

Research Article

Solvability of Second-Order m -Point Boundary Value Problems with Impulses

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By Leray-Schauder continuation theorem and the nonlinear alternative of Leray-Schauder type, the existence of a solution for an m -point boundary value problem with impulses is proved.

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1. Introduction

The main purpose of this paper is to get results on the solvability of the following boundary value problem (BVP):

$$\begin{aligned} x''(t) &= f(t, x(t), x'(t)), \\ \Delta x'(t_k) &= b_k x'(t_k), \quad \Delta x(t_k) = c_k x(t_k), \\ x'(0) &= 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{aligned} \tag{1.1}$$

where $\xi_i \in (0, 1)$, $i = 1, 2, \dots, m-2$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $a_i \in R$, $i = 1, 2, \dots, m-2$, $\sum_{i=1}^{m-2} a_i \neq 1$, $0 = t_0 < t_1 < t_2 < \dots < t_T < t_{T+1} = 1$.

Such problems without impulses effects have been solved before, for example, in [1–3]. But as far as we know the publication on the solvability of m -point problems with impulses is fewer [4]. Our main goal is to find condition for f, b_k, c_k , $1 \leq k \leq T$, which guarantees the existence of at least one solution of problem (1.1). The proofs are based on the Leray-Schauder continuation theorem [5] and the nonlinear alternative of Leray-Schauder type [6].

2 Boundary Value Problems

In order to define the concept of solution for BVP (1.1), we introduce the following spaces of functions:

- (i) $PC[0, 1] = \{u : [0, 1] \rightarrow R, u \text{ is continuous at } t \neq t_k, u(t_k^+), u(t_k^-) \text{ exist, and } u(t_k^-) = u(t_k^+)\}$;
- (ii) $PC^1[0, 1] = \{u \in PC[0, 1] : u \text{ is continuously differentiable at } t \neq t_k, u'(0^+), u'(t_k^+), u'(t_k^-) \text{ exist and } u'(t_k^-) = u'(t_k^+)\}$;
- (iii) $PC^2[0, 1] = \{u \in PC^1[0, 1] : u \text{ is twice continuously differentiable at } t \neq t_k\}$.

Note that $PC[0, 1]$ and $PC^1[0, 1]$ are Banach spaces with the norms

$$\|u\|_\infty = \sup \{|u(t)| : t \in [0, 1]\}, \quad \|u\|_1 = \max \{\|u\|_\infty, \|u'\|_\infty\}, \quad (1.2)$$

respectively.

Definition 1.1. The set $\mathcal{F}$ is said to be quasiequicontinuous in $[0, c]$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathcal{F}$, $k \in Z$, $t^*, t^{**} \in (t_{k-1}, t_k) \cap [0, c]$, and $|t^* - t^{**}| < \delta$, then $|x(t^*) - x(t^{**})| < \varepsilon$.

LEMMA 1.2 (compactness criterion [7]). *The set $\mathcal{F} \subset PC([0, c], R^n)$ is relatively compact if and only if one has the following:*

- (1) $\mathcal{F}$ is bounded;
- (2) $\mathcal{F}$ is quasiequicontinuous in $[0, c]$.

LEMMA 1.3 [7]. *Let $s \in [0, T]$, $c_k \geq 0$, α_k , $k = 1, \dots, p$, are constants and let $p, q \in PC(J, R)$, $x \in PC^1(J, R)$. If*

$$\begin{aligned} x'(t) &\leq p(t)x(t) + q(t), \quad t \in [s, T], \quad t \neq t_k, \\ x(t_k^+) &\leq c_k x(t_k) + \alpha_k, \quad t_k \in [s, T], \end{aligned} \quad (1.3)$$

then for $t \in [s, T]$,

$$\begin{aligned} x(t) &\leq x(s^+) \left(\prod_{s < t_k < t} c_k \right) \exp \left(\int_s^t p(u) du \right) \\ &\quad + \int_s^t \left(\prod_{u < t_k < t} c_k \right) \exp \left(\int_u^t p(\tau) d\tau \right) q(u) du \\ &\quad + \sum_{s < t_k < t} \left(\prod_{t_k < t_i < t} c_i \right) \exp \left(\int_{t_k}^t p(\tau) d\tau \right) \alpha_k. \end{aligned} \quad (1.4)$$

The result also holds if the above inequalities are reversed.

2. Main results

THEOREM 2.1. *Let $f : [0, 1] \times R^2 \rightarrow R$ be a continuous function. Assume that there exist $p(t)$, $q(t)$, and $r(t) : [0, 1] \rightarrow [0, \infty)$ such that*

$$|f(t, u, v)| \leq p(t)|u| + q(t)|v| + r(t) \quad (2.1)$$

for $t \in [0, 1]$ and all $(u, v) \in \mathbb{R}^2$. Then the BVP (1.1) has at least one solution in $PC^1[0, 1]$ provided

$$Q + B < 1, \quad (2.2)$$

$$\left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|}\right) \left(\frac{P}{1 - Q - B} + C\right) < 1, \quad (2.3)$$

where $P = \int_0^1 p(t)dt$, $Q = \int_0^1 q(t)dt$, $B = \sum_{k=1}^T |b_k|$, $C = \sum_{k=1}^T |c_k|$.

Proof. Let $Y = X = PC^1[0, 1]$. Define a linear operator $L : D(L) \subset X \rightarrow Y$ by setting

$$D(L) = \left\{ x \in PC^2[0, 1], x'(0) = 0, x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i) \right\}, \quad (2.4)$$

and for $x \in D(L) : Lx = (x'', \Delta x'(t_k), \Delta x(t_k))$. We also define a nonlinear mapping $F : X \rightarrow Y$ by setting

$$(Fx)(t) = (f(t, x(t), x'(t)), b_k x'(t_k), c_k x(t_k)). \quad (2.5)$$

From the assumption on f , we see that F is a bounded mapping from X to Y . Next, it is easy to see that $L : D(L) \rightarrow Y$ is one-to-one mapping. Moreover, it follows easily using Lemma 1.2 that $L^{-1}F : X \rightarrow X$ is a compact mapping.

We note that $x \in PC^1[0, 1]$ is a solution of (1.1) if and only if x is a fixed point of the equation

$$x = L^{-1}Fx. \quad (2.6)$$

We apply the Leray-Schauder continuation theorem to obtain the existence of a solution for $x = L^{-1}Fx$.

To do this, it suffices to verify that the set of all possible solutions of the family of equations:

$$\begin{aligned} x''(t) &= \lambda f(t, x(t), x'(t)), \\ \Delta x'(t_k^+) &= \lambda b_k x'(t_k), \quad \Delta x(t_k) = \lambda c_k x(t_k), \\ x'(0) &= 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i). \end{aligned} \quad (2.7)$$

Integrate (2.7) from 0 to t to obtain

$$x'(t) = \lambda \int_0^t f(s, x(s), x'(s)) ds + \lambda \sum_{0 < t_k < t} b_k x'(t_k). \quad (2.8)$$

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By condition (2.1), we have

$$\begin{aligned} |x'(t)| &\leq \int_0^t [p(s)\|x\| + q(s)\|x'\| + r(s)]ds + \sum_{k=1}^T |b_k| \|x'\| \\ &\leq (Q+B)\|x'\| + P\|x\| + R_1, \end{aligned} \quad (2.9)$$

where $R_1 = \int_0^1 r(t)dt$. Thus,

$$\|x'\| \leq \frac{1}{1-Q-B} (P\|x\| + R_1). \quad (2.10)$$

Integrate (2.8) from t to 1 to obtain

$$\begin{aligned} &-x(t) \\ &= \lambda \left\{ \int_0^1 H(t,s)f(s,x(s),x'(s))ds + \int_t^1 \sum_{0 < t_k < s} b_k x'(t_k)ds + \sum_{t < t_k < 1} c_k x(t_k) \right. \\ &\quad \left. + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \left[\int_0^1 H(\xi_i, s)f(s,x(s),x'(s))ds + \int_{\xi_i}^1 \sum_{0 < t_k < s} b_k x'(t_k)ds + \sum_{\xi_i < t_k < 1} c_k x(t_k) \right] \right\}, \end{aligned} \quad (2.11)$$

where

$$H(t,s) = \begin{cases} 1-t, & 0 \leq s \leq t \leq 1, \\ 1-s, & 0 \leq t \leq s \leq 1. \end{cases} \quad (2.12)$$

So

$$\|x\| \leq \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) [(P+C)\|x\| + (Q+B)\|x'\| + R_1]. \quad (2.13)$$

Equations (2.10) and (2.13) imply

$$\|x\| \leq \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \left[\left(\frac{P}{1-Q-B} + C \right) \|x\| + R_1 \right]. \quad (2.14)$$

It follows from the assumption (2.3) that there is a constant M_1 in dependent of $\lambda \in [0, 1]$ such that $\|x\| \leq M_1$. Furthermore, by (2.10), there is a constant M_2 such that $\|x'\| \leq M_2$. It is now immediate that the set of solutions of the family of equations (2.7) is, a priori, bounded in $PC^1[0, 1]$ by a constant independent of $\lambda \in [0, 1]$. This completes the proof of the theorem.

THEOREM 2.2. *Let $f : [0, 1] \times R^2 \rightarrow R$. Assume that the following conditions hold:*

(H₁) *$|f(t, u, v)| \leq q(t)w(\max\{|u|, |v|\})$ on $[0, 1] \times R^2$ with $w > 0$ continuous and non-decreasing on $[0, \infty)$, $q(t) : [0, 1] \rightarrow [0, \infty)$ is continuous;*

(H₂) $b_k \geq 0$, and

$$C \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) < 1, \quad (2.15)$$

$$\sup_{r \geq 0} \frac{r}{w(r)} > M_3 = \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \left[1 - C \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \right]^{-1} Q,$$

where $Q = \int_0^1 \prod_{0 < t_k < 1} (1 + b_k) q(s) ds$.

Then (1.1) has at least one solution.

Choose $\widetilde{M} > 0$ such that

$$\frac{\widetilde{M}}{w(\widetilde{M})} > M_3. \quad (2.16)$$

To show that (1.1) has at least one solution, we consider the operator

$$x = \lambda L^{-1} Fx, \quad \lambda \in [0, 1], \quad (2.17)$$

which is equivalent to (2.7). Let $x \in PC^1[0, 1]$ be any solution of (2.7), from (H₁), we have

$$-q(t)w(\|x\|_1) \leq x''(t) \leq q(t)w(\|x\|_1). \quad (2.18)$$

Consider the inequalities

$$\begin{aligned} x''(t) &\leq q(t)w(\|x\|_1), \\ x'(t_k) &= (1 + b_k)x(t_k), \\ x'(0) &= 0, \\ x''(t) &\geq -q(t)w(\|x\|_1), \\ x'(t_k) &= (1 + b_k)x(t_k), \\ x'(0) &= 0. \end{aligned} \quad (2.19)$$

By Lemma 1.3, we have

$$\begin{aligned} x'(t) &\leq w(\|x\|_1) \int_0^t \prod_{0 < t_k < t} (1 + b_k) q(s) ds \\ &\leq Qw(\|x\|_1), \\ x'(t) &\geq -w(\|x\|_1) \int_0^t \prod_{0 < t_k < t} (1 + b_k) q(s) ds \\ &\geq -Qw(\|x\|_1). \end{aligned} \quad (2.20)$$

From (2.20), we can deduce

$$|x'(t)| \leq Qw(\|x\|_1), \quad (2.21)$$

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and so

$$\|x'\| \leq Qw(\|x\|_1). \quad (2.22)$$

Using $x(t) = x(1) - \int_t^1 x'(s)ds - \sum_{t < t_k < 1} c_k x(t_k)$ and $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$, we have

$$x(t) = -\frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \left[\int_{\xi_i}^1 x'(s)ds + \sum_{\xi_i < t_k < 1} c_k x(t_k) \right] - \int_t^1 x'(s)ds - \sum_{t < t_k < 1} c_k x(t_k), \quad (2.23)$$

which implies

$$|x(t)| \leq \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) (\|x'\| + C\|x\|), \quad (2.24)$$

and so

$$\begin{aligned} \|x\| &\leq \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \left[1 - C \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \right]^{-1} \|x'\| \\ &\leq \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \left[1 - C \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \right]^{-1} Qw(\|x\|_1). \end{aligned} \quad (2.25)$$

Now, (2.22) together with (2.25) imply $\|x\|_1 \neq \widetilde{M}$. Set

$$U = \{u \in PC^1[0, 1] : \|u\|_1 < \widetilde{M}\}, \quad K = E = PC^1[0, 1], \quad (2.26)$$

then the nonlinear alternative of Leray-Schauder type [6] guarantees that $L^{-1}F$ has a fixed point, that is, (1.1) has a solution $x \in PC^1[0, 1]$, which completes the proof. $\square$

3. Examples

Example 3.1. Consider the boundary value problem

$$\begin{aligned} x'' &= f(t, x, x'), \quad t \in [0, 1], \quad t \neq \frac{1}{2}, \\ \Delta x'(t_k) &= \frac{1}{6} x'(t_k), \quad \Delta x(t_k) = \frac{1}{4} x(t_k), \quad t_k = \frac{1}{2}, \\ x'(0) &= 0, \quad x(1) = \frac{1}{2} x\left(\frac{1}{3}\right) - \frac{1}{3} x\left(\frac{2}{3}\right), \end{aligned} \quad (3.1)$$

where

$$f(t, u, v) = t^5 u + \frac{1}{2} t^3 v + t^2 [1 + \cos(u^{200} + v^{30})]. \quad (3.2)$$

It is easy to see that

$$|f(t, u, v)| \leq p(t)|u| + q(t)|v| + r(t) \quad (3.3)$$

with $p(t) = t^5$, $q(t) = (1/2)t^3$, $r(t) = 2t^2$. Clearly, $P = 1/6$, $Q = 1/8$, $B = 1/6$, $C = 1/4$, and

$$Q + B = \frac{7}{24} < 1, \quad \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|}\right) \left(\frac{P}{1 - Q - B} + C\right) = \frac{33}{34} < 1. \quad (3.4)$$

By Theorem 2.1, (3.1) has at least one solution.

Example 3.2. Consider the boundary value problem

$$\begin{aligned} x'' &= f(t, x, x'), \quad t \in [0, 1], \quad t \neq \frac{1}{2}, \\ \Delta x'(t_k) &= x'(t_k), \quad \Delta x(t_k) = \frac{1}{3}x(t_k), \quad t_k = \frac{1}{2}, \\ x'(0) &= 0, \quad x(1) = \frac{1}{2}x\left(\frac{1}{3}\right) - \frac{1}{2}x\left(\frac{2}{3}\right), \end{aligned} \quad (3.5)$$

where

$$f(t, u, v) = e^{-t}(u^\alpha + v^\beta) + \mu e^{-t} \quad (3.6)$$

with $\alpha \in [0, 1]$, $\beta \in [0, 1]$, $\mu > 0$. It is easy to see that

$$|f(t, u, v)| \leq q(t)w(\max\{|u|, |v|\}) \quad (3.7)$$

with $q(t) = e^{-t}$, $w(s) = s^\alpha + s^\beta + \mu$. Clearly

$$C \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|}\right) = \frac{2}{3} < 1, \quad (3.8)$$

$$\sup_{r \geq 0} \frac{r}{w(r)} = \sup_{r \geq 0} \frac{r}{r^\alpha + r^\beta + \mu} = \infty,$$

so (H_2) is true. Theorem 2.2 shows that (3.5) has at least one solution.

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