

*Research Article*

## On Comparison Principles for Parabolic Equations with Nonlocal Boundary Conditions

Yuandi Wang and Hamdi Zorgati

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A generalization of the comparison principle for a semilinear and a quasilinear parabolic equations with nonlocal boundary conditions including changing sign kernels is obtained. This generalization uses a positivity result obtained here for a parabolic problem with nonlocal boundary conditions.

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### 1. Introduction

The positivity of solutions for parabolic problems is the base of comparison principle which is important in monotonic methods used for these problems. Recently, Yin [1] developed several results in applications of the comparison principle, especially on nonlocal problems. Earlier works on problems with nonlocal boundary conditions can be found in [2], and some of references can be found in [1, 3]. In the literature, for example [2, 4–6], a restriction on the boundary condition (see (2.1)) of the kind

$$\int_{\Omega} |k(x, y)| dy < 1, \quad k(x, y) \geq 0, \quad (\text{AK})$$

where  $k$  represents the kernel of the nonlocal boundary condition, is sufficient to obtain the comparison principles. Recent results show that this restriction is not necessary for problems with lower regularity (see [3, Theorem 3.11] for problem with Dirichlet-type nonlocal boundary value). Moreover, in [7], an existence result for classical solutions of a parabolic problem with nonlocal boundary condition was obtained. In [8] we find an illustration of how the boundary kernel influences some results such as those on the eigenvalues problem and on the decay of solutions for evolution equation with a special kernel. In this paper, we give some general comparison results without the restriction

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(AK). Then, we use these results to discuss nonlocal boundary problems for a semilinear and a fully nonlinear equations.

### 2. Case of a semilinear equation

In this section, we are interested in the positivity of solution of the following problem:

$$\begin{aligned} u_t + A(t, x)u &\geq 0, \quad t > 0, x \in \Omega, \\ (\beta(t, x)\partial_\nu u + \alpha(t, x)u) &\geq \int_{\Omega} k(t, x; y)u(t, y)dy, \quad t > 0, x \in \Gamma, \\ u(0, x) &= u_0(x), \quad x \in \overline{\Omega}, \end{aligned} \quad (2.1)$$

where

$$A(t, x)u := -\mathbf{a}\nabla^2 u + \vec{b}\nabla u + cu \quad (2.2)$$

with  $\mathbf{a} := (a_{ij})_{n \times n}$ ,  $\vec{b} := \{b_1, \dots, b_n\}^T$ ,  $((\mathbf{a}, \vec{b}, c), (\alpha, \beta), k, u_0) \in C([0, T], \mathbb{E})$ ,  $\mathbb{E} := C(\overline{\Omega}, \mathbb{R}^{n^2+n+1}) \times C(\Gamma, \mathbb{R}^2) \times C(\Gamma \times \overline{\Omega}, \mathbb{R}) \times C^2(\overline{\Omega}, \mathbb{R})$ ,

$$\mathbf{a}\nabla^2 u = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \vec{b}\nabla u = \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i}, \quad (2.3)$$

and the elliptic operator  $A$  satisfies the following: there exists a  $\delta_0 > 0$  such that

$$\xi^T \mathbf{a} \xi \geq \delta_0 |\xi|^2, \quad \forall \xi \in \mathbb{R}^n. \quad (2.4)$$

The boundary  $\Gamma = \partial\Omega$  of the bounded domain  $\Omega \subset \mathbb{R}^n$  is a smooth  $(n-1)$ -dimensional manifold and  $\nu$  is the outward unit normal vector to  $\Gamma$ .

We also assume the following hypotheses.

(H\*)  $\alpha(t, x) \geq 1$ ,  $\beta(t, x) \geq 0$ ,  $k(t, x, y)$ , and  $u_0(x)$  satisfy the compatibility condition

$$\beta(0, x)\partial_\nu u + \alpha(0, x)u \geq \int_{\Omega} k(0, x; y)u_0(y)dy \quad \text{on } \Gamma. \quad (2.5)$$

Let  $Q_T = (0, T] \times \Omega$ . A (classical) solution  $u(t, x)$  of (2.1) should be in  $C^{1,2}(Q_T) \cap C^{0,1}(\overline{Q_T})$ . We have the following result.

**THEOREM 2.1.** *If  $u_0$  is nonnegative, then the solution  $u(t, x)$  of problem (2.1) is nonnegative.*

*Proof.* We can find a positive function  $\phi(x) \in C^2(\overline{\Omega})$  such that

$$\begin{aligned} \phi(x) &\equiv 1, \quad \partial_\nu \phi(x) \geq 0 \quad \text{on } \Gamma, \\ \min_{\overline{\Omega}} \phi(x) &\geq \varepsilon > 0, \\ \int_{\Omega} |k(t, x, y)\phi(y)| dy &< 1, \quad t \in [0, T], x \in \Gamma. \end{aligned} \quad (2.6)$$

Let us consider the function  $v := u/\phi$ . We have

$$\begin{aligned} v_t + \tilde{A}(t, x)v &\geq 0, \quad t > 0, x \in \Omega, \\ (\beta \partial_\nu v + \tilde{\alpha}v) &\geq \int_{\Omega} \tilde{k}(t, x; y)v(t, y)dy, \quad t > 0, x \in \Gamma, \\ v(0, x) &= v_0(x) := u_0(x)/\phi(x), \quad x \in \overline{\Omega}, \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} \tilde{A}(t, x)v &:= -\mathbf{a}\nabla^2 v + \vec{b}\nabla v + \tilde{c}v, \\ \tilde{\alpha} &:= \beta\partial_\nu\phi + \alpha, \\ \tilde{k}(t, x; y) &:= k(t, x; y)\phi(y), \end{aligned} \quad (2.8)$$

with

$$\vec{b} := -\frac{2}{\phi}(\nabla\phi)^T\mathbf{a} + \vec{b}, \quad \tilde{c} := -\frac{1}{\phi}[\mathbf{a}\nabla^2\phi - \vec{b}\nabla\phi] + c. \quad (2.9)$$

Without loss of generality, we can suppose that  $\tilde{c} > 0$ , otherwise, we replace  $v$  by  $e^{\lambda t}v$  with a  $\lambda > 0$  large enough to have  $\lambda + \tilde{c} > 0$ . Following the same approach in [2] and using (2.6) we show that  $v(t, x) \geq 0$ . In fact, suppose there exists a  $(t^*, x^*) \in (0, T] \times \overline{\Omega}$  such that  $v(t^*, x^*) < 0$ . If  $x^* \in \Gamma$  and  $v(t^*, x^*) = \min\{v(t, x) : (t, x) \in Q_{t^*}\} < 0$ , then using (2.6) we get

$$\begin{aligned} 0 > v(t^*, x^*) &\geq (\tilde{\alpha}v)|_{x^*} \geq (\beta\partial_\nu v + \tilde{\alpha}v)|_{x^*} \geq \int_{\Omega} \tilde{k}(t^*, x^*; y)v(t^*, y)dy \\ &\geq \int_{\Omega} |\tilde{k}(t^*, x^*; y)| dy v(t^*, x^*) > v(t^*, x^*), \end{aligned} \quad (2.10)$$

which is impossible. And if  $x^* \in \Omega$ , then using the first inequality in (2.7) we get

$$0 \leq (v_t + \tilde{A}v)|_{(t^*, x^*)} \leq \tilde{c}(t^*, x^*)v(t^*, x^*) < 0, \quad (2.11)$$

which is also impossible.

Therefore, we conclude that  $v(t, x) \geq 0$  on  $\overline{Q}_T$  and thus  $u \geq 0$  in  $\overline{Q}_T$ .  $\square$

*Remark 2.2.* The existence of the function  $\phi$  can be obtained by means of the function

$$\phi_{\varepsilon, \vartheta} = \begin{cases} 1, & x \in \Omega, \text{ dist}(x, \Gamma) < \vartheta, \\ \varepsilon, & x \in \Omega, \text{ dist}(x, \Gamma) > \vartheta. \end{cases} \quad \text{for small positive numbers } \varepsilon, \vartheta. \quad (2.12)$$

We define  $\phi$  by

$$\phi(x) = r^{-n} \int_{\Omega} \rho\left(\frac{x-y}{r}\right) \phi_{\varepsilon, \vartheta}(y) dy, \quad (2.13)$$

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where the constants  $\varepsilon$  and  $\vartheta$  are small enough so that (2.6) holds. Here  $r = \vartheta/4$  and

$$\rho(x) = \begin{cases} \left[ \int_{|y| \leq 1} e^{1/(|y|^2-1)} dy \right]^{-1} \cdot e^{1/(|x|^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases} \quad (2.14)$$

It is obvious that

$$\varepsilon \leq \phi(x) \leq 1, \quad \text{for } x \in \Omega, \quad \partial_\nu \phi|_\Gamma \equiv 0. \quad (2.15)$$

Let  $M = \sup\{|k(t, x, y)| : (t, x, y) \in [0, T] \times \partial\Omega \times \overline{\Omega}\}$ . If  $\theta$  and  $\varepsilon$  satisfy  $M(|\Gamma|(5\theta/4) + \varepsilon|\Omega|) < 1$ , where  $|\Omega|$  denotes the measure of  $\Omega$ , then (2.6) holds.

More generally, if  $\alpha \geq \alpha_0 > 0$ , we can get a similar result replacing  $k$  by  $k/(\alpha_0)$ .

In addition, for some special domains  $\Omega$ , we can construct  $\phi$  according to the geometry of  $\Omega$  as in the following example.

*Example 2.3.* Let us consider the following problem on  $B_R := \{x \in \mathbb{R}^n, |x| < R\}$ :

$$\begin{aligned} u_t - \Delta u &= 0, \quad x \in B_R, t > 0, \\ \partial_\nu u + \alpha u &= k \int_{B_R} u(t, y) dy, \quad |x| = R, t > 0, \\ u(0, x) &= u_0(x), \quad x \in \overline{B}_R, \end{aligned} \quad (2.16)$$

with the corresponding compatibility condition. In (2.16),  $\alpha$  and  $k$  are constants. Then,  $\phi$  can be chosen as the following:

$$\phi(x) = \begin{cases} \varepsilon + (1 - \varepsilon)(R^2 - \vartheta^2)^{-4}(|x|^2 - \vartheta^2)^4, & R - \vartheta \leq |x| \leq R, \\ \varepsilon, & |x| \leq R - \vartheta \end{cases} \quad (2.17)$$

with  $\varepsilon$  and  $\vartheta$  verifying

$$\partial_\nu \phi = \frac{8R(1 - \varepsilon)}{R^2 - \vartheta^2} \geq 0, \quad |k|((\varepsilon - 1)|B_{R-\vartheta}| + |B_R|) < 1. \quad (2.18)$$

*Remark 2.4.* The condition  $\alpha(t, x) \geq 1$  in (H\*) is not necessary. We can just assume that  $\alpha > 0$  on  $[0, T] \times \Gamma$  and we replace  $\beta$  and  $k$ , respectively, by  $\beta/\alpha$  and  $k/\alpha$ . This means that we can prove Theorem 2.1 without assuming  $\alpha(t, x) \geq 1$ .

Let us now consider the decay behavior of the following control problem:

$$\begin{aligned} u_t + A(x)u + \omega(x)u &= 0, \quad t > 0, x \in \Omega, \\ \beta(x)\partial_\nu u + \alpha(x)u &= \int_{\Omega} k(x; y)u(t, y) dy, \quad t > 0, x \in \Gamma, \\ u(0, x) &= u_0(x), \quad x \in \overline{\Omega}, \end{aligned} \quad (P_\omega)$$

where  $A$  is an elliptic operator defined as in (2.2) with  $((\mathbf{a}, \vec{b}, c), (\alpha, \beta), k, u_0) \in \mathbb{E}$ . Following the same approach as in [4], we obtain that the  $C$ -norm  $U(t) := \max_{\overline{\Omega}} |u(t, x)|$ ,  $u$  being the classical solution of problem  $(P_0)$  ( $\omega \equiv 0$  in  $(P_\omega)$ ) decays to zero exponentially provided that  $\int_{\Omega} |k(x; y)| dy < 1$ .

For any  $k(x, y) \in C(\Gamma \times \overline{\Omega})$ , we can find  $\omega$  and  $\phi$  such that

$$\tilde{c} + \omega \geq 0, \quad \int_{\Omega} |k(x; y)\phi(y)| dy < 1, \quad (2.19)$$

where  $\tilde{c}$  and  $\phi$  are defined in (2.6) and (2.9), and the functions  $\beta$ ,  $\alpha$ , and  $k$  also satisfy some corresponding conditions as in  $(H^*)$ . Hence, by using the same method as in [4], we have the following theorem.

**THEOREM 2.5.** *For any fixed  $k(x, y)$ , there exist a function  $\omega$  and positive constants  $M$  and  $\lambda$  such that the solution  $u$  of problem  $(P_\omega)$  satisfies*

$$\|u(t, \cdot)\|_{C(\overline{\Omega})} \leq M e^{-\lambda t}, \quad \forall t \geq 0. \quad (2.20)$$

We can look at the following one-dimensional example.

*Example 2.6.* Let  $\Omega = [a, 3\pi - a]$  with  $a \in (0, \pi/2)$ . The following problem

$$\begin{aligned} u_t - u_{xx} - u + \omega u &= 0, \quad \text{in } Q_T, \\ u(t, a) &= u(t, 3\pi - a) = \frac{1}{2} \tan a \int_a^{3\pi - a} u(t, y) dy, \\ u(0, x) &= \sin x \end{aligned} \quad (E_\omega)$$

has a solution  $u(t, x) \equiv \sin x$  when  $\omega = 0$ . But when  $\omega = 1$ ,  $(E_1)$  has a decay solution  $u = e^{-t} \sin x$ . We can see that  $\int_{\Omega} k dy = ((3\pi - 2a)/2) \tan a > 1$  when  $a \in (\arctan 1/\pi, \pi/2)$ .

We propose to use a positivity result of Theorem 2.1 in order to establish a comparison principle for a semilinear parabolic equation with nonlinear nonlocal boundary condition. Let us consider the following problem:

$$\begin{aligned} u_t - \mathbf{a} \nabla^2 u &= f(t, x, u, \nabla u) \quad \text{in } Q_T, \\ \beta \partial_{\nu} u + u &= \int_{\Omega} k(t, x, y; u(t, y)) dy \quad \text{on } (0, T) \times \Gamma, \\ u(0, x) &= u_0(x), \quad x \in \Omega, \end{aligned} \quad (\text{SP})$$

where  $\mathbf{a}$ ,  $\beta$ , and  $u_0$  satisfy the hypotheses above, and  $f$  and  $k$  satisfying the following hypotheses:

- (i)  $k(\cdot; u) \in C([0, T] \times \Gamma \times \overline{\Omega})$  and  $k(t, x, y; \cdot) \in C^1(\mathbb{R})$ ;
- (ii)  $f$  satisfies the following Lipschitz condition: there exists  $L_1, L_2 > 0$  such that

$$\begin{aligned} f(t, x, u, P) - f(t, x, v, P) &\leq L_1(u - v), \quad \text{if } u \geq v; \\ |f(t, x, u, P) - f(t, x, u, Q)| &\leq L_2|P - Q|. \end{aligned} \quad (2.21)$$

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A function  $u(t, x) \in C^{1,2}(Q_T) \cap C^{0,1}(\overline{Q}_T)$  is called an *upper solution* of (SP) on  $\overline{Q}_T$  if it satisfies

$$\begin{aligned} u_t - \mathbf{a} \nabla^2 u &\geq f(t, x, u, \nabla u) \quad \text{in } Q_T, \\ \beta \partial_\nu u + u &\geq \int_{\Omega} k(t, x, y; u(t, y)) dy \quad \text{on } (0, T) \times \Gamma, \\ u(0, x) &\geq u_0(x), \quad x \in \Omega. \end{aligned} \quad (2.22)$$

A *lower solution* is defined analogously by reversing the inequalities in (2.22). A *solution*  $u$  of problem (SP) means that  $u$  is both an upper and a lower solutions.

**THEOREM 2.7.** *If  $u, v$  are, respectively, an upper and a lower solutions of the problem (SP), then  $u \geq v$  for all  $(t, x) \in \overline{Q}_T$ .*

*Proof.* Let us consider the function  $w(t, x) = u(t, x) - v(t, x)$ . This function verifies

$$\begin{aligned} w_t - \mathbf{a} \nabla^2 w &\geq f(t, x, u, \nabla u) - f(t, x, v, \nabla v) \quad \text{in } Q_T, \\ \beta \partial_\nu w + w &\geq \int_{\Omega} k_u(t, x, y; \xi(t, y)) w(t, y) dy \quad \text{on } (0, T) \times \Gamma, \\ w(0, x) &= u_0(x) - v_0(x) \geq 0, \quad x \in \Omega \end{aligned} \quad (2.23)$$

with  $\xi$  situated between  $u$  and  $v$ .

We note that the right-hand side of the first inequality in (2.23) depends on  $u$  and  $\nabla u$ , thus, Theorem 2.1 cannot be applied directly. We introduce

$$w(t, x) = V(t, x) \phi(x) e^{\lambda t}, \quad (2.24)$$

where  $\phi(x)$  satisfies (2.6) with  $k(t, x, y)$  replaced by  $k_u(t, x, y, \xi(t, y))$  and

$$\lambda > L_1 + \max_{\overline{\Omega}} \left\{ \frac{L_2 |\nabla \phi(x)| + \mathbf{a} \nabla^2 \phi(x)}{\phi(x)} \right\}. \quad (2.25)$$

If there is a point  $(t, x) \in (0, T] \times \overline{\Omega}$  such that  $w(t, x) < 0$ , then  $V$  will attain its negative minimum at some point  $(t_1, x_1)$  with

$$V(t_1, x_1) < 0, \quad V_t(t_1, x_1) \leq 0, \quad \nabla V(t_1, x_1) = 0. \quad (2.26)$$

Hence, using the hypotheses on  $f$ , we obtain a contradiction since we have

$$0 \geq V_t \geq - \left( \lambda - L_1 - \frac{L_2 |\nabla \phi|}{\phi} - \frac{\mathbf{a} \nabla^2 \phi}{\phi} \right) V > 0 \quad \text{at } (t_1, x_1) \text{ if } x_1 \in \Omega. \quad (2.27)$$

We obtain also a contradiction if  $x_1 \in \Gamma$  since we have

$$\int_{\Omega} |k_u(t_1, x_1, y, \xi(t_1, y))| \phi(y) dy < 1. \quad (2.28)$$

We thus conclude that  $V \geq 0$ , and therefore,  $w(t, x) \geq 0$  on  $\overline{Q}_T$ .  $\square$

A similar result can be obtained for parabolic systems with changing-sign kernels. Note that in [9, Example 2.1], the kernel  $K_{ij}$  appearing in the boundary condition is assumed to be positive.

*Remark 2.8.* From the above discussion, the result of Theorem 2.7 holds true if we just assume  $k$  and  $f$  to be locally (one side) Lipschitz continuous, respectively, on  $u$  and  $\nabla u$ , that is,  $k(\cdot, u) \in C([0, T] \times \Gamma \times \bar{\Omega})$  for any fixed  $u$  and there exists  $L, L_1, L_2 > 0$  such that

$$\left. \begin{aligned} & |k(t, x, y, u) - k(t, x, y, v)| \leq L(\rho)|u - v|; \\ & f(t, x, u, P) - f(t, x, v, P) \leq L_1(\rho)(u - v), \quad \text{if } u \geq v; \\ & |f(t, x, u, P) - f(t, x, u, Q)| \leq L_2(\rho)|P - Q| \end{aligned} \right\} \text{when } |u|, |v| \leq \rho. \quad (2.29)$$

The uniqueness of the solution of problem (SP) is a direct consequence of Theorem 2.7. Using the upper and lower solutions, some existence theorems of the solutions for problem (SP) will be obtained by monotonicity methods (see [2]). We can also discuss the quadric convergence of iterative series constructed using upper and lower solutions (see [10]). Here we do not give more details about that.

### 3. A fully nonlinear equation

Let us consider a general nonlinear parabolic equation with nonlinear and nonlocal boundary conditions

$$\begin{aligned} u_t &= f(t, x, u, \nabla u, \nabla^2 u) \quad \text{in } Q_T, \\ \beta \partial_\nu u + u &= \int_{\Omega} k(t, x, y; u) dy \quad \text{on } (0, T] \times \Gamma, \\ u(0, x) &= u_0(x) \quad \text{in } \Omega, \end{aligned} \quad (\text{Pf})$$

where  $f \in C(\bar{Q}_T \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}, \mathbb{R})$ ,  $\nabla u = (u_{x_1}, \dots, u_{x_n})$ , and  $\nabla^2 u = (u_{x_1 x_1}, u_{x_1 x_2}, \dots, u_{x_n x_n})$ .

In order to establish the comparison principle, we give a definition of elliptic function. We say that  $f \in C(\bar{Q}_T \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}, \mathbb{R})$  is *elliptic* at point  $(t_0, x_0)$  if for any  $u, P, R, S$  with  $R = (R_{ij})_{n \times n}$ ,  $S = (S_{ij})_{n \times n}$ , verifying  $\Lambda^T (R - S) \Lambda \geq 0$  for any vector  $\Lambda \in \mathbb{R}^n$ , we have  $f(t_0, x_0, u, P, R) \geq f(t_0, x_0, u, P, S)$ . If  $f$  is elliptic for every  $(t, x) \in Q_T$ , then  $f$  is said to be *elliptic* in  $Q_T$ . In the remainder of this paper, we assume  $f$  to be elliptic in  $Q_T$ .

A function  $u(t, x) \in C^{1,2}(Q_T) \cap C^{0,1}(\bar{Q}_T)$  is said to be an upper solution (resp., a lower solution) of problem (Pf) on  $\bar{Q}_T$  if  $u$  satisfies the following system:

$$\begin{aligned} u_t &\geq (\leq) f(t, x, u, \nabla u, \nabla^2 u) \quad \text{in } Q_T, \\ \beta \partial_\nu u + u &\geq (\leq) \int_{\Omega} k(t, x, y; u) dy \quad \text{on } (0, T] \times \Gamma, \\ u(0, x) &\geq (\leq) u_0(x) \quad \text{in } \Omega. \end{aligned} \quad (3.1)$$

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Assuming  $\beta$  to be positive,  $k$  to be continuous, and there exists a nonnegative  $C([0, T] \times \Gamma \times \bar{\Omega})$ -function  $L_2$  verifying

$$k(t, x, y, u) - k(t, x, y, v) \geq L_2(t, x, y)(u - v) \quad \text{if } u \geq v, \quad (3.2)$$

we get the following theorem.

**THEOREM 3.1.** *Let  $u$  and  $v$  be, respectively, an upper and lower solutions of problem (Pf). Suppose  $u(0, x) > v(0, x)$  and one of the first two inequalities in (3.1) to be strict. Then  $u(t, x) > v(t, x)$  on  $\bar{Q}_T$ .*

*Proof.* Let us consider the function  $U(t, x) = u(t, x) - v(t, x)$ . If the conclusion was not true, then the initial condition implies that  $U(t, x) > 0$  for some  $t > 0$  and there exists  $(t_1, x_1) \in \bar{Q}_T$  such that  $U(t_1, x_1) = 0$ . We can assume that  $(t_1, x_1)$  is the first nonnegative maximum point, that is,

$$U(t, x) > 0, \quad \forall t < t_1, x \in \bar{\Omega}. \quad (3.3)$$

We have that  $(t_1, x_1) \notin Q_T$ . In fact, if  $(t_1, x_1) \in Q_T$ , then we have

$$U_t \leq 0, \quad \nabla U = 0, \quad \Lambda^T (U_{x_i x_j})_{n \times n} \Lambda \geq 0 \quad \text{at } (t_1, x_1). \quad (3.4)$$

Using the ellipticity of  $f$ , we obtain that

$$U_t(t_1, x_1) > f(t_1, x_1, u, \nabla u, \nabla^2 u) - f(t_1, x_1, v, \nabla v, \nabla^2 v) \geq 0, \quad (3.5)$$

which is in contradiction with (3.4). Hence,  $U(t, x) > 0$  in  $Q_{t_1}$ . We have also  $(t_1, x_1) \notin (0, T] \times \Gamma$ . Otherwise,

$$0 \geq \beta \partial_\nu U + U \geq \int_{\Omega} L_2 U \, dy > 0, \quad \text{at } (t_1, x_1), \quad (3.6)$$

which leads to a contradiction again.

Finally, we conclude that  $U(t, x) > 0$ , that is,  $u(t, x) > v(t, x)$  on  $\bar{Q}_T$ .  $\square$

Let us now assume  $\beta$  to be positive,  $f$  satisfying locally one-side Lipschitz conditions, that is, for  $|u| \leq \rho$  and  $|v| \leq \rho$ , there exists a constant  $L_1(\rho)$  such that

$$f(t, x, u, P, R) - f(t, x, v, P, R) \leq L_1(u - v), \quad \text{if } u \geq v. \quad (3.7)$$

We also assume  $k$  to be continuous and there exist two nonnegative  $C([0, T] \times \Gamma \times \bar{\Omega})$ -functions,  $L_2$  and  $\bar{L}_2$ , such that

$$L_2(t, x, y)(u - v) \leq k((t, x, y); u) - k((t, x, y); v) \leq \bar{L}_2(t, x, y)(u - v), \quad \text{if } u \geq v. \quad (3.8)$$

Then, for  $\varepsilon > 0$ , it is obvious that

$$(\varepsilon e^{\delta t})_t = \delta \varepsilon e^{\delta t} > f(t, x, u + \varepsilon e^{\delta t}, \nabla(u + \varepsilon e^{\delta t}), \nabla^2(u + \varepsilon e^{\delta t})) - f(t, x, u, \nabla u, \nabla^2 u) \quad (3.9)$$

whenever  $\delta > L_1$ .

Let  $\tilde{u} = u + \varepsilon e^{\delta t}$  with  $\delta > L_1$  and suppose  $\bar{L}_2 |\Omega| < 1$ , then

$$\begin{aligned} \tilde{u}_t &= u_t + \delta \varepsilon e^{\delta t} > f(t, x, \tilde{u}, \nabla \tilde{u}, \nabla^2 \tilde{u}), \quad \text{in } Q_T, \\ \beta \partial_\nu \tilde{u} + \tilde{u} &\geq \varepsilon e^{\delta t} + \int_{\Omega} k(t, x, y; u) dy > \int_{\Omega} k(t, x, y; \tilde{u}) dy, \quad \text{on } (0, T] \times \Gamma, \\ \tilde{u}(0, x) &= u(0, x) + \varepsilon, \quad \text{in } \Omega. \end{aligned} \quad (3.10)$$

This means that  $\tilde{u}$  is a (strict) upper solution as well as  $u$ . Letting  $\varepsilon \rightarrow 0^+$  and using Theorem 3.1, we obtain the following corollary.

**COROLLARY 3.2.** *Under the above assumptions, if  $u$  and  $v$  are, respectively, the upper and the lower solutions of problem (Pf) and if  $\bar{L}_2 |\Omega| < 1$ , then  $u(t, x) \geq v(t, x)$  on  $\bar{Q}_T$ .*

The uniqueness of the solution for problem (Pf) can be easily obtained and an extension to a fully nonlinear system can be derived.

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Yuandi Wang: Department of Mathematics, Shanghai University, Shanghai 200444, China  
*Email address:* ydwang@mail.shu.edu.cn

Hamdi Zorgati: Institut für Mathematik, Universität Zürich, Winterthurerstr. 190, CH-8057 Zürich, Switzerland

*Current address:* Department of Mathematics, Campus Universitaire, University of Tunis, Elmanar 2092, Tunisia  
*Email address:* hamdi.zorgati@fst.rnu.tn

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