrary $x \in C$, $\{(1/(n+1)) \sum_{i=0}^n T^i x\}_{n=0}^\infty$ converges weakly to a fixed point of T . Wittmann [9] studied the following iteration scheme, which has first been considered by Halpern [3]:

$$\begin{aligned} x_0 &= x \in C, \\ x_{n+1} &= \alpha_{n+1}x + (1 - \alpha_{n+1})Tx_n, \quad n \geq 0, \end{aligned} \quad (1.2)$$

where a sequence $\{\alpha_n\}$ in $[0,1]$ is chosen so that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, and $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$; see also Reich [7]. Wittmann proved that for any $x \in C$, the sequence

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$\{x_n\}$ defined by (1.2) converges strongly to the unique element $Px \in F(T)$, where P is the metric projection of H onto $F(T)$.

Recall that two mappings S and T of H into itself are called commutative if

$$ST = TS, \quad (1.3)$$

for all $x, y \in H$.

Recently, Shimizu and Takahashi [8] have first considered an iteration scheme for two commutative nonexpansive mappings S and T and proved that the iterations converge strongly to a common fixed point of S and T . They obtained the following result.

THEOREM 1.1 (see [8]). *Let H be a Hilbert space, and let C be a nonempty closed convex subset of H . Let S and T be nonexpansive mappings of C into itself such that $ST = TS$ and $F(S) \cap F(T)$ is nonempty. Suppose that $\{\alpha_n\}_{n=0}^{\infty} \subseteq [0, 1]$ satisfies*

(i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, and

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, for an arbitrary $x \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by $x_0 = x$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n, \quad n \geq 0, \quad (1.4)$$

converges strongly to a common fixed point Px of S and T , where P is the metric projection of H onto $F(S) \cap F(T)$.

Remark 1.2. At this point, we note that the authors have imposed the commutativity on the mappings S and T . But there are many mappings, that do not satisfy $ST = TS$. For example, if $X = [-1/2, 1/2]$, and S and T of X into itself are defined by

$$S = x^2, \quad T = \sin x, \quad (1.5)$$

then $ST = \sin^2 x$, whereas $TS = \sin x^2$.

In this paper, we deal with the strong convergence to common fixed points of two nonexpansive mappings in a Hilbert space. We consider an iteration scheme for nonexpansive mappings without commutativity assumption and prove that the iterations converge strongly to a common fixed point of the mappings T_i , $i = 1, 2$.

2. Preliminaries

Let C be a closed convex subset of a Hilbert space H and let S and T be nonexpansive mappings of C into itself. Then we consider the iteration scheme

$$\begin{aligned} x_0 &= x \in C, \\ x_{n+1} &= \alpha_n x + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j y_n, \\ y_n &= \beta_n x_n + (1 - \beta_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} T^i S^j x_n, \quad n \geq 0, \end{aligned} \quad (2.1)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$. We know that a Hilbert space H satisfies Opial's condition [6], that is, if a sequence $\{x_n\}$ in H converges weakly to an element y of H and $y \neq z$, then

$$\liminf_{n \rightarrow \infty} \|x_n - y\| < \liminf_{n \rightarrow \infty} \|x_n - z\|. \quad (2.2)$$

In what follows, we will use P_C to denote the metric projection from H onto C ; that is, for each $x \in H$, P_C is the only point in C with the property

$$\|x - P_C x\| = \min_{u \in C} \|u - x\|. \quad (2.3)$$

It is known that P_C is nonexpansive and characterized by the following inequality: given $x \in H$ and $v \in H$, then $v = P_C x$ if and only if

$$\langle x - v, v - y \rangle \geq 0, \quad y \in C. \quad (2.4)$$

Now, we introduce several lemmas for our main result in this paper. The first lemma can be found in [4, 5, 10].

LEMMA 2.1. *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \quad (2.5)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (2) $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

LEMMA 2.2. *Let C be a nonempty bounded closed convex subset of a Hilbert H , and let S, T be nonexpansive mappings of C into itself. For $x \in C$ and $n \in \mathbb{N} \cup \{0\}$, put*

$$\begin{aligned} G_n(x) &= \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x, \\ \overline{G}_n(x) &= \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} T^i S^j x. \end{aligned} \quad (2.6)$$

Then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{x \in C} \|G_n(x) - SG_n(x)\| &= 0, \\ \limsup_{n \rightarrow \infty} \sup_{x \in C} \|\overline{G}_n(x) - T\overline{G}_n(x)\| &= 0. \end{aligned} \quad (2.7)$$

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Proof. We first prove $\lim_{n \rightarrow \infty} \sup_{x \in C} \|G_n(x) - SG_n(x)\| = 0$.

By an idea in [2], for $\{x_{i,j}\}_{i,j=0}^\infty$, $\{\bar{x}_{i,j}\}_{i,j=0}^\infty \subseteq C$ and $z_n = (1/l_n) \sum_{k=0}^n \sum_{i+j=k} x_{i,j}$, $\bar{z}_n = (1/l_n) \sum_{k=0}^n \sum_{i+j=k} \bar{x}_{i,j} \in C$, with $l_n = (n+1)(n+2)/2$, we have

$$\|z_n - v\|^2 = \frac{1}{l_n} \sum_{k=0}^n \sum_{i+j=k} \|x_{i,j} - v\|^2 - \frac{1}{l_n} \sum_{k=0}^n \sum_{i+j=k} \|x_{i,j} - z_n\|^2 \quad (2.8)$$

for each $v \in H$. For $x \in C$, put $x_{i,j} = S^i T^j x$, $\bar{x}_{i,j} = T^i S^j x$ and $v = Sz_n, \bar{v} = T\bar{z}_n$. Then, we have

$$\begin{aligned} \|G_n(x) - SG_n(x)\|^2 &= \frac{1}{l_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - Sz_n\|^2 - \frac{1}{l_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - z_n\|^2 \\ &= \frac{1}{l_n} \sum_{k=0}^n \|T^k x - Sz_n\|^2 + \frac{1}{l_n} \sum_{k=1}^n \sum_{i+j=k, i \geq 1} \|S^i T^j x - Sz_n\|^2 \\ &\quad - \frac{1}{l_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - z_n\|^2 \\ &\leq \frac{1}{l_n} \sum_{k=0}^n \|T^k x - Sz_n\|^2 + \frac{1}{l_n} \sum_{k=1}^n \sum_{i+j=k, i \geq 1} \|S^{i-1} T^j x - z_n\|^2 \\ &\quad - \frac{1}{l_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - z_n\|^2 \quad (2.9) \\ &= \frac{1}{l_n} \sum_{k=0}^n \|T^k x - Sz_n\|^2 + \frac{1}{l_n} \sum_{k=0}^{n-1} \sum_{i+j=k} \|S^i T^j x - z_n\|^2 \\ &\quad - \frac{1}{l_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - z_n\|^2 \\ &= \frac{1}{l_n} \sum_{k=0}^n \|T^k x - Sz_n\|^2 - \frac{1}{l_n} \sum_{i+j=n} \|S^i T^j x - z_n\|^2 \\ &\leq \frac{1}{l_n} \sum_{k=0}^n \|T^k x - Sz_n\|^2 \leq \frac{2}{n+2} \{ \text{diam}(C) \}^2, \end{aligned}$$

where $\text{diam}(C)$ is the diameter of C . So, we have, for each $n \in \mathbb{N} \cup \{0\}$,

$$\sup_{x \in C} \|G_n(x) - SG_n(x)\|^2 \leq \frac{2}{n+2} \{ \text{diam}(C) \}^2, \quad (2.10)$$

and hence

$$\lim_{n \rightarrow \infty} \sup_{x \in C} \|G_n(x) - SG_n(x)\| = 0. \quad (2.11)$$

Similarly, we have

$$\limsup_{n \rightarrow \infty} \|\bar{G}_n(x) - T\bar{G}_n(x)\| = 0. \quad (2.12)$$

□

3. Convergence theorem

Now we can prove a strong convergence theorem in a Hilbert space.

THEOREM 3.1. *Let H be a Hilbert space, and let C be a nonempty closed convex subset of H . Let S and T be nonexpansive mappings of C into itself such that $F(S) \cap F(T)$ is nonempty. Suppose that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are two sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, and

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For an arbitrary $x \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ is generated by $x_0 = x$ and

$$\begin{aligned} x_{n+1} &= \alpha_n x + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j y_n, \\ y_n &= \beta_n x_n + (1 - \beta_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} T^i S^j x_n, \quad n \geq 0. \end{aligned} \quad (3.1)$$

Let

$$z_n = \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j y_n, \quad \bar{z}_n = \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} T^i S^j x_n, \quad (3.2)$$

for each $n \in \mathbb{N} \cup \{0\}$. If there exist subsequences $\{z_{n_i}\}_{i=0}^{\infty}$ of $\{z_n\}_{n=0}^{\infty}$ and $\{\bar{z}_{n_j}\}_{j=0}^{\infty}$ of $\{\bar{z}_n\}_{n=0}^{\infty}$, respectively, which converge weakly to some common point z in some bounded subset D of C , then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (3.1) converges strongly to $P_{F(S) \cap F(T)}x$.

Proof. Let $x \in C$ and $w \in F(S) \cap F(T)$. Putting $r = \|x - w\|$, then the set

$$D = \{y \in H : \|y - w\| \leq r\} \cap C \quad (3.3)$$

is a nonempty bounded closed convex subset of C which is S - and T -invariant and contains $x_0 = x$. So we may assume, without loss of generality, that S and T are the mappings of D into itself. Since P is the metric projection of H onto $F(S) \cap F(T)$, we have

$$\langle y - Px, x - Px \rangle \leq 0 \quad (3.4)$$

for each $y \in F(S) \cap F(T)$.

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From (3.4), we have

$$\limsup_{n \rightarrow \infty} \langle z_n - Px, x - Px \rangle \leq 0, \quad \limsup_{n \rightarrow \infty} \langle \bar{z}_n - Px, x - Px \rangle \leq 0. \quad (3.5)$$

In fact, assume that, there exist two positive real numbers r_0 and r_1 such that

$$\limsup_{n \rightarrow \infty} \langle z_n - Px, x - Px \rangle > r_0, \quad \limsup_{n \rightarrow \infty} \langle \bar{z}_n - Px, x - Px \rangle > r_1. \quad (3.6)$$

Since $\{z_n\}_{n=0}^\infty$ and $\{\bar{z}_n\}_{n=0}^\infty \subseteq D$ are bounded, from (3.6), there exist subsequences $\{z_{n_i}\}_{i=0}^\infty$ of $\{z_n\}_{n=0}^\infty$ and $\{\bar{z}_{n_j}\}_{j=0}^\infty$ of $\{\bar{z}_n\}_{n=0}^\infty$, respectively, such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle z_n - Px, x - Px \rangle &= \lim_{i \rightarrow \infty} \langle z_{n_i} - Px, x - Px \rangle > r_0, \\ \limsup_{n \rightarrow \infty} \langle \bar{z}_n - Px, x - Px \rangle &= \lim_{j \rightarrow \infty} \langle \bar{z}_{n_j} - Px, x - Px \rangle > r_1. \end{aligned} \quad (3.7)$$

By the assumption, we know that $\{z_{n_i}\}_{i=0}^\infty$ and $\{\bar{z}_{n_j}\}_{j=0}^\infty$ converge weakly to some common point $z \in D$. Thus from Lemma 2.2 and Opial's condition, we have $z \in F(S) \cap F(T)$. In fact, if $z \neq Sz$, we have

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|z_{n_i} - z\| &< \liminf_{i \rightarrow \infty} \|z_{n_i} - Sz\| \\ &\leq \liminf_{i \rightarrow \infty} (\|z_{n_i} - Sz_{n_i}\| + \|Sz_{n_i} - Sz\|) \\ &\leq \liminf_{i \rightarrow \infty} \|z_{n_i} - z\|. \end{aligned} \quad (3.8)$$

This is a contradiction. Therefore, we have $z = Sz$.

Similarly, we have $z = Tz$. So, we have

$$\langle z - Px, x - Px \rangle \leq 0. \quad (3.9)$$

On the other hand, since $\{z_{n_i}\}$ converges weakly to z , we obtain

$$\langle z - Px, x - Px \rangle \geq r_0. \quad (3.10)$$

This is a contradiction. Hence, we have

$$\limsup_{n \rightarrow \infty} \langle z_n - Px, x - Px \rangle \leq 0, \quad \limsup_{n \rightarrow \infty} \langle \bar{z}_n - Px, x - Px \rangle \leq 0. \quad (3.11)$$

Since

$$\begin{aligned}
\|\bar{z}_n - Px\| &\leq \left\{ \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} \|T^i S^j x_n - Px\| \right\}^2 \\
&\leq \left\{ \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} \|x_n - Px\| \right\}^2 = \|x_n - Px\|^2, \\
\|y_n - Px\|^2 &= \|\beta_n x_n + (1 - \beta_n) \bar{z}_n - Px\|^2 \\
&= \|\beta_n (x_n - Px) + (1 - \beta_n) (\bar{z}_n - Px)\|^2 \\
&= \beta_n^2 \|x_n - Px\|^2 + 2\beta_n (1 - \beta_n) (x_n - Px, \bar{z}_n - Px) + (1 - \beta_n)^2 \|\bar{z}_n - Px\|^2 \\
&\leq \beta_n^2 \|x_n - Px\|^2 + 2\beta_n (1 - \beta_n) \frac{\|x_n - Px\|^2 + \|\bar{z}_n - Px\|^2}{2} \\
&\quad + (1 - \beta_n)^2 \|\bar{z}_n - Px\|^2 \leq \|x_n - Px\|^2.
\end{aligned} \tag{3.12}$$

Then, we have

$$\begin{aligned}
\|x_{n+1} - Px\|^2 &= \|\alpha_n x + (1 - \alpha_n) z_n - Px\|^2 \\
&= \alpha_n^2 \|x - Px\|^2 + (1 - \alpha_n)^2 \|z_n - Px\|^2 + 2\alpha_n (1 - \alpha_n) \langle z_n - Px, x - Px \rangle \\
&\leq (1 - \alpha_n)^2 \left\{ \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j y_n - Px\| \right\}^2 \\
&\quad + \alpha_n^2 \|x - Px\|^2 + 2\alpha_n (1 - \alpha_n) \langle z_n - Px, x - Px \rangle \\
&\leq (1 - \alpha_n)^2 \left\{ \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} \|y_n - Px\| \right\}^2 \\
&\quad + \alpha_n^2 \|x - Px\|^2 + 2\alpha_n (1 - \alpha_n) \langle z_n - Px, x - Px \rangle \\
&= (1 - \alpha_n)^2 \|y_n - Px\|^2 + \alpha_n^2 \|x - Px\|^2 + 2\alpha_n (1 - \alpha_n) \langle z_n - Px, x - Px \rangle \\
&\leq (1 - \alpha_n) \|x_n - Px\|^2 + \alpha_n \{ \alpha_n \|x - Px\|^2 + 2(1 - \alpha_n) \langle z_n - Px, x - Px \rangle \}.
\end{aligned} \tag{3.13}$$

Putting $a_n = \|x_n - Px\|^2$, from (3.13), we have

$$a_{n+1} \leq (1 - \alpha_n) a_n + \delta_n, \tag{3.14}$$

where $\delta_n = \alpha_n \{ \alpha_n \|x - Px\|^2 + 2(1 - \alpha_n) \langle z_n - Px, x - Px \rangle \}$.

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It is easily seen that

$$\limsup_{n \rightarrow \infty} \delta_n/\alpha_n = \limsup_{n \rightarrow \infty} \{\alpha_n \|x - Px\|^2 + 2(1 - \alpha_n) \langle z_n - Px, x - Px \rangle\} \leq 0. \quad (3.15)$$

Now applying Lemma 2.1 with (3.15) to (3.14) concludes that $\|x_n - Px\| \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof. \square

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