

# MULTIVALUED $p$ -LIENARD SYSTEMS

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Received 7 October 2003 and in revised form 9 March 2004

We examine  $p$ -Lienard systems driven by the vector  $p$ -Laplacian differential operator and having a multivalued nonlinearity. We consider Dirichlet systems. Using a fixed point principle for set-valued maps and a nonuniform nonresonance condition, we establish the existence of solutions.

## 1. Introduction

In this paper, we use fixed point theory to study the following multivalued  $p$ -Lienard system:

$$\begin{aligned} & (||x'(t)||^{p-2}x'(t))' + \frac{d}{dt} \nabla G(x(t)) + F(t, x(t), x'(t)) \ni 0 \quad \text{a.e. on } T = [0, b], \\ & x(0) = x(b) = 0, \quad 1 < p < \infty. \end{aligned} \tag{1.1}$$

In the last decade, there have been many papers dealing with second-order multivalued boundary value problems. We mention the works of Erbe and Krawcewicz [5, 6], Frigon [7, 8], Halidias and Papageorgiou [9], Kandilakis and Papageorgiou [11], Kyritsi et al. [12], Palmucci and Papalini [17], and Pruszko [19]. In all the above works, with the exception of Kyritsi et al. [12],  $p = 2$  (linear differential operator),  $G = 0$ , and  $g = 0$ . Moreover, in Frigon [7, 8] and Palmucci and Papalini [17], the inclusions are scalar (i.e.,  $N = 1$ ). Finally we should mention that recently single-valued  $p$ -Lienard systems were studied by Mawhin [14] and Manásevich and Mawhin [13].

In this work, for problem (1.1), we prove an existence theorem under conditions of nonuniform nonresonance with respect to the first weighted eigenvalue of the negative vector ordinary  $p$ -Laplacian with Dirichlet boundary conditions [15, 20]. Our approach is based on the multivalued version of the Leray-Schauder alternative principle due to Bader [1] (see Section 2).

## 2. Mathematical background

In this section, we recall some basic definitions and facts from multivalued analysis, the spectral properties of the negative vector  $p$ -Laplacian, and the multivalued fixed point principles mentioned in the introduction. For details, we refer to Denkowski et al. [3] and Hu and Papageorgiou [10] (for multivalued analysis), to Denkowski et al. [2] and Zhang [20] (for the spectral properties of the  $p$ -Laplacian), and to Bader [1] (for the multivalued fixed point principle; similar results can also be found in O'Regan and Precup [16] and Precup [18]).

Let  $(\Omega, \Sigma)$  be a measurable space and  $X$  a separable Banach space. We introduce the following notations:

$$\begin{aligned} P_{f(c)}(X) &= \{A \subseteq X : \text{nonempty, closed (and convex)}\}, \\ P_{(w)k(c)}(X) &= \{A \subseteq X : \text{nonempty, (weakly) compact (and convex)}\}. \end{aligned} \quad (2.1)$$

A multifunction  $F : \Omega \rightarrow P_f(X)$  is said to be measurable if, for all  $x \in X$ ,  $\omega \rightarrow d(x, F(\omega)) = \inf [\|x - y\| : y \in F(\omega)]$  is measurable. A multifunction  $F : \Omega \rightarrow 2^X \setminus \{\emptyset\}$  is said to be “graph measurable” if  $\text{Gr}F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$ , with  $B(X)$  being the Borel  $\sigma$ -field of  $X$ . For  $P_f(X)$ -valued multifunctions, measurability implies graph measurability and the converse is true if  $\Sigma$  is complete (i.e.,  $\Sigma = \hat{\Sigma}$  = the universal  $\sigma$ -field). Let  $\mu$  be a finite measure on  $(\Omega, \Sigma)$ ,  $1 \leq p \leq \infty$ , and  $F : \Omega \rightarrow 2^X \setminus \{\emptyset\}$ . We introduce the set  $S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega) \text{ } \mu\text{-a.e.}\}$ . This set may be empty. For a graph-measurable multifunction, it is nonempty if and only if  $\inf [\|y\| : y \in F(\omega)] \leq \varphi(\omega) \text{ } \mu\text{-a.e.}$  on  $\Omega$ , with  $\varphi \in L^p(\Omega)_+$ .

Let  $Y, Z$  be Hausdorff topological spaces. A multifunction  $G : Y \rightarrow 2^Z \setminus \{\emptyset\}$  is said to be “upper semicontinuous” (usc for short) if, for all  $C \subseteq Z$  closed,  $G^-(C) = \{y \in Y : G(y) \cap C \neq \emptyset\}$  is closed or equivalently for all  $U \subseteq Z$  open,  $G^+ \{y \in Y : G(y) \subseteq U\}$  is open. If  $Z$  is a regular space, then a  $P_f(Z)$ -valued multifunction which is usc has a closed graph. The converse is true if the multifunction  $G$  is locally compact (i.e., for every  $y \in Y$ , there exists a neighborhood  $U$  of  $y$  such that  $\overline{G(U)}$  is compact in  $Z$ ). A  $P_k(Z)$ -valued multifunction which is usc maps compact sets to compact sets.

Consider the following weighted nonlinear eigenvalue problem in  $\mathbb{R}^N$ :

$$\begin{aligned} -(\|x'(t)\|^{p-2} x'(t))' &= \lambda \theta(t) \|x(t)\|^{p-2} x(t) \quad \text{a.e. on } T = [0, b], \\ x(0) = x(b) &= 0, \quad 1 < p < \infty, \quad \theta \in L^\infty(T), \quad |\{\theta > 0\}|_1 > 0, \quad \lambda \in \mathbb{R}. \end{aligned} \quad (2.2)$$

Here by  $|\cdot|_1$  we denote the 1-dimensional Lebesgue measure. The real parameters  $\lambda$ , for which problem (2.3) has a nontrivial solution, are called eigenvalues of the negative vector  $p$ -Laplacian with Dirichlet boundary conditions denoted by  $(-\Delta_p, W_0^{1,p}(T, \mathbb{R}^N))$ , with weight  $\theta \in L^\infty(T)$ . The corresponding nontrivial solutions are known as eigenfunctions. We know that the eigenvalues of problem (2.3) are the same as those of the corresponding scalar problem [13]. Then from Denkowski et al. [2] and Zhang [20], we know that there exist two sequences  $\{\lambda_n(\theta)\}_{n \geq 1}$  and  $\{\lambda_{-n}(\theta)\}_{n \geq 1}$  such that  $\lambda_n(\theta) > 0$ ,  $\lambda_n(\theta) \rightarrow +\infty$  and  $\lambda_{-n}(\theta) < 0$ ,  $\lambda_{-n}(\theta) \rightarrow -\infty$  as  $n \rightarrow \infty$ . Moreover, if  $\theta(t) \geq 0$  a.e. on  $T$  with strict inequality on a set of positive Lebesgue measure, then we have only the positive

sequence  $\{\lambda_n(\theta)\}_{n \geq 1}$ . Also, for  $\lambda_1(\theta) > 0$ , we have the following variational characterization:

$$\lambda_1(\theta) = \inf \left[ \frac{\|x'\|_p^p}{\int_0^b \theta(t) \|x(t)\|^p dt} : x \in W_0^{1,p}(T, \mathbb{R}^N), x \neq 0 \right]. \quad (2.3)$$

The infimum is attained at the normalized principal eigenfunction  $u_1$  ( $\lambda_1(\theta) > 0$  is simple) and  $u_1(t) \neq 0$  a.e. on  $T$ . Also,  $\lambda_1(\theta)$  is strictly monotone with respect to  $\theta$ , namely, if  $\theta_1(t) \leq \theta_2(t)$  a.e. on  $T$  with strict inequality on a set of positive measure, then  $\lambda_1(\theta_2) < \lambda_1(\theta_1)$  (see (3.2)).

Finally we state the multivalued fixed point principle that we will use in the study of problem (1.1). So let  $Y, Z$  be two Banach spaces and  $C \subseteq Y, D \subseteq Z$  two nonempty closed and convex sets. We consider multifunctions  $G : C \rightarrow 2^C \setminus \{\emptyset\}$  which have a decomposition  $G = K \circ N$ , satisfying the following:  $K : D \rightarrow C$  is completely continuous, namely, if  $z_n \xrightarrow{w} z$  in  $D$ , then  $K(z_n) \rightarrow K(z)$  in  $C$  and  $N : C \rightarrow P_{wkc}(D)$  is usc from  $C$ , furnished with the strong topology into  $D$ , furnished with the weak topology.

**THEOREM 2.1.** If  $C, D$ , and  $G = K \circ N$  are as above,  $0 \in C$ , and  $G$  is compact (namely,  $G$  maps bounded subsets of  $C$  into relatively compact subsets of  $D$ ), then one of the following alternatives holds:

- (a)  $S = \{y \in C : y \in \mu G(y) \text{ for some } \mu \in (0, 1)\}$  is unbounded or
- (b)  $G$  has a fixed point, that is, there exists  $y \in C$  such that  $y \in G(y)$ .

*Remark 2.2.* Evidently this is a multivalued version of the classical Leray-Schauder alternative principle [2, page 206]. In contrast to previous multivalued extensions of the Leray-Schauder alternative principle [4, page 61], **Theorem 2.1** does not require  $G$  to have convex values, which is important when dealing with nonlinear problems such as (1.1).

### 3. Nonuniform nonresonance

In this section, we deal with problem (1.1) using a condition of nonuniform nonresonance with respect to the first eigenvalue  $\lambda_1(\theta) > 0$ . Our hypotheses on the multivalued nonlinearity  $F(t, x, y)$  are as follows.

( $H(F)_1$ )  $F : T \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow P_{kc}(\mathbb{R}^N)$  is a multifunction such that

- (i) for all  $x, y \in \mathbb{R}^N$ ,  $t \rightarrow F(t, x, y)$  is graph measurable;
- (ii) for almost all  $t \in T$ ,  $(x, y) \rightarrow F(t, x, y)$  is usc;
- (iii) for every  $M > 0$ , there exists  $\gamma_M \in L^1(T)_+$  such that, for almost all  $t \in T$ , all  $\|x\|, \|y\| \leq M$ , and all  $u \in F(t, x, y)$ , we have  $\|u\| \leq \gamma_M(t)$ ;
- (iv) there exists  $\theta \in L^\infty(T)$ ,  $\theta(t) \geq 0$  a.e. on  $T$ , with strict inequality on a set of positive measure and

$$\limsup_{\|x\| \rightarrow +\infty} \frac{\sup \{(u, x)_{\mathbb{R}^N} : u \in F(t, x, y), y \in \mathbb{R}^N\}}{\|x\|^p} \leq \theta(t) \quad (3.1)$$

uniformly for almost all  $t \in T$  and  $\lambda_1(\theta) > 1$ .

*Remark 3.1.* Hypothesis  $(H(F)_1)$ (iv) is the nonuniform nonresonance condition. In the literature [15, 20], we encounter the condition  $\theta(t) \leq \lambda_1$  a.e. on  $T$  with strict inequality on a set of positive measure. Here  $\lambda_1 > 0$  is the principal eigenvalue corresponding to the unit weight  $\theta = 1$  (i.e.,  $\lambda_1 = \lambda_1(1)$ ). Then by virtue of the strict monotonicity property, we have  $\lambda_1(\lambda_1) = 1 < \lambda_1(\theta)$ , which is the condition assumed in hypothesis  $(H(F)_1)$ (iv).

$(H(G)_1)$   $G \in C^2(\mathbb{R}^N, \mathbb{R})$ .

Given  $h \in L^1(T, \mathbb{R}^N)$ , we consider the following Dirichlet problem:

$$\begin{aligned} -(\|x'(t)\|^{p-2}x'(t))' &= h(t) \quad \text{a.e. on } T = [0, b], \\ x(0) &= x(b) = 0. \end{aligned} \quad (3.2)$$

From Manásevich and Mawhin [13, Lemma 4.1], we know that problem (3.3) has a unique solution  $K(h) \in C_0^1(T, \mathbb{R}^N) = \{x \in C^1(T, \mathbb{R}^N) : x(0) = x(b) = 0\}$ . So we can define the solution map  $K : L^1(T, \mathbb{R}^N) \rightarrow C_0^1(T, \mathbb{R}^N)$ .

**PROPOSITION 3.2.**  $K : L^1(T, \mathbb{R}^N) \rightarrow C_0^1(T, \mathbb{R}^N)$  is completely continuous, that is, if  $h_n \xrightarrow{w} h$  in  $L^1(T, \mathbb{R}^N)$ , then  $K(h_n) \rightarrow K(h)$  in  $C_0^1(T, \mathbb{R}^N)$ .

*Proof.* Let  $h_n \xrightarrow{w} h$  in  $L^1(T, \mathbb{R}^N)$  and set  $x_n = K(h_n)$ ,  $n \geq 1$ . We have

$$-(\|x'_n(t)\|^{p-2}x'_n(t))' = h_n(t) \quad \text{a.e. on } T, \quad x_n(0) = x_n(b) = 0, \quad n \geq 1. \quad (3.3)$$

Taking the inner product with  $x_n(t)$ , integrating over  $T$ , and performing integration by parts, we obtain

$$\|x'_n\|_p^p \leq \|h_n\|_1 \|x_n\|_\infty \leq c_1 \|x'_n\|_p \quad \text{for some } c_1 > 0 \text{ and all } n \geq 1. \quad (3.4)$$

Here we have used Hölder and Poincare inequalities. It follows that

$$\begin{aligned} \{x'_n\}_{n \geq 1} &\subseteq L^p(T, \mathbb{R}^N) \text{ is bounded (since } p > 1\text{)} \\ \implies \{x_n\}_{n \geq 1} &\subseteq W_0^{1,p}(T, \mathbb{R}^N) \text{ is bounded (by the Poincare inequality).} \end{aligned} \quad (3.5)$$

So from (3.22) we infer that

$$\begin{aligned} \{\|x'_n\|^{p-2}x'_n\}_{n \geq 1} &\subseteq W^{1,q}(T, \mathbb{R}^N) \left( \frac{1}{p} + \frac{1}{q} = 1 \right) \text{ is bounded} \\ \implies \{\|x'_n\|^{p-2}x'_n\}_{n \geq 1} &\subseteq C(T, \mathbb{R}^N) \text{ is relatively compact} \end{aligned} \quad (3.6)$$

(recall that  $W^{1,q}(T, \mathbb{R}^N)$  is embedded compactly in  $C(T, \mathbb{R}^N)$ ). The map  $\varphi_p : \mathbb{R}^N \rightarrow \mathbb{R}^N$ , defined by  $\varphi_p(y) = \|y\|^{p-2}y$ ,  $y \in \mathbb{R}^N \setminus \{\emptyset\}$ , and  $\varphi_p(0) = 0$ , is a homeomorphism and so  $\varphi_p^{-1} : C(T, \mathbb{R}^N) \rightarrow C(T, \mathbb{R}^N)$ , defined by  $\varphi_p^{-1}(y)(\cdot) = \varphi_p^{-1}(y(\cdot))$ , is continuous and bounded. Thus it follows that

$$\begin{aligned} \{x'_n\}_{n \geq 1} &\subseteq C(T, \mathbb{R}^N) \text{ is relatively compact} \\ \implies \{x_n\}_{n \geq 1} &\subseteq C_0^1(T, \mathbb{R}^N) \text{ is relatively compact.} \end{aligned} \quad (3.7)$$

Therefore we may assume that  $x_n \rightarrow x$  in  $C_0^1(T, \mathbb{R}^N)$ . Also  $\{\|x_n'\|^{p-2}x_n'\}_{n \geq 1} \subseteq W^{1,q}(T, \mathbb{R}^N)$  is bounded and so we may assume that  $\|x_n'\|^{p-2}x_n' \xrightarrow{w} u$  in  $W^{1,q}(T, \mathbb{R}^N)$  and  $\|x_n'\|^{p-2}x_n' \rightarrow u$  in  $C(T, \mathbb{R}^N)$  (because  $W^{1,q}(T, \mathbb{R}^N)$  is embedded compactly in  $C(T, \mathbb{R}^N)$ ). It follows that  $u = \|x'\|^{p-2}x'$ . Hence if in (3.22) we pass to the limit as  $n \rightarrow \infty$ , we obtain

$$\begin{aligned} -(\|x'(t)\|^{p-2}x'(t))' &= h(t) \quad \text{a.e. on } T = [0, b], \quad x(0) = x(b) = 0 \\ \implies K(h) &= x. \end{aligned} \quad (3.8)$$

Since every subsequence of  $\{x_n\}_{n \geq 1}$  has a further subsequence which converges to  $x$  in  $C_0^1(T, \mathbb{R}^N)$ , we conclude that the original sequence converges too. This proves the complete continuity of  $K$ .  $\square$

Let  $N_F : C_0^1(T, \mathbb{R}^N) \rightarrow 2^{L^1(T, \mathbb{R}^N)}$  be the multivalued Nemitsky operator corresponding to  $F$ , that is,

$$N_F(x) = \{u \in L^1(T, \mathbb{R}^N) : u(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T\}. \quad (3.9)$$

Also let  $N : C_0^1(T, \mathbb{R}^N) \rightarrow 2^{L^1(T, \mathbb{R}^N)}$  be defined by

$$N(x) = \frac{d}{dx} \nabla G(x(\cdot)) + N_F(x). \quad (3.10)$$

This multifunction has the following structure.

**PROPOSITION 3.3.** *If hypotheses  $(H(F)_1)$  and  $(H(G)_1)$  hold, then  $N$  has values in  $P_{wkc}(L^1(T, \mathbb{R}^N))$  and it is usc from  $C_0^1(T, \mathbb{R}^N)$  with the norm topology into  $L^1(T, \mathbb{R}^N)$  with the weak topology.*

*Proof.* Clearly  $N$  has closed, convex values which are uniformly integrable (see hypothesis  $(H(F)_1)$ (iii)). Therefore for every  $x \in C_0^1(T, \mathbb{R}^N)$ ,  $N(x)$  is convex and  $w$ -compact in  $L^1(T, \mathbb{R}^N)$ . What is not immediately clear is that  $N(x) \neq \emptyset$ , since hypotheses  $(H(F)_1)$ (i) and (ii) in general do not imply the graph measurability of  $(t, x, y) \rightarrow F(t, x, y)$  [10, page 227]. To see that  $N(x) \neq \emptyset$ , we proceed as follows. Let  $\{s_n\}_{n \geq 1}$ ,  $\{r_n\}_{n \geq 1}$  be step functions such that  $s_n \rightarrow x$  and  $r_n \rightarrow x'$  a.e. on  $T$  and  $\|s_n(t)\| \leq \|x(t)\|$ ,  $\|r_n(t)\| \leq \|x'(t)\|$  a.e. on  $T$ ,  $n \geq 1$ . Then by virtue of hypothesis  $(H(F)_1)$ (i), for every  $n \geq 1$ , the multifunction  $t \rightarrow F(t, s_n(t), r_n(t))$  is measurable and so by the Yankon-von Neumann-Aumann selection theorem [10, page 158], we can find  $u_n : T \rightarrow \mathbb{R}^N$  a measurable map such that  $u_n(t) \in F(t, s_n(t), r_n(t))$  for all  $t \in T$ . Note that  $\|s_n\|_\infty, \|r_n\|_\infty \leq M_1$  for some  $M_1 > 0$  and all  $n \geq 1$ . So  $\|u_n(t)\| \leq \gamma_{M_1}(t)$  a.e. on  $T$ , with  $\gamma_{M_1} \in L^1(T)_+$  (see hypothesis  $(H(F)_1)$ (iii)). Thus by virtue of the Dunford-Pettis theorem, we may assume that  $u_n \xrightarrow{w} u$  in  $L^1(T, \mathbb{R}^N)$  as  $n \rightarrow \infty$ . From Hu and Papageorgiou [10, page 694], we have

$$u(t) \in \overline{\text{conv}} \limsup_{n \rightarrow \infty} F(t, s_n(t), r_n(t)) \subseteq F(t, x(t), x'(t)) \quad \text{a.e. on } T, \quad (3.11)$$

with the last inclusion being a consequence of hypothesis  $(H(F)_1)$ (ii). So we have  $u \in S_{F(\cdot, x(\cdot), x'(\cdot))}^q$ , hence  $N(x) \neq \emptyset$ .

Next we check the upper semicontinuity of  $N$  into  $L^1(T, \mathbb{R}^N)_w$  ( $L^1(T, \mathbb{R}^N)_w$  equals the Banach space  $L^1(T, \mathbb{R}^N)$  furnished with the weak topology). Because of hypothesis  $(H(F)_1)(iii)$ ,  $N$  is locally compact into  $L^1(T, \mathbb{R}^N)_w$  (recall that uniformly integrable sets are relatively compact in  $L^1(T, \mathbb{R}^N)_w$ ). Also on weakly compact subsets of  $L^1(T, \mathbb{R}^N)$ , the relative weak topology is metrizable. Therefore to check the upper semicontinuity of  $N$ , it suffices to show that  $\text{GrN}$  is sequentially closed in  $C_0^1(T, \mathbb{R}^N) \times L^1(T, \mathbb{R}^N)_w$  (see [Section 2](#)). To this end, let  $(x_n, f_n) \in \text{GrN}$ ,  $n \geq 1$ , and suppose that  $x_n \rightarrow x$  in  $C_0^1(T, \mathbb{R}^N)$  and  $f_n \xrightarrow{w} f$  in  $L^1(T, \mathbb{R}^N)$ . For every  $n \geq 1$ , we have

$$f_n(t) = \frac{d}{dt} \nabla G(x_n(t)) + u_n(t) \quad \text{a.e. on } T, \text{ with } u_n \in S_{F(\cdot, x_n(\cdot), x'_n(\cdot))}^1. \quad (3.12)$$

Because of hypothesis  $(H(F)_1)(iii)$ , we may assume (at least for a subsequence) that  $u_n \xrightarrow{w} u$  in  $L^1(T, \mathbb{R}^N)$ . As before, from Hu and Papageorgiou [10, page 694], we have

$$u(t) \in \overline{\text{conv}} \limsup_{n \rightarrow \infty} F(t, x_n(t), x'_n(t)) \subseteq F(t, x(t), x'(t)) \quad \text{a.e. on } T \quad (3.13)$$

(again the last inclusion follows from hypothesis  $(H(F)_1)(ii)$ ). So  $u \in S_{F(\cdot, x(\cdot), x'(\cdot))}^1$ . Also by virtue of hypothesis  $(H(G)_1)$ , we have

$$\begin{aligned} \frac{d}{dt} \nabla G(x_n(t)) &= G''(x_n(t))x'_n(t) \longrightarrow G''(x(t))x'(t) = \frac{d}{dt} \nabla G(x(t)), \quad \forall t \in T \\ \implies \frac{d}{dt} \nabla G(x_n(\cdot)) &\longrightarrow \frac{d}{dt} \nabla G(x(\cdot)) \quad \text{in } L^1(T, \mathbb{R}^N) \\ &\quad (\text{by the dominated convergence theorem}). \end{aligned} \quad (3.14)$$

So in the limit as  $n \rightarrow \infty$ , we have

$$\begin{aligned} f &= \frac{d}{dt} \nabla G(x(\cdot)) + u \quad \text{with } u \in N_F(x) \\ \implies (x, f) &\in \text{GrN}. \end{aligned} \quad (3.15)$$

This proves the desired upper semicontinuity of  $N$ .  $\square$

**PROPOSITION 3.4.** *There exists  $\xi > 0$  such that, for all  $x \in W_0^{1,p}(T, \mathbb{R}^N)$ ,*

$$\|x'\|_p^p - \int_0^b \theta(t) \|x(t)\|^p dt \geq \xi \|x'\|_p^p. \quad (3.16)$$

*Proof.* Let  $\eta: W_0^{1,p}(T, \mathbb{R}^N) \rightarrow \mathbb{R}$  be the functional defined by

$$\eta(x) = \|x'\|_p^p - \int_0^b \theta(t) \|x(t)\|^p dt. \quad (3.17)$$

From the variational characterization of  $\lambda_1(\theta) > 1$ , we see that  $\eta(x) > 0$  for all  $x \in W_0^{1,p}(T, \mathbb{R}^N)$ ,  $x \neq 0$ . Suppose that the proposition was not true. Then by virtue of the  $p$ -homogeneity of  $\eta$ , we can find  $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(T, \mathbb{R}^N)$  such that  $\|x'_n\|_p = 1$  and  $\eta(x_n) \downarrow 0$ .

By the Poincare inequality, the sequence  $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(T, \mathbb{R}^N)$  is bounded and so we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W_0^{1,p}(T, \mathbb{R}^N), \quad x_n \rightarrow x \quad \text{in } C_0(T, \mathbb{R}^N). \quad (3.18)$$

Also exploiting the weak lower semicontinuity of the norm functional in a Banach space, we obtain

$$\|x'\|_p^p \leq \int_0^b \theta(t) \|x(t)\|^p dt \Rightarrow \lambda_1(\theta) \leq 1, \quad (3.19)$$

a contradiction to our hypothesis that  $\lambda_1(\theta) > 1$ .  $\square$

We introduce the set

$$S = \{x \in C_0^1(T, \mathbb{R}^N) : x \in \lambda KN(x), 0 < \lambda < 1\}. \quad (3.20)$$

**PROPOSITION 3.5.** If hypotheses  $(H(F)_1)$  and  $(H(G)_1)$  hold, then  $S \subseteq C_0^1(T, \mathbb{R}^N)$  is bounded.

*Proof.* Let  $x \in S$ . We have

$$\begin{aligned} & \frac{1}{\lambda} x \in KN(x) \quad \text{with } 0 < \lambda < 1 \\ & \Rightarrow \frac{1}{\lambda^{p-1}} (||x'(t)||^{p-2} x'(t))' + \frac{d}{dt} \nabla G(x(t)) + u(t) = 0 \quad \text{a.e. on } T, \text{ with } u \in S_{F(\cdot, x(\cdot), x'(\cdot))}^1 \\ & \Rightarrow (||x'(t)||^{p-2} x'(t))' + \lambda^{p-1} \frac{d}{dt} \nabla G(x(t)) + \lambda^{p-1} u(t) = 0 \quad \text{a.e. on } T. \end{aligned} \quad (3.21)$$

Taking the inner product with  $x(t)$ , integrate over  $T$ , and perform integration by parts, we obtain

$$-\|x'\|_p^p - \lambda^{p-1} \int_0^b (\nabla G(x(t)), x'(t))_{\mathbb{R}^N} dt + \lambda^{p-1} \int_0^b (u(t), x(t))_{\mathbb{R}^N} dt = 0. \quad (3.22)$$

Remark that

$$\int_0^b (\nabla G(x(t)), x'(t))_{\mathbb{R}^N} dt = \int_0^b \frac{d}{dt} G(x(t)) dt = G(x(b)) - G(x(0)) = 0. \quad (3.23)$$

By virtue of hypotheses  $(H(F)_1)$ (iii) and (iv), given  $\varepsilon > 0$ , we can find  $\gamma_\varepsilon \in L^1(T)_+$  such that for almost all  $t \in T$ , all  $x, y \in \mathbb{R}^N$ , and all  $u \in F(t, x, y)$ , we have

$$(u, x)_{\mathbb{R}^N} \leq (\theta(t) + \varepsilon) \|x\|^p + \gamma_\varepsilon(t). \quad (3.24)$$

So we have

$$\int_0^b (u(t), x(t))_{\mathbb{R}^N} dt \leq \int_0^b \theta(t) \|x(t)\|^p dt + \varepsilon \|x\|_p^p + \|\gamma_\varepsilon\|_1. \quad (3.25)$$

Using (3.24) and (3.27) in (3.23), we obtain

$$\begin{aligned} \|x'\|_p^p &\leq \int_0^b \theta(t) \|x(t)\|^p dt + \varepsilon \|x\|_p^p + \|\gamma_\varepsilon\|_1 \\ &\Rightarrow \xi \|x'\|_p^p - \frac{\varepsilon}{\lambda_1} \|x'\|_p^p \leq \|\gamma_\varepsilon\|_1 \end{aligned} \tag{3.26}$$

(see Proposition 3.5 and recall that  $\lambda_1 \|x\|_p^p \leq \|x'\|_p^p$ ,  $\lambda_1 = \lambda_1(1)$ ).

Choose  $\varepsilon > 0$  so that  $\varepsilon < \lambda_1 \xi$ . Then from the last inequality, we infer that

$$\begin{aligned} \{x'\}_{x \in S} &\subseteq L^p(T, \mathbb{R}^N) \text{ is bounded} \\ &\Rightarrow S \subseteq W_0^{1,p}(T, \mathbb{R}^N) \text{ is bounded (by Poincaré's inequality)} \\ &\Rightarrow S \subseteq C_0(T, \mathbb{R}^N) \text{ is relatively compact.} \end{aligned} \tag{3.27}$$

Also we have

$$\begin{aligned} &\|(|x'(t)|^{p-2} x'(t))'\| \\ &\leq \|G''(x(t))\|_{\mathcal{L}} \|x'(t)\| + \|u(t)\| \text{ a.e. on } T \\ &\leq M_2 (\|x'(t)\| + \theta(t) + \varepsilon + \gamma_\varepsilon(t)) \text{ a.e. on } T \text{ for some } M_2 > 0 \text{ (see (3.25))} \\ &\Rightarrow \{\|x'\|^{p-2} x'\}_{x \in S} \subseteq W^{1,1}(T, \mathbb{R}^N) \text{ is bounded} \\ &\Rightarrow \{\|x'\|^{p-2} x'\}_{x \in S} \subseteq C(T, \mathbb{R}^N) \text{ is bounded} \\ &\quad (\text{since } W^{1,1}(T, \mathbb{R}^N) \text{ is embedded continuously but not compactly in } C(T, \mathbb{R}^N)) \\ &\Rightarrow \{x'\}_{x \in S} \subseteq C(T, \mathbb{R}^N) \text{ is bounded.} \end{aligned} \tag{3.28}$$

From (3.28) and (3.29), we conclude that  $S \subseteq C_0^1(T, \mathbb{R}^N)$  is bounded.  $\square$

Propositions 3.2, 3.3, and 3.5 permit the use of Theorem 2.1. So we obtain the following existence result for problem (1.1).

**THEOREM 3.6.** If hypotheses  $(H(F)_1)$  and  $(H(G)_1)$  hold, then problem (1.1) has a solution  $x \in C_0^1(T, \mathbb{R}^N)$  with  $\|x'\|^{p-2} x' \in W^{1,1}(T, \mathbb{R}^N)$ .

As an application of this theorem, we consider the following system:

$$\begin{aligned} &(|x'(t)|^{p-2} x'(t))' + \|x(t)\|^{p-2} A x(t) + F(t, x(t)) \ni e(t) \quad \text{a.e. on } T = [0, b], \\ &x(0) = x(b) = 0, \quad e \in L^1(T, \mathbb{R}^N). \end{aligned} \tag{3.29}$$

Our hypotheses on the data of problem (3.29) are the following.

$(H(A))$   $A$  is an  $N \times N$  matrix such that for all  $x \in \mathbb{R}^N$  we have  $(Ax, x)_{\mathbb{R}^N} \leq \theta \|x\|^2$  with  $\theta < (\pi_p/b)^p$ .

**Remark 3.7.** The quantity  $\pi_p$  is defined by  $\pi_p = 2(p-1)^{1/p} \int_0^1 (1/(1-t)^{1/p}) dt = 2(p-1)^{1/p} ((\pi/p)/\sin(\pi/p))$ . If  $p = 2$ , then  $\pi_2 = \pi$ . Recall that the eigenvalues of  $(-\Delta_p, W_0^{1,p}(T, \mathbb{R}^N))$  are  $\lambda_n = (n\pi_p/b)^p$ ,  $n \geq 1$  [13]. So in hypothesis  $(H(A))$ , we have  $\theta < \lambda_1$ .

( $H(F)_1'$ )  $F : T \times \mathbb{R}^N \rightarrow P_{kc}(\mathbb{R}^N)$  is a multifunction such that

- (i) for all  $x \in \mathbb{R}^N$ ,  $t \rightarrow F(t, x)$  is graph measurable;
- (ii) for almost all  $t \in T$ ,  $x \rightarrow F(t, x)$  is usc;
- (iii) for every  $M > 0$ , there exists  $\gamma_M \in L^1(T)_+$  such that for almost all  $t \in T$ , all  $\|x\| \leq M$ , and all  $u \in F(t, x)$ , we have  $\|u\| \leq \gamma_M(t)$ ;
- (iv)  $\lim_{\|x\| \rightarrow \infty} ((u, x)_{\mathbb{R}^N} / \|x\|^p) = 0$  uniformly for almost all  $t \in T$  and all  $u \in F(t, x)$ .

Invoking [Theorem 3.6](#), we obtain the following existence result for problem (3.29).

**THEOREM 3.8.** *If hypotheses  $(H(A))$  and  $(H(F)_1')$  hold, then for every  $e \in L^1(T, \mathbb{R}^N)$ , problem (3.29) has a solution  $x \in C_0^1(T, \mathbb{R}^N)$  with  $\|x'\|^{p-2}x' \in W^{1,1}(T, \mathbb{R}^N)$ .*

*Remark 3.9.* [Theorem 3.8](#) extends Theorem 7.1 of Manásevich and Mawhin [13].

## Acknowledgments

The authors wish to thank a very knowledgeable referee for pointing out an error in the first version of the paper and for constructive remarks. Michael E. Filippakis was supported by a grant from the National Scholarship Foundation of Greece (IKY).

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