

Research Article

Iteration Scheme with Perturbed Mapping for Common Fixed Points of a Finite Family of Nonexpansive Mappings

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We propose an iteration scheme with perturbed mapping for approximation of common fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$. We show that the proposed iteration scheme converges to the common fixed point $x^* \in \bigcap_{i=1}^N \text{Fix}(T_i)$ which solves some variational inequality.

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1. Introduction and preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. A mapping T with domain $D(T)$ and range $R(T)$ in H is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(T). \quad (1.1)$$

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-maps of H . Denote the common fixed points set of $\{T_i\}_{i=1}^N$ by $\bigcap_{i=1}^N \text{Fix}(T_i)$. Let $F : H \rightarrow H$ be a mapping such that for some constants $k, \eta > 0$, F is k -Lipschitzian and η -strongly monotone. Let $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$, $\{\lambda_n\}_{n=1}^\infty \subset [0, 1)$ and take a fixed number $\mu \in (0, 2\eta/k^2)$. The iterative schemes concerning nonlinear operators have been studied extensively by many authors, you may refer to [1–12]. Especially, in [13], Zeng and Yao introduced the following implicit iteration process with perturbed mapping F .

For an arbitrary initial point $x_0 \in H$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) [T_n x_n - \lambda_n \mu F(T_n x_n)], \quad n \geq 1, \quad (1.2)$$

where $T_n := T_{n \bmod N}$.

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Using this iteration process, they proved the following weak and strong convergence theorems for nonexpansive mappings in Hilbert spaces.

THEOREM 1.1 (see [13]). *Let H be a real Hilbert space and let $F : H \rightarrow H$ be a mapping such that for some constants $k, \eta > 0$, F is k -Lipschitzian and η -strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that $\bigcap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/k^2)$ and $x_0 \in H$. Let $\{\lambda_n\}_{n=1}^\infty \subset [0, 1)$ and $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$ satisfying the conditions $\sum_{n=1}^\infty \lambda_n < \infty$ and $\alpha \leq \alpha_n \leq \beta$, $n \geq 1$, for some $\alpha, \beta \in (0, 1)$. Then the sequence $\{x_n\}_{n=1}^\infty$ defined by (1.2) converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.*

THEOREM 1.2 (see [13]). *Let H be a real Hilbert space and let $F : H \rightarrow H$ be a mapping such that for some constants $k, \eta > 0$, F is k -Lipschitzian and η -strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that $\bigcap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/k^2)$ and $x_0 \in H$. Let $\{\lambda_n\}_{n=1}^\infty \subset [0, 1)$ and $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$ satisfying the conditions $\sum_{n=1}^\infty \lambda_n < \infty$ and $\alpha \leq \alpha_n \leq \beta$, $n \geq 1$, for some $\alpha, \beta \in (0, 1)$. Then the sequence $\{x_n\}_{n=1}^\infty$ defined by (1.2) converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$ if and only if*

$$\liminf_{n \rightarrow \infty} d\left(x_n, \bigcap_{i=1}^N \text{Fix}(T_i)\right) = 0. \quad (1.3)$$

Very recently, Wang [14] considered an explicit iterative scheme with perturbed mapping F and obtained the following result.

THEOREM 1.3. *Let H be a Hilbert space, let $T : H \rightarrow H$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and let $F : H \rightarrow H$ be an η -strongly monotone and k -Lipschitzian mapping. For any given $x_0 \in H$, $\{x_n\}$ is defined by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^{\lambda_{n+1}} x_n, \quad n \geq 0, \quad (1.4)$$

where $T^{\lambda_{n+1}} x_n = T x_n - \lambda_{n+1} \mu F(T x_n)$, $\{\alpha_n\}$ and $\{\lambda_n\} \subset [0, 1)$ satisfy the following conditions:

- (1) $\alpha \leq \alpha_n \leq \beta$ for some $\alpha, \beta \in (0, 1)$;
- (2) $\sum_{n=1}^\infty \lambda_n < \infty$;
- (3) $0 < \mu < 2\eta/k^2$.

Then

- (1) $\{x_n\}$ converges weakly to a fixed point of T ,
- (2) $\{x_n\}$ converges strongly to a fixed point of T if and only if

$$\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0. \quad (1.5)$$

This naturally brings us the following questions.

Questions 1.4. Let $T_i : H \rightarrow H$ ($i = 1, 2, \dots, N$) be a finite family of nonexpansive mappings and F is k -Lipschitzian and η -strongly monotone.

- (i) Could we construct an explicit iterative algorithm to approximate the common fixed points of the mappings $\{T_i\}_{i=1}^N$?
- (ii) Could we remove the assumption (2) imposed on the sequence $\{x_n\}$?

Motivated and inspired by the above research work of Zeng and Yao [13] and Wang [14], in this paper, we will propose a new explicit iteration scheme with perturbed mapping for approximation of common fixed points of a finite family of nonexpansive self-mappings of H . We will establish strong convergence theorem for this explicit iteration scheme. To be more specific, let $\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nN} \in (0, 1]$, $n \in N$. Given the mappings T_1, T_2, \dots, T_N , following [15], one can define, for each n , mappings $U_{n1}, U_{n2}, \dots, U_{nN}$ by

$$\begin{aligned} U_{n1} &= \alpha_{n1} T_1 + (1 - \alpha_{n1}) I, \\ U_{n2} &= \alpha_{n2} T_2 U_{n1} + (1 - \alpha_{n2}) I, \\ &\vdots \\ U_{n,N-1} &= \alpha_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \alpha_{n,N-1}) I, \\ W_n &:= U_{nN} = \alpha_{nN} T_N U_{n,N-1} + (1 - \alpha_{nN}) I. \end{aligned} \tag{1.6}$$

Such a mapping W_n is called the W -mapping generated by T_1, \dots, T_N and $\alpha_{n1}, \dots, \alpha_{nN}$.

First we introduce the following explicit iteration scheme with perturbed mapping F .

For an arbitrary initial point $x_0 \in H$, the sequence $\{x_n\}_{n=1}^\infty$ is generated iteratively by

$$x_{n+1} = \beta x_n + (1 - \beta) [W_n x_n - \lambda_n \mu F(W_n x_n)], \quad n \geq 0, \tag{1.7}$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$, β is a constant in $(0, 1)$, F is k -Lipschitzian and η -strongly monotone, and W_n is the W -mapping defined by (1.6).

We have the following crucial conclusion concerning W_n .

PROPOSITION 1.5 (see [15]). *Let C be a nonempty closed convex subset of a Banach space E . Let T_1, T_2, \dots, T_N be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N \text{Fix}(T_i)$ is nonempty, and let $\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nN}$ be real numbers such that $0 < \alpha_{ni} \leq b < 1$ for any $i \in N$. For any $n \in N$, let W_n be the W -mapping of C into itself generated by T_N, T_{N-1}, \dots, T_1 and $\alpha_{nN}, \alpha_{n,N-1}, \dots, \alpha_{n1}$. Then W_n is nonexpansive. Further, if E is strictly convex, then $\text{Fix}(W_n) = \bigcap_{i=1}^N \text{Fix}(T_i)$.*

Now we recall some basic notations. Let $T : H \rightarrow H$ be nonexpansive mapping and $F : H \rightarrow H$ be a mapping such that for some constants $k, \eta > 0$, F is k -Lipschitzian and η -strongly monotone; that is, F satisfies the following conditions:

$$\begin{aligned} \|Fx - Fy\| &\leq k\|x - y\|, \quad \forall x, y \in H, \\ \langle Fx - Fy, x - y \rangle &\geq \eta\|x - y\|^2, \quad \forall x, y \in H, \end{aligned} \tag{1.8}$$

respectively. We may assume, without loss of generality, that $\eta \in (0, 1)$ and $k \in [1, \infty)$. Under these conditions, it is well known that the variational inequality problem—find $x^* \in \bigcap_{i=1}^N \text{Fix}(T_i)$ such that

$$VI \left(F, \bigcap_{i=1}^N \text{Fix}(T_i) \right) : \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^N \text{Fix}(T_i), \tag{1.9}$$

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has a unique solution $x^* \in \bigcap_{i=1}^N \text{Fix}(T_i)$. [Note: the unique existence of the solution $x^* \in \bigcap_{i=1}^N \text{Fix}(T_i)$ is guaranteed automatically because F is k -Lipschitzian and η -strongly monotone over $\bigcap_{i=1}^N \text{Fix}(T_i)$.]

For any given numbers $\lambda \in [0, 1]$ and $\mu \in (0, 2\eta/k^2)$, we define the mapping $T^\lambda : H \rightarrow H$ by

$$T^\lambda x := Tx - \lambda \mu F(Tx), \quad \forall x \in H. \quad (1.10)$$

Concerning the corresponding result of $T^\lambda x$, you can find it in [16].

LEMMA 1.6 (see [16]). *If $0 \leq \lambda < 1$ and $0 < \mu < 2\eta/k^2$, then there holds for $T^\lambda : H \rightarrow H$,*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in H, \quad (1.11)$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0, 1)$.

Next, let us state four preliminary results which will be needed in the sequel. Lemma 1.7 is very interesting and important, you may find it in [17], the original prove can be found in [18]. Lemmas 1.8 and 1.9 well-known demiclosedness principle and subdifferential inequality, respectively. Lemma 1.10 is basic and important result, please consult it in [19].

LEMMA 1.7 (see [17]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1. \quad (1.12)$$

Suppose

$$x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n, \quad (1.13)$$

for all integers $n \geq 0$ and

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \quad (1.14)$$

Then, $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

LEMMA 1.8 (see [20]). *Assume that T is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H . If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y , it follows that $(I - T)x = y$. Here, I is the identity operator of H .*

LEMMA 1.9 (see [21]). $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$ for all $x, y \in H$.

LEMMA 1.10 (see [19]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad (1.15)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (1) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
- (2) $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

2. Main result

Now we state and prove our main result.

THEOREM 2.1. *Let H be a real Hilbert space and let $F : H \rightarrow H$ be a k -Lipschitzian and η -strongly monotone mapping. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of H such that $\bigcap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/k^2)$. Suppose the sequences $\{\alpha_{n,i}\}_{i=1}^N$ satisfy $\lim_{n \rightarrow \infty} (\alpha_{n,i} - \alpha_{n-1,i}) = 0$, for all $i = 1, 2, \dots, N$. If $\{\lambda_n\}_{n=1}^{\infty} \subset [0, 1)$ satisfy the following conditions:*

- (i) $\lim_{n \rightarrow \infty} \lambda_n = 0$;
- (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$,

then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (1.7) converges strongly to a common fixed point $x^ \in \bigcap_{i=1}^N \text{Fix}(T_i)$ which solves the variational inequality (1.9).*

Proof. Let x^* be an arbitrary element of $\bigcap_{i=1}^N \text{Fix}(T_i)$. Observe that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\beta x_n + (1 - \beta)W_n^{\lambda_n} x_n - x^*\| \\ &\leq \beta \|x_n - x^*\| + (1 - \beta) \|W_n^{\lambda_n} x_n - x^*\|, \end{aligned} \quad (2.1)$$

where $W_n^{\lambda_n} x := W_n x - \lambda_n \mu F(W_n x)$. Note that

$$W_n^{\lambda_n} x^* = x^* - \lambda_n \mu F(x^*). \quad (2.2)$$

Utilizing Lemma 1.6, we have

$$\begin{aligned} \|W_n^{\lambda_n} x_n - x^*\| &= \|W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^* + W_n^{\lambda_n} x^* - x^*\| \\ &\leq \|W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^*\| + \|W_n^{\lambda_n} x^* - x^*\| \\ &\leq (1 - \lambda_n \tau) \|x_n - x^*\| + \lambda_n \mu \|F(x^*)\|. \end{aligned} \quad (2.3)$$

From (2.1) and (2.3), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq [\beta + (1 - \beta)(1 - \lambda_n \tau)] \|x_n - x^*\| + (1 - \beta) \lambda_n \mu \|F(x^*)\| \\ &= [1 - (1 - \beta) \lambda_n \tau] \|x_n - x^*\| + (1 - \beta) \lambda_n \mu \|F(x^*)\| \\ &\leq \max \left\{ \|x_0 - x^*\|, \left(\frac{\mu}{\tau} \right) \|F(x^*)\| \right\}. \end{aligned} \quad (2.4)$$

Hence, $\{x_n\}$ is bounded. We also can obtain that $\{W_n x_n\}$, $\{T_i U_{n,j} x_n\}$ ($i = 1, \dots, N$; $j = 1, \dots, N$), and $\{F(W_n x_n)\}$ are all bounded.

We will use M to denote the possible different constants appearing in the following reasoning.

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We note that

$$\begin{aligned}
 & \|W_{n+1}^{\lambda_{n+1}}x_{n+1} - W_n^{\lambda_n}x_n\| \\
 &= \|W_{n+1}x_{n+1} - W_nx_n - \lambda_{n+1}\mu F(W_{n+1}x_{n+1}) + \lambda_n\mu F(W_nx_n)\| \\
 &\leq \|W_{n+1}x_{n+1} - W_nx_n\| + \lambda_{n+1}\mu \|F(W_{n+1}x_{n+1})\| + \lambda_n\mu \|F(W_nx_n)\| \\
 &\leq \|W_{n+1}x_{n+1} - W_{n+1}x_n\| + \|W_{n+1}x_n - W_nx_n\| + (\lambda_{n+1} + \lambda_n)M \\
 &\leq \|x_{n+1} - x_n\| + \|W_{n+1}x_n - W_nx_n\| + (\lambda_{n+1} + \lambda_n)M.
 \end{aligned} \tag{2.5}$$

From (1.6), since T_N and $U_{n,N}$ are nonexpansive,

$$\begin{aligned}
 & \|W_{n+1}x_n - W_nx_n\| \\
 &= \|\alpha_{n+1,N}T_NU_{n+1,N-1}x_n + (1 - \alpha_{n+1,N})x_n - \alpha_{n,N}T_NU_{n,N-1}x_n - (1 - \alpha_{n,N})x_n\| \\
 &\leq \|\alpha_{n+1,N}T_NU_{n+1,N-1}x_n - \alpha_{n,N}T_NU_{n,N-1}x_n\| + |\alpha_{n+1,N} - \alpha_{n,N}| \|x_n\| \\
 &\leq \|\alpha_{n+1,N}(T_NU_{n+1,N-1}x_n - T_NU_{n,N-1}x_n)\| + |\alpha_{n+1,N} - \alpha_{n,N}| \|T_NU_{n,N-1}x_n\| \\
 &\quad + |\alpha_{n+1,N} - \alpha_{n,N}| \|x_n\| \\
 &\leq \alpha_{n+1,N} \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + 2M |\alpha_{n+1,N} - \alpha_{n,N}|.
 \end{aligned} \tag{2.6}$$

Again, from (1.6), we have

$$\begin{aligned}
 & \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
 &= \|\alpha_{n+1,N-1}T_{N-1}U_{n+1,N-2}x_n + (1 - \alpha_{n+1,N-1})x_n \\
 &\quad - \alpha_{n,N-1}T_{N-1}U_{n,N-2}x_n - (1 - \alpha_{n,N-1})x_n\| \\
 &\leq \|\alpha_{n+1,N-1}T_{N-1}U_{n+1,N-2}x_n - \alpha_{n,N-1}T_{N-1}U_{n,N-2}x_n\| \\
 &\quad + |\alpha_{n+1,N-1} - \alpha_{n,N-1}| \|x_n\| \\
 &\leq |\alpha_{n+1,N-1} - \alpha_{n,N-1}| \|x_n\| + |\alpha_{n+1,N-1} - \alpha_{n,N-1}| M \\
 &\quad + \alpha_{n+1,N-1} \|T_{N-1}U_{n+1,N-2}x_n - T_{N-1}U_{n,N-2}x_n\| \\
 &\leq 2M |\alpha_{n+1,N-1} - \alpha_{n,N-1}| + \alpha_{n+1,N-1} \|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| \\
 &\leq 2M |\alpha_{n+1,N-1} - \alpha_{n,N-1}| + \|U_{n+1,N-2}x_n - U_{n,N-2}x_n\|.
 \end{aligned} \tag{2.7}$$

Therefore, we have

$$\begin{aligned}
& \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
& \leq 2M|\alpha_{n+1,N-1} - \alpha_{n,N-1}| + 2M|\alpha_{n+1,N-2} - \alpha_{n,N-2}| \\
& \quad + \|U_{n+1,N-3}x_n - U_{n,N-3}x_n\| \\
& \leq 2M \sum_{i=2}^{N-1} |\alpha_{n+1,i} - \alpha_{n,i}| + \|U_{n+1,1}x_n - U_{n,1}x_n\| \\
& = \|\alpha_{n+1,1}T_1x_n + (1 - \alpha_{n+1,1})x_n - \alpha_{n,1}T_1x_n - (1 - \alpha_{n,1})x_n\| \\
& \quad + 2M \sum_{i=2}^{N-1} |\alpha_{n+1,i} - \alpha_{n,i}|,
\end{aligned} \tag{2.8}$$

then

$$\begin{aligned}
& \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
& \leq |\alpha_{n+1,1} - \alpha_{n,1}| \|x_n\| + \|\alpha_{n+1,1}T_1x_n - \alpha_{n,1}T_1x_n\| \\
& \quad + 2M \sum_{i=2}^{N-1} |\alpha_{n+1,i} - \alpha_{n,i}| \leq 2M \sum_{i=1}^{N-1} |\alpha_{n+1,i} - \alpha_{n,i}|.
\end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.6), we have

$$\begin{aligned}
\|W_{n+1}x_n - W_nx_n\| & \leq 2M|\alpha_{n+1,N} - \alpha_{n,N}| + 2\alpha_{n+1,N}M \sum_{i=1}^{N-1} |\alpha_{n+1,i} - \alpha_{n,i}| \\
& \leq 2M \sum_{i=1}^N |\alpha_{n+1,i} - \alpha_{n,i}|.
\end{aligned} \tag{2.10}$$

Substituting (2.10) into (2.5), we have

$$\|W_{n+1}^{\lambda_{n+1}}x_{n+1} - W_n^{\lambda_n}x_n\| \leq \|x_{n+1} - x_n\| + 2M \sum_{i=1}^N |\alpha_{n+1,i} - \alpha_{n,i}| + (\lambda_{n+1} + \lambda_n)M, \tag{2.11}$$

which implies that

$$\limsup_{n \rightarrow \infty} (\|W_{n+1}^{\lambda_{n+1}}x_{n+1} - W_n^{\lambda_n}x_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{2.12}$$

We note that $x_{n+1} = \beta x_n + (1 - \beta)W_n^{\lambda_n}x_n$ and $0 < \beta < 1$, then from Lemma 1.7 and (2.12), we have $\lim_{n \rightarrow \infty} \|W_n^{\lambda_n}x_n - x_n\| = 0$. It follows that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \beta) \|W_n^{\lambda_n}x_n - x_n\| = 0. \tag{2.13}$$

On the other hand,

$$\begin{aligned}
\|x_n - W_nx_n\| & \leq \|x_{n+1} - x_n\| + \|x_{n+1} - W_nx_n\| \\
& \leq \|x_{n+1} - x_n\| + \beta \|x_n - W_nx_n\| + (1 - \beta)\lambda_n\mu \|F(W_nx_n)\|,
\end{aligned} \tag{2.14}$$

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that is,

$$\|x_n - W_n x_n\| \leq \frac{1}{1-\beta} \|x_{n+1} - x_n\| + \lambda_n \mu \|F(W_n x_n)\|, \quad (2.15)$$

this together with (i) and (2.13) imply

$$\lim_{n \rightarrow \infty} \|x_n - W_n x_n\| = 0. \quad (2.16)$$

We next show that

$$\limsup_{n \rightarrow \infty} \langle -F(x^*), x_n - x^* \rangle \leq 0. \quad (2.17)$$

To prove this, we pick a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle -F(x^*), x_n - x^* \rangle = \lim_{i \rightarrow \infty} \langle -F(x^*), x_{n_i} - x^* \rangle. \quad (2.18)$$

Without loss of generality, we may further assume that $x_{n_i} \rightarrow z$ weakly for some $z \in H$.

By Lemma 1.8 and (2.16), we have

$$z \in \text{Fix}(W_n), \quad (2.19)$$

this together with Proposition 1.5 imply that

$$z \in \bigcap_{i=1}^N \text{Fix}(T_i). \quad (2.20)$$

Since x^* solves the variational inequality (1.9), then we obtain

$$\limsup_{n \rightarrow \infty} \langle -F(x^*), x_n - x^* \rangle = \langle -F(x^*), z - x^* \rangle \leq 0. \quad (2.21)$$

Finally, we show that $x_n \rightarrow x^*$. Indeed, from Lemma 1.9, we have

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ &= \|\beta(x_n - x^*) + (1-\beta)(W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^*) + (1-\beta)(W_n^{\lambda_n} x^* - x^*)\|^2 \\ &\leq \|\beta(x_n - x^*) + (1-\beta)(W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^*)\|^2 + 2(1-\beta)\langle W_n^{\lambda_n} x^* - x^*, x_{n+1} - x^* \rangle \\ &\leq [\beta\|x_n - x^*\| + (1-\beta)\|W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^*\|]^2 + 2(1-\beta)\lambda_n \mu \langle -F(x^*), x_{n+1} - x^* \rangle \\ &\leq [\beta\|x_n - x^*\| + (1-\beta)(1-\lambda_n \tau)\|x_n - x^*\|]^2 + 2(1-\beta)\lambda_n \mu \langle -F(x^*), x_{n+1} - x^* \rangle \\ &\leq [1 - (1-\beta)\tau\lambda_n]\|x_n - x^*\|^2 + (1-\beta)\tau\lambda_n \left\{ 2\frac{\mu}{\tau} \langle -F(x^*), x_{n+1} - x^* \rangle \right\}. \end{aligned} \quad (2.22)$$

Now applying Lemma 1.10 and (2.21) to (2.22) concludes that $x_n \rightarrow x^*$ ($n \rightarrow \infty$). This completes the proof. \square

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