

Research Article

On the Sets of Regularity of Solutions for a Class of Degenerate Nonlinear Elliptic Fourth-Order Equations with L^1 Data

S. Bonafede and F. Nicolosi

Received 24 January 2007; Accepted 29 January 2007

Recommended by V. Lakshmikantham

We establish Hölder continuity of generalized solutions of the Dirichlet problem, associated to a degenerate nonlinear fourth-order equation in an open bounded set $\Omega \subset \mathbb{R}^n$, with L^1 data, on the subsets of Ω where the behavior of weights and of the data is regular enough.

Copyright © 2007 S. Bonafede and F. Nicolosi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

In this paper, we will deal with equations involving an operator $A : \overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega) \rightarrow (\overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega))^*$ of the form

$$Au = \sum_{|\alpha|=1,2} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \nabla_2 u), \quad (1.1)$$

where Ω is a bounded open set of \mathbb{R}^n , $n > 4$, $2 < p < n/2$, $\max(2p, \sqrt{n}) < q < n$, ν and μ are positive functions in Ω with properties precised later, $\overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$ is the Banach space of all functions $u : \Omega \rightarrow \mathbb{R}$ with the properties $|u|^q, \nu|D^\alpha u|^q, \mu|D^\beta u|^p \in L^1(\Omega)$, $|\alpha| = 1$, $|\beta| = 2$, and “zero” boundary values; $\nabla_2 u = \{D^\alpha u : |\alpha| \leq 2\}$.

The functions A_α satisfy growth and monotonicity conditions, and in particular, the following strengthened ellipticity condition (for a.e. $x \in \Omega$ and $\xi = \{\xi_\alpha : |\alpha| = 1, 2\}$):

$$\sum_{|\alpha|=1,2} A_\alpha(x, \xi) \xi_\alpha \geq c_2 \left\{ \sum_{|\alpha|=1} \nu(x) |\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x) |\xi_\alpha|^p \right\} - g_2(x), \quad (1.2)$$

where $c_2 > 0$, $g_2(x) \in L^1(\Omega)$.

2 Boundary Value Problems

We will assume that the right-hand sides of our equations, depending on unknown function, belong to $L^1(\Omega)$.

A model representative of the given class of equations is the following:

$$\begin{aligned} - \sum_{|\alpha|=1} D^\alpha \left[\nu \left(\sum_{|\beta|=1} |D^\beta u|^2 \right)^{(q-2)/2} D^\alpha u \right] + \sum_{|\alpha|=2} D^\alpha \left[\mu \left(\sum_{|\beta|=2} |D^\beta u|^2 \right)^{(p-2)/2} D^\alpha u \right] \\ = -|u|^{\sigma-1} u + f \quad \text{in } \Omega, \end{aligned} \quad (1.3)$$

where $\sigma > 1$ and $f \in L^1(\Omega)$.

The assumed conditions and known results of the theory of monotone operators allow us to prove existence of generalized solutions of the Dirichlet problem associated to our operator (see, e.g., [1]), bounded on the sets $G \subset \Omega$ where the behavior of weights and of the data of the problem is regular enough (see [2]).

In our paper, following the approach of [3], we establish on such sets a result on Hölder continuity of generalized solutions of the same Dirichlet problem.

We note that for one high-order equation with degenerate nonlinear operator satisfying a strengthened ellipticity condition, regularity of solutions was studied in [4, 5] (non-degenerate case) and in [6, 7] (degenerate case). However, it has been made for equations with right-hand sides in L^t with $t > 1$.

2. Hypotheses

Let $n \in \mathbb{N}$, $n > 4$, and let Ω be a bounded open set of \mathbb{R}^n . Let p, q be two real numbers such that $2 < p < n/2$, $\max(2p, \sqrt{n}) < q < n$.

Let $\nu : \Omega \rightarrow \mathbb{R}^+$ be a measurable function such that

$$\nu \in L^1_{\text{loc}}(\Omega), \quad \left(\frac{1}{\nu} \right)^{1/(q-1)} \in L^1_{\text{loc}}(\Omega). \quad (2.1)$$

$W^{1,q}(\nu, \Omega)$ is the space of all functions $u \in L^q(\Omega)$ such that their derivatives, in the sense of distribution, $D^\alpha u$, $|\alpha| = 1$, are functions for which the following properties hold: $\nu^{1/q} D^\alpha u \in L^q(\Omega)$ if $|\alpha| = 1$; $W^{1,q}(\nu, \Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,q,\nu} = \left(\int_{\Omega} |u|^q dx + \sum_{|\alpha|=1} \int_{\Omega} \nu |D^\alpha u|^q dx \right)^{1/q}. \quad (2.2)$$

$\overset{\circ}{W}{}^{1,q}(\nu, \Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\nu, \Omega)$.

Let $\mu(x) : \Omega \rightarrow \mathbb{R}^+$ be a measurable function such that

$$\mu \in L^1_{\text{loc}}(\Omega), \quad \left(\frac{1}{\mu} \right)^{1/(p-1)} \in L^1_{\text{loc}}(\Omega). \quad (2.3)$$

$W_{2,p}^{1,q}(\nu, \mu, \Omega)$ is the space of all functions $u \in W^{1,q}(\nu, \Omega)$, such that their derivatives, in the sense of distribution, $D^\alpha u$, $|\alpha| = 2$, are functions with the following properties:

$\mu^{1/p} D^\alpha u \in L^p(\Omega)$, $|\alpha| = 2$; $W_{2,p}^{1,q}(\nu, \mu, \Omega)$ is a Banach space with respect to the norm

$$\|u\| = \|u\|_{1,q,\nu} + \left(\sum_{|\alpha|=2} \int_{\Omega} \mu |D^\alpha u|^p dx \right)^{1/p}. \quad (2.4)$$

$\overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W_{2,p}^{1,q}(\nu, \mu, \Omega)$.

Hypothesis 2.1. Let $\nu(x)$ be a measurable positive function:

$$\begin{aligned} \frac{1}{\nu} &\in L^t(\Omega) \quad \text{with } t > \frac{nq}{q^2 - n}, \\ \nu &\in L^{\bar{t}}(\Omega) \quad \text{with } \bar{t} > \frac{nt}{qt - n}. \end{aligned} \quad (2.5)$$

We put $\tilde{q} = nqt/(n(1+t) - qt)$. We can easily prove that a constant $c_0 > 0$ exists such that if $u \in W^{1,q}(\nu, \Omega)$, the following inequality holds:

$$\int_{\Omega} |u|^{\tilde{q}} dx \leq c_0 \left\{ \int_{\text{supp } u} \left(\frac{1}{\nu} \right)^t dx \right\}^{\tilde{q}/qt} \left\{ \sum_{|\alpha|=1} \int_{\Omega} \nu |D^\alpha u|^q dx \right\}^{\tilde{q}/q}. \quad (2.6)$$

We set $\tilde{\nu} = \mu^{q/(q-2p)} (1/\nu)^{2p/(q-2p)}$.

Hypothesis 2.2. $\tilde{\nu} \in L^1(\Omega)$.

Hypothesis 2.3. There exists a real number $r > \tilde{q}(q-1)/(\tilde{q}(q-1)(p-1) - q)$ such that

$$\frac{1}{\mu} \in L^r(\Omega). \quad (2.7)$$

For more details about weight functions, see [8, 9].

Let Ω_1 be a nonempty open set of \mathbb{R}^n such that $\Omega_1 \subset \Omega$.

Definition 2.4. It is said that G closed set of \mathbb{R}^n is a “regular set” if $\overset{\circ}{G}$ is nonempty and $G \subset \Omega_1$.

Denote by $\mathbb{R}^{n,2}$ the space of all sets $\xi = \{\xi_\alpha \in \mathbb{R} : |\alpha| = 1, 2\}$ of real numbers; if a function $u \in L^1_{\text{loc}}(\Omega)$ has the weak derivatives $D^\alpha u$, $|\alpha| = 1, 2$ then $\nabla_2 u = \{D^\alpha u : |\alpha| = 1, 2\}$. Suppose that $A_\alpha : \Omega \times \mathbb{R}^{n,2} \rightarrow \mathbb{R}$ are Carathéodory functions.

Hypothesis 2.5. There exist $c_1, c_2 > 0$ and $g_1(x), g_2(x)$ nonnegative functions such that $g_1, g_2 \in L^1(\Omega)$ and, for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^{n,2}$, the following inequalities

4 Boundary Value Problems

hold:

$$\begin{aligned} \sum_{|\alpha|=1} [\nu(x)]^{-1/(q-1)} |A_\alpha(x, \xi)|^{q/(q-1)} + \sum_{|\alpha|=2} [\mu(x)]^{-1/(p-1)} |A_\alpha(x, \xi)|^{p/(p-1)} \\ \leq c_1 \left\{ \sum_{|\alpha|=1} \nu(x) |\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x) |\xi_\alpha|^p \right\} + g_1(x), \end{aligned} \quad (2.8)$$

$$\sum_{|\alpha|=1,2} A_\alpha(x, \xi) \xi_\alpha \geq c_2 \left\{ \sum_{|\alpha|=1} \nu(x) |\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x) |\xi_\alpha|^p \right\} - g_2(x). \quad (2.9)$$

Moreover, we will assume that for almost every $x \in \Omega$ and every $\xi, \xi' \in \mathbb{R}^{n,2}$, $\xi \neq \xi'$,

$$\sum_{|\alpha|=1,2} [A_\alpha(x, \xi) - A_\alpha(x, \xi')] (\xi_\alpha - \xi'_\alpha) > 0. \quad (2.10)$$

Let $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that

- (a) for almost every $x \in \Omega$, the function $F(x, \cdot)$ is nonincreasing in \mathbb{R} ;
- (b) for every $x \in \Omega$, the function $F(\cdot, s)$ belongs to $L^1(\Omega)$.

Let $A : \overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega) \rightarrow (\overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega))^*$ be the operator such that for every $u, v \in \overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$,

$$\langle Au, v \rangle = \int_{\Omega} \left\{ \sum_{|\alpha|=1,2} A_\alpha(x, \nabla_2 u) D^\alpha v \right\} dx. \quad (2.11)$$

We consider the following Dirichlet problem:

$$(P) = \begin{cases} Au = F(x, u) & \text{in } \Omega \\ D^\alpha u = 0, \quad |\alpha| = 0, 1, & \text{on } \partial\Omega. \end{cases} \quad (2.12)$$

Definition 2.6. A W -solution of problem (P) is a function $u \in \overset{\circ}{W}^{2,1}(\Omega)$ such that

- (i) $F(x, u) \in L^1(\Omega)$;
- (ii) $A_\alpha(x, \nabla_2 u) \in L^1(\Omega)$, for every $\alpha : |\alpha| = 1, 2$;
- (iii) $\langle Au, \phi \rangle = \langle F(x, u), \phi \rangle$ in distributional sense.

It is well known that Hypotheses 2.1–2.3, 2.5, and assumptions on $F(x, s)$ imply the existence of a W -solution of problem (P) (see [1]). Moreover, a boundedness local result for such solution has been established in [2] under more restrictive hypotheses on data and weight functions.

More precisely, the following holds (see [2, Theorem 5.1]).

THEOREM 2.7. *Suppose that Hypotheses 2.1–2.3 and 2.5 are satisfied. Let $q_1 \in (q, \tilde{q}(q-1)/q)$, $\tau > \tilde{q}/(\tilde{q}-q_1)$. Assume that restrictions of the functions $\nu^{q_1/(q_1-q)}$, $\tilde{\nu}$, g_1 , g_2 , and $|F(\cdot, 0)|^{q_1/(q_1-1)}$ on G belong to $L^\tau(G)$, for every “regular set” G .*

Then there exists \bar{u} W -solution of problem (P) such that for every G , $\text{ess}_G \sup |\bar{u}| \leq M_G < +\infty$, with M_G positive constant depending only on known values.

3. Main result

In the sequel of paper, G will be a “*regular set*.” In order to obtain our regularity result on G , we need the following further hypotheses.

Hypothesis 3.1. There exists a constant $c' > 0$ such that for all $y \in \overset{\circ}{G}$ and for all $\rho > 0$, with $\overline{B(y, \rho)} \subset G$, we have

$$\left\{ \rho^{-n} \int_{B(y, \rho)} \left(\frac{1}{\nu} \right)^t dx \right\}^{1/t} \left\{ \rho^{-n} \int_{B(y, \rho)} \nu^\tau dx \right\}^{1/\tau} \leq c'. \quad (3.1)$$

With regard to this assumption, see [3].

Hypothesis 3.2. There exist a real positive number σ and two real functions $h(x)(\geq 0)$, $f(x)(> 0)$ defined on G , such that

$$|F(x, s)| \leq h(x)|s|^\sigma + f(x), \quad \text{for almost every } x \in G \text{ and every } s \in \mathbb{R}. \quad (3.2)$$

Moreover, we assume that

$$h(x), f(x) \in L^\tau(G), \quad (3.3)$$

with τ defined as above.

Using considerations stated in [1], following the approach of [3], we establish the following result.

THEOREM 3.3. *Let all above-stated hypotheses hold and let conditions of Theorem 2.7 be satisfied. Then, the W -solution \bar{u} of Dirichlet problem (P), essentially bounded on G , is also locally Hölderian on G .*

More precisely, there exist positive constant C and λ ($0 < \lambda < 1$) such that for every open set $\Omega' \subset \overset{\circ}{G}$, and every $x, y \in \Omega'$

$$|\bar{u}(x) - \bar{u}(y)| \leq C[d(\Omega', \partial G)]^{-\lambda} |x - y|^\lambda, \quad (3.4)$$

where C and λ depend only on $c_1, c_2, c_0, c', n, q, p, t, \tau, \sigma, M_G, \text{diam } G, \text{meas } G, \|f\|_{L^\tau(G)}, \|h\|_{L^\tau(G)}, \|g_1\|_{L^\tau(G)}, \|g_2\|_{L^\tau(G)}, \|\tilde{\nu}\|_{L^\tau(G)}$, and $\|1/\nu\|_{L^t(\Omega)}$.

Proof. For every $l \in \mathbb{N}$, we define the function $F_l : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$F_l(x, s) = \begin{cases} -l & \text{if } F(x, 0) - F(x, s) < -l, \\ F(x, 0) - F(x, s) & \text{if } |F(x, 0) - F(x, s)| \leq l, \\ l & \text{if } F(x, 0) - F(x, s) > l, \end{cases} \quad (3.5)$$

and the function $f_l : \Omega \rightarrow \mathbb{R}$ by

$$f_l(x) = \begin{cases} F(x, 0) & \text{if } |F(x, 0)| \leq l, \\ 0 & \text{if } |F(x, 0)| > l. \end{cases} \quad (3.6)$$

6 Boundary Value Problems

By Lebesgue's theorem and property (b) of $F(x, s)$, we have that $f_l(x)$ goes to $F(x, 0)$ in $L^1(\Omega)$.

Next, inequalities (2.6), (2.8)–(2.10), property (a) of $F(x, s)$, and known results of the theory of monotone operators (see, e.g., [10]) imply that for any $l \in \mathbb{N}$, there exists $u_l \in \overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$ such that

$$\int_{\Omega} \left\{ \sum_{|\alpha|=1,2} A_{\alpha}(x, \nabla_2 u_l) D^{\alpha} \nu + F_l(x, u_l) \nu \right\} dx = \int_{\Omega} f_l \nu dx, \quad (3.7)$$

for every $\nu \in \overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$.

From considerations stated in [1, Section 3], we deduce that there exists a W -solution \bar{u} of problem (P) such that

$$u_l \rightharpoonup \bar{u} \quad \text{a.e. in } \Omega. \quad (3.8)$$

Moreover, see proof of Theorem 2.7,

$$\operatorname{ess\,sup}_G |u_l| \leq M_G, \quad \text{for every } l \in \mathbb{N}. \quad (3.9)$$

We set $\bar{n} = q^2/(q - 2p)$, $a = (1/\bar{n})(q - \bar{n}/t - n/\tau)$.

Let us fix $y \in G$, $\rho > 0$ and $B(y, 2\rho) \subset G$. Let us put

$$\begin{aligned} \omega_{1,l} &= \operatorname{ess\,inf}_{B(y, 2\rho)} u_l, & \omega_{2,l} &= \operatorname{ess\,sup}_{B(y, 2\rho)} u_l, \\ \omega_l &= \omega_{2,l} - \omega_{1,l}. \end{aligned} \quad (3.10)$$

We will show that

$$\operatorname{osc}_{B(y, \rho)} \{u_l\} \leq \tilde{c} \omega_l + \rho^a, \quad (3.11)$$

with $\tilde{c} \in]0, 1[$ independent of $l \in \mathbb{N}$.

To this aim, we fix $l \in \mathbb{N}$ and we set

$$\begin{aligned} \Phi_l &= \sum_{|\alpha|=1} \nu |D^{\alpha} u_l|^q + \sum_{|\alpha|=2} \mu |D^{\alpha} u_l|^p, \\ \psi(x) &= \rho^{-a\bar{n}} (1 + f(x) + h(x) + g_1(x) + g_2(x) + \tilde{\nu}(x)) + \rho^{-q} \nu. \end{aligned} \quad (3.12)$$

Obviously, we will assume that

$$\omega_l \geq \rho^a \quad (\text{otherwise, it is clear that (3.11) is true}). \quad (3.13)$$

We introduce now the following functions:

$$F_{1,l}(x) = \begin{cases} \frac{2e\omega_l}{u_l(x) - \omega_{1,l} + \rho^a} & \text{if } x \in B(y, 2\rho), \\ e & \text{if } x \in \Omega \setminus B(y, 2\rho); \end{cases} \quad (3.14)$$

$\varphi \in C_0^\infty(\Omega)$: $0 \leq \varphi \leq 1$ in Ω , $\varphi = 0$ in $\Omega \setminus B(y, 2\rho)$ and satisfying

$$|D^\alpha \varphi| \leq \bar{c} \rho^{-|\alpha|}, \quad |\alpha| = 1, 2, \quad (3.15)$$

where the positive constant \bar{c} depends only on n .

Let us fix $s > q$ and $r \geq 0$ and define

$$\begin{aligned} v_l &= (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^s, \\ z_l &= -\frac{1}{2e\omega_l} [r(\lg F_{1,l})^{r-1} + (q-1)(\lg F_{1,l})^r] F_{1,l}^q \varphi^s. \end{aligned} \quad (3.16)$$

From Hypothesis 2.2 and (3.15), we have that $v_l \in \overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$ and the next inequalities are true:

$$|D^\alpha v_l - z_l D^\alpha u_l| \leq \bar{c} s \varphi^{s-1} (\lg F_{1,l})^r F_{1,l}^{q-1} \rho^{-1} \quad \text{if } |\alpha| = 1 \text{ a.e. in } B(y, 2\rho), \quad (3.17)$$

$$\begin{aligned} |D^\alpha v_l - z_l D^\alpha u_l| &\leq 5q^2 s(r+1)^2 (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^s \left\{ \sum_{|\beta|=1} \frac{|D^\beta u_l|^2}{(u_l - \omega_{1,l} + \rho^q)^2} \right\} \\ &\quad + 2nq s^2 \bar{c}^2 \rho^{-2} (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^{s-2} \quad \text{if } |\alpha| = 2 \text{ a.e. in } B(y, 2\rho). \end{aligned} \quad (3.18)$$

Since $u_l(x)$ satisfies (3.7), for $\nu = v_l$, we obtain

$$\int_{\Omega} \left\{ \sum_{|\alpha|=1,2} A_\alpha(x, \nabla_2 u_l) D^\alpha v_l + F_l(x, u_l) v_l \right\} dx = \int_{\Omega} f_l v_l dx. \quad (3.19)$$

From this, taking into account (3.9) and Hypothesis 3.2, we have

$$\int_{\Omega} \sum_{|\alpha|=1,2} A_\alpha(x, \nabla_2 u_l) D^\alpha v_l dx \leq (3 + M_G^\sigma) \int_{\Omega} \{1 + f(x) + h(x)\} v_l dx. \quad (3.20)$$

Hence

$$\int_{\Omega} \sum_{|\alpha|=1,2} \{A_\alpha(x, \nabla_2 u_l) D^\alpha u_l\} (-z_l) dx \leq (3 + M_G^\sigma) \int_{\Omega} \{1 + f(x) + h(x)\} v_l dx + I_1 + I_2, \quad (3.21)$$

where

$$I_i = \int_{\Omega} \sum_{|\alpha|=i} |A_\alpha(x, \nabla_2 u_l)| |D^\alpha v_l - z_l D^\alpha u_l| dx, \quad i = 1, 2. \quad (3.22)$$

Using Hypothesis 2.5 and definition of z_l , we have

$$\begin{aligned} \frac{(q-1)c_2}{2e\omega_l} \int_{\Omega} \Phi_l(\lg F_{1,l})^r F_{1,l}^q \varphi^s dx &\leq (3 + M_G^\sigma) \int_{\Omega} \{1 + f(x) + h(x)\} (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^s dx \\ &\quad + \int_{\Omega} g_2(x) (-z_l) dx + I_1 + I_2. \end{aligned} \quad (3.23)$$

8 Boundary Value Problems

Note that

$$\begin{aligned} F_{1,l}^{q-1} &\leq (\text{diam } G)^a (2e\omega_l)^{q-1} \rho^{-aq}, \\ -z_l &\leq (q-1)(r+1) (2e\omega_l)^{q-1} \rho^{-aq} \varphi^s (\lg F_{1,l})^r \quad \text{a.e. in } B(y, 2\rho), \end{aligned} \quad (3.24)$$

consequently, from (3.23), we obtain

$$\begin{aligned} &\frac{c_2}{2e\omega_l} \int_{B(y, 2\rho)} \Phi_l (\lg F_{1,l})^r F_{1,l}^q \varphi^s dx \\ &\leq c_3 (r+1) (2e\omega_l)^{q-1} \int_{B(y, 2\rho)} \rho^{-aq} \{1 + f(x) + h(x) + g_2(x)\} (\lg F_{1,l})^r \varphi^s dx + I_1 + I_2, \end{aligned} \quad (3.25)$$

where $c_3 = (q-1)(3+M_G^\sigma)(\text{diam } G+1)$.

Let us fix $|\alpha| = 1$. Let $\epsilon > 0$, then, applying Young's inequality and using (2.8) and (3.17), we establish

$$\begin{aligned} I_1 &\leq \frac{c_1 \epsilon}{2e\omega_l} \int_{B(y, 2\rho)} \Phi_l F_{1,l}^q (\lg F_{1,l})^r \varphi^s dx \\ &\quad + c_1 \epsilon (2e\omega_l)^{q-1} \int_{B(y, 2\rho)} \rho^{-aq} g_1(x) (\lg F_{1,l})^r \varphi^s dx \\ &\quad + \epsilon^{1-q} (2e\omega_l)^{q-1} n (\bar{c}s)^q \int_{B(y, 2\rho)} \rho^{-q} \nu (\lg F_{1,l})^r \varphi^{s-q} dx. \end{aligned} \quad (3.26)$$

Let us fix $|\alpha| = 2$ and estimate I_2 . To this aim, it will be useful to observe that the following equalities are true:

$$\frac{p-1}{p} + \frac{2}{q} + \frac{q-2p}{qp} = 1, \quad q-1 = \frac{p-1}{p}q + \left(\frac{q}{p} - 1\right). \quad (3.27)$$

Moreover,

$$\rho^{-aq-2p} \mu \leq \rho^{-a\bar{n}} \tilde{\nu} + \rho^{-q} \nu \quad \text{in } \Omega. \quad (3.28)$$

Furthermore, due to (2.8), (3.18), and Young's inequality, we have

$$\begin{aligned} I_2 &\leq \frac{c_4 \epsilon}{2e\omega_l} \int_{B(y, 2\rho)} \Phi_l F_{1,l}^q (\lg F_{1,l})^r \varphi^s dx \\ &\quad + c_5 (2e\omega_l)^{q-1} \epsilon \left(1 + \frac{1}{\epsilon}\right)^{\bar{n}} \bar{s}^{\bar{n}} (r+1)^{\bar{n}} \int_{B(y, 2\rho)} \{\rho^{-a\bar{n}} (g_1(x) + \tilde{\nu}(x)) + \rho^{-q} \nu\} (\lg F_{1,l})^r \varphi^{s-q} dx, \end{aligned} \quad (3.29)$$

where c_4 depends only on c_1, n, q ; and c_5 depends only on c_1, n, q, p, \bar{c} , and $\text{diam } G$.

From (3.25), (3.26), and (3.29), we get

$$\begin{aligned} & \frac{c_2}{2e\omega_l} \int_{B(y,2\rho)} \Phi_l(\lg F_{1,l})^r F_{1,l}^q \varphi^s dx \\ & \leq \frac{(c_1 + c_4)\epsilon}{2e\omega_l} \int_{B(y,2\rho)} \Phi_l F_{1,l}^q (\lg F_{1,l})^r \varphi^s dx \\ & \quad + (2e\omega_l)^{q-1} c_6 (r+1)^{\bar{n}} s^{\bar{n}} \left(1 + \epsilon + \frac{1}{\epsilon}\right)^{\bar{n}+1} \int_{B(y,2\rho)} \psi (\lg F_{1,l})^r \varphi^{s-q} dx, \end{aligned} \quad (3.30)$$

where the constant c_6 depends only on $c_1, \bar{c}, n, q, p, M_G, \sigma$, and $\text{diam } G$.

Setting

$$\epsilon = \frac{c_2}{2(c_1 + c_4)}, \quad (3.31)$$

from the last inequality, we deduce

$$\int_{B(y,2\rho)} \Phi_l (\lg F_{1,l})^r F_{1,l}^q \varphi^s dx \leq c_7 (2e\omega_l)^q (r+1)^{\bar{n}} s^{\bar{n}} \int_{B(y,2\rho)} \psi (\lg F_{1,l})^r \varphi^{s-q} dx, \quad (3.32)$$

where the constant c_7 depends only on $c_1, c_2, \bar{c}, n, q, p, M_G, \sigma$, and $\text{diam } G$.

Now, if we choose φ such that $\varphi = 1$ in $B(y, (4/3)\rho)$, from (3.32), with $r = 0$ and $s = q+1$, we get

$$\int_{B(y, (4/3)\rho)} \left\{ \sum_{|\alpha|=1} \nu |D^\alpha u_l|^q \right\} F_{1,l}^q dx \leq c_7 (2e\omega_l)^q (q+1)^{\bar{n}} \int_{B(y,2\rho)} \psi dx. \quad (3.33)$$

Moreover, if we take in (3.32) instead of φ the function $\varphi_1 \in C_0^\infty(\Omega)$ with the properties $0 \leq \varphi_1 \leq 1$ in Ω , $\varphi_1 = 0$ in $\Omega \setminus B(y, (4/3)\rho)$, $\varphi_1 = 1$ in $B(y, \rho)$, and $|D^\alpha \varphi| \leq \bar{c} \rho^{-|\alpha|}$ in Ω , $|\alpha| = 1, 2$, we obtain that for every $r > 0$ and $s > q$,

$$\int_{B(y,2\rho)} \left\{ \sum_{|\alpha|=1} \nu |D^\alpha u_l|^q \right\} (\lg F_{1,l})^r F_{1,l}^q dx \leq c_7 (2e\omega_l)^q s^{\bar{n}} (r+1)^{\bar{n}} \int_{B(y,2\rho)} \psi (\lg F_{1,l})^r \varphi_1^{s-q} dx. \quad (3.34)$$

We fix arbitrary $r > 0$ and $s > \tilde{q}$, and let

$$z_l = (\lg F_{1,l})^{r/\tilde{q}} \varphi_1^{s/\tilde{q}}. \quad (3.35)$$

By means of Hypothesis 2.1, we establish that $z_l \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega)$ and for $|\alpha| = 1$,

$$\begin{aligned} \nu |D^\alpha z_l|^q & \leq 2^{q-1} \left(\frac{r}{\tilde{q}} \right)^q (\lg F_{1,l})^{(r/\tilde{q}-1)q} (F_{1,l})^q \frac{1}{(2e\omega_l)^q} |D^\alpha u_l|^q \nu \varphi_1^{sq/\tilde{q}} \\ & \quad + 2^{q-1} \left(\frac{s}{\tilde{q}} \right)^q (\lg F_{1,l})^{rq/\tilde{q}} \varphi_1^{(s/\tilde{q}-1)q} \bar{c}^q \rho^{-q} \nu. \end{aligned} \quad (3.36)$$

10 Boundary Value Problems

Now, it is convenient to observe that $\tilde{q}/(\tilde{q} - q_1) > nt/(qt - n)$, then $\tau > nt/(qt - n)$; moreover, $\psi(x) \in L^\tau(G)$. From (3.34) and (3.36), we deduce

$$\begin{aligned} & \int_{\Omega} \nu |D^\alpha z_l|^q dx \\ & \leq c_8 s^{\bar{n}} (r+1)^{\bar{n}+q} \left(\int_{B(y,2\rho)} \psi^\tau dx \right)^{1/\tau} \left(\int_{B(y,2\rho)} (\lg F_{1,l})^{r(q/\tilde{q})(\tau/(\tau-1))} \varphi_1^{(s/\tilde{q}-1)q(\tau/(\tau-1))} dx \right)^{(\tau-1)/\tau}, \end{aligned} \quad (3.37)$$

where the constant c_8 depends only on $c_1, c_2, \bar{c}, n, q, p, M_G, \sigma$, and $\text{diam } G$.

We set

$$\theta = \frac{\tilde{q}(\tau-1)}{q\tau}, \quad m = \frac{q\tau}{\tau-1}, \quad (3.38)$$

and for every $r, s > 0$, we define

$$I(r, s) = \int_{B(y,2\rho)} (\lg F_{1,l})^r \varphi_1^s dx. \quad (3.39)$$

Consequently, last inequality can be rewritten in this manner:

$$\int_{\Omega} \nu |D^\alpha z_l|^q dx \leq c_8 s^{\bar{n}} (r+1)^{\bar{n}+q} \left(\int_{B(y,2\rho)} \psi^\tau dx \right)^{1/\tau} \left[I\left(\frac{r}{\theta}, \frac{s}{\theta} - m\right) \right]^{(\tau-1)/\tau}. \quad (3.40)$$

Due to Hypothesis 2.1,

$$I(r, s) = \int_{B(y,2\rho)} z_l^{\tilde{q}} dx \leq c_0 \left[\int_{B(y,2\rho)} \left(\frac{1}{\nu}\right)^t dx \right]^{\tilde{q}/qt} \left[\sum_{|\alpha|=1} \int_{\Omega} \nu |D^\alpha z_l|^q dx \right]^{\tilde{q}/q}. \quad (3.41)$$

Let us denote by \prod_G the norm of $(1 + f(x) + h(x) + g_1(x) + g_2(x) + \tilde{\nu}(x))$ in $L^\tau(G)$. By simple computation, we have

$$\left(\int_{B(y,2\rho)} \psi^\tau dx \right)^{1/\tau} \leq \rho^{-q} \left(\int_{B(y,2\rho)} \nu^\tau dx \right)^{1/\tau} + \prod_G \rho^{-a\bar{n}}. \quad (3.42)$$

Now, it is convenient to observe that $(q - n/t - n/\tau)(\tilde{q}/q) = n(\theta - 1)$.

Then, from (3.40)–(3.42), using Hypothesis 3.1, we get

$$I(r, s) \leq M(r+s)^{\bar{m}} \rho^{n(1-\theta)} \left[I\left(\frac{r}{\theta}, \frac{s}{\theta} - m\right) \right]^\theta, \quad \text{for every } r > 0, s > \tilde{q}, \quad (3.43)$$

where $\bar{m} = 2(q + \bar{n})\tilde{q}$ and the positive constant M depends only on $c_1, c_2, \bar{c}, c_0, c', n, q, p, t, \|1/\nu\|_{L^t(\Omega)}, M_G, \sigma$, $\text{meas } G$, $\text{diam } G$, and \prod_G .

We set for $i = 0, 1, 2, \dots$ that

$$r_i = \frac{tq}{t+1} \theta^i, \quad s_i = \frac{m\theta}{\theta-1} (\theta^{i+1} - 1). \quad (3.44)$$

Then by (3.43), it is trivial to establish the following iterative relation:

$$I(r_i, s_i) \leq Mc_9 \rho^{n(1-\theta)} \theta^{i\bar{m}} [I(r_{i-1}, s_{i-1})]^\theta \quad \text{for every } i \in \mathbb{N}, \quad (3.45)$$

where c_9 depends only on n, q, p, t , and τ .

Using this recurrent relation, we obtain that for every $i \in \mathbb{N}$,

$$I(r_i, s_i) \leq \left[(Mc_9 + 1)^{1/(1-\theta)} \theta^{S\bar{m}} (\text{diam } G + 1)^n \rho^{-n} I(r_0, s_0) \right]^{\theta^i}, \quad (3.46)$$

where S is a positive constant depending only on n, q, t , and τ .

Now, we assume that

$$\text{meas} \left\{ x \in B\left(y, \frac{4}{3}\rho\right) : u_l(x) \geq \frac{\omega_{1,l} + \omega_{2,l}}{2} \right\} \geq \frac{1}{2} \text{meas} B\left(y, \frac{4}{3}\rho\right). \quad (3.47)$$

We observe that if $x \in B(y, (4/3)\rho)$ satisfies $u_l(x) \geq (\omega_{1,l} + \omega_{2,l})/2$, then $F_{1,l}(x) \leq 4e$, so by [11, Lemma 4], we deduce

$$\int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx \leq c\rho^n + \frac{c\rho r_0}{2e\omega_l} \int_{B(y, (4/3)\rho)} \left\{ \sum_{|\alpha|=1} |D^\alpha u_l| (\lg F_{1,l})^{r_0-1} F_{1,l} \right\} dx, \quad (3.48)$$

where c depends only on n .

Then, using Young's inequality, we get

$$\int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx \leq c r_0 \rho^n + r_0 \left(\frac{c r_0 \rho}{2e\omega_l} \right)^{r_0} \int_{B(y, (4/3)\rho)} \left\{ \sum_{|\alpha|=1} |D^\alpha u_l| \right\}^{r_0} F_{1,l}^{r_0} dx. \quad (3.49)$$

Last inequality, using Hölder's inequality and (3.33), gives

$$\begin{aligned} \int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx &\leq c r_0 \rho^n + r_0 [c r_0]^{r_0} 2^{r_0-1} [c_7 (q+1)^{\bar{n}}]^{t/(t+1)} \rho^{r_0} \\ &\quad \times \left(\int_{B(y, 2\rho)} \psi dx \right)^{t/(t+1)} \left(\int_{B(y, 2\rho)} \left(\frac{1}{\nu} \right)^t dx \right)^{1/(t+1)}. \end{aligned} \quad (3.50)$$

Observe that due to (3.42) and Hypothesis 3.1,

$$\left(\int_{B(y, 2\rho)} \psi dx \right)^{t/(t+1)} \left(\int_{B(y, 2\rho)} \left(\frac{1}{\nu} \right)^t dx \right)^{1/(t+1)} \leq c_{10} (1+M) \rho^{n-r_0}, \quad (3.51)$$

where c_{10} depends only on measure of the unit ball in \mathbb{R}^n .

12 Boundary Value Problems

Consequently, from (3.50), we obtain

$$\int_{B(y,(4/3)\rho)} (\lg F_{1,l})^{r_0} dx \leq (c_{10}(1+M)r_0[cr_0]^{r_0} 2^{r_0-1} [c_7(q+1)^{\bar{n}}]^{t/(t+1)} + cr_0)\rho^n. \quad (3.52)$$

Taking into account that

$$I(r_0, s_0) \leq \int_{B(y,(4/3)\rho)} (\lg F_{1,l})^{r_0} dx, \quad (3.53)$$

from (3.46) we get

$$I(r_i, s_i) \leq [c_{11}]^{\theta^i}, \quad \text{for every } i \in \mathbb{N}. \quad (3.54)$$

Last inequality allow us to conclude that

$$\operatorname{ess\ sup}_{B(y,\rho)} F_{1,l}(x) \leq (1 + c_{11}), \quad (3.55)$$

and so

$$\operatorname{osc} \{u_l, B(y, \rho)\} \leq (1 - 2e^{-1-c_{11}})\omega_l + \rho^a. \quad (3.56)$$

Recall that we proved (3.11) under assumption (3.47). If (3.47) is not true, we take instead of $F_{1,l}$ the function $F_{2,l} : \Omega \rightarrow \mathbb{R}^n$ such that $F_{2,l} = 2e\omega_l(\omega_{2,l} - u_l + \rho^a)^{-1}$ in $B(y, 2\rho)$, and arguing as above, we establish (3.11) again.

It is important to observe that the positive constant c_{11} depends only on $c_1, c_2, c, \bar{c}, c_0, c', n, q, p, t, \|1/\nu\|_{L^t(\Omega)}, M_G, \sigma, \operatorname{diam} G$, and \prod_G , and is independent of $l \in \mathbb{N}$.

Now from (3.11), taking into account [12, Chapter 2, Lemma 4.8], we deduce that there exist positive constant C and $\lambda (< 1)$ depending on c_{11} and a but independent of $l \in \mathbb{N}$ such that

$$\operatorname{osc} \{u_l, B(y, \rho)\} \leq C[d(y, \partial \overset{\circ}{G})]^{-\lambda} \rho^\lambda, \quad \text{for every } \rho \in]0, d(y, \partial \overset{\circ}{G})]. \quad (3.57)$$

This and (3.8) imply that

$$\operatorname{osc} \{\bar{u}, B(y, \rho)\} \leq C[d(y, \partial \overset{\circ}{G})]^{-\lambda} \rho^\lambda, \quad \text{for every } \rho \in]0, d(y, \partial \overset{\circ}{G})]. \quad (3.58)$$

The proof is complete. \square

4. An example

Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, $0 < \gamma < \min(q - n/q, q/2)$, and let ν, μ be the restriction in $\Omega \setminus \{0\}$ of real functions

$$|x|^\gamma, \quad |x|^{2p\gamma/q}. \quad (4.1)$$

According to considerations stated in [3, Section 7], we have that functions ν, μ satisfy Hypotheses 2.1 and 2.3.

Now, we will verify that $\nu(x)$ satisfies Hypothesis 3.1, for all $t: nq/(q^2 - n) < t < n/\gamma$. To this aim, let $G \subset \Omega \setminus \{0\}$ be a “regular set,” and fix $y \in G, \rho > 0: \overline{B(y, \rho)} \subset G$.

If $|y| < 2\rho$, it follows that $B(y, \rho) \subset B(0, 3\rho)$. Hence, we have

$$\begin{aligned} \int_{B(y, \rho)} \frac{1}{|x|^{\gamma t}} dx &\leq \int_{B(0, 3\rho)} \frac{1}{|x|^{\gamma t}} dx = n\chi_n \int_0^{3\rho} r^{n-1-\gamma t} dr = n\chi_n \frac{3^{n-\gamma t}}{n-\gamma t} \rho^{n-\gamma t}, \\ \int_{B(y, \rho)} |x|^{\gamma \tau} dx &\leq \int_{B(0, 3\rho)} |x|^{\gamma \tau} dx = n\chi_n \frac{3^{n+\gamma \tau}}{n+\gamma \tau} \rho^{n+\gamma \tau}. \end{aligned} \quad (4.2)$$

From (4.2), taking into account that $\tau > nt/(qt - n)$, we get

$$\left(\rho^{-n} \int_{B(y, \rho)} \frac{1}{|x|^{\gamma t}} dx \right)^{1/t} \left(\rho^{-n} \int_{B(y, \rho)} |x|^{\gamma \tau} dx \right)^{1/\tau} \leq (n\chi_n + 1) 3^n \left(\frac{1}{n-\gamma t} + 1 \right) \quad \text{if } |y| < 2\rho. \quad (4.3)$$

Instead if $|y| \geq 2\rho$, we denote by Ξ that

$$\Xi = \left\{ k \in \mathbb{N} : \frac{|y|}{\rho} \geq k \right\}. \quad (4.4)$$

Note that $\Xi \neq \emptyset$ and is bounded from above. Consequently, if we denote $\bar{k} = \max \Xi$, we obtain

$$\bar{k}\rho \leq |y| < \rho(\bar{k} + 1). \quad (4.5)$$

Last inequality implies that for every $x \in B(y, \rho)$, it results that

$$(\bar{k} - 1)\rho \leq |x| \leq (\bar{k} + 2)\rho. \quad (4.6)$$

From (4.6), we obtain

$$\begin{aligned} \int_{B(y, \rho)} \frac{1}{|x|^{\gamma t}} dx &\leq \frac{\chi_n}{(\bar{k} - 1)^{\gamma t}} \rho^{n-\gamma t}, \\ \int_{B(y, \rho)} |x|^{\gamma \tau} dx &\leq \chi_n (\bar{k} + 2)^{\gamma \tau} \rho^{n+\gamma \tau}, \end{aligned} \quad (4.7)$$

where χ_n is the measure of the unit ball in \mathbb{R}^n .

14 Boundary Value Problems

Therefore, we get

$$\left(\rho^{-n} \int_{B(y, \rho)} \frac{1}{|x|^{\gamma t}} \right)^{1/t} \left(\rho^{-n} \int_{B(y, \rho)} |x|^{\gamma \tau} dx \right)^{1/\tau} \leq 4^n (\chi_n + 1) \quad \text{if } |y| \geq 2\rho. \quad (4.8)$$

We can conclude that (3.1) holds with $c' = 4^n (n\chi_n + 1)(1/(n - \gamma t) + 1)$.

Next, let $f : \Omega \rightarrow \mathbb{R}$ be the function such that for every $x \in \Omega \setminus \{0\}$,

$$f(x) = \frac{|x|^{-n}}{(1 - \lg|x|)^2} + \frac{1}{\sqrt{1 - |x|}}. \quad (4.9)$$

Observe that $f(x) \in L^1(\Omega)$ but $f(x)$ does not belong to $L^\gamma(\Omega)$, for every $\gamma > 1$.

Let $\sigma > 1$, we consider the following Dirichlet problem:

$$\begin{aligned} - \sum_{|\alpha|=1} D^\alpha \left[\nu \left(\sum_{|\beta|=1} |D^\beta u|^2 \right)^{(q-2)/2} D^\alpha u \right] + \sum_{|\alpha|=2} D^\alpha \left[\mu \left(\sum_{|\beta|=2} |D^\beta u|^2 \right)^{(p-2)/2} D^\alpha u \right] \\ = -|u|^{\sigma-1} u + f \quad \text{in } \Omega, \\ D^\alpha u = 0, \quad |\alpha| = 0, 1, \text{ on } \partial\Omega. \end{aligned} \quad (4.10)$$

By Theorem 2.7, we establish that there exists a W -solution \bar{u} of problem (4.10), bounded in every “regular set” $G \subset \Omega \setminus \{0\}$, and moreover, applying our result, Hölderian in every open set $A : \bar{A} \subset \Omega \setminus \{0\}$.

References

- [1] A. Kovalevsky and F. Nicolosi, “Existence of solutions of some degenerate nonlinear elliptic fourth-order equations with L^1 -data,” *Applicable Analysis*, vol. 81, no. 4, pp. 905–914, 2002.
- [2] A. Kovalevsky and F. Nicolosi, “On the sets of boundedness of solutions for a class of degenerate nonlinear elliptic fourth-order equations with L^1 -data,” *Fundamentalnaya I Priladnaya Matematika*, vol. 12, no. 4, pp. 99–112, 2006.
- [3] A. Kovalevsky and F. Nicolosi, “Existence and regularity of solutions to a system of degenerate nonlinear fourth-order equations,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 61, no. 3, pp. 281–307, 2005.
- [4] I. V. Skrypnik, “Higher order quasilinear elliptic equations with continuous generalized solutions,” *Differential Equations*, vol. 14, no. 6, pp. 786–795, 1978.
- [5] S. Bonafede and S. D’Asero, “Hölder continuity of solutions for a class of nonlinear elliptic variational inequalities of high order,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 44, no. 5, pp. 657–667, 2001.
- [6] I. V. Skrypnik and F. Nicolosi, “On the regularity of solutions of higher-order degenerate nonlinear elliptic equations,” *Dopovidi Natsional’noi Akademii Nauk Ukrayini*, no. 3, pp. 24–28, 1997.
- [7] A. Kovalevsky and F. Nicolosi, “On Hölder continuity of solutions of equations and variational inequalities with degenerate nonlinear elliptic high order operators,” in *Problemi Attuali dell’Analisi e della Fisica Matematica*, pp. 205–220, Aracne Editrice, Rome, Italy, 2000.
- [8] F. Guglielmino and F. Nicolosi, “ W -solutions of boundary value problems for degenerate elliptic operators,” *Ricerche di Matematica*, vol. 36, supplement, pp. 59–72, 1987.
- [9] F. Guglielmino and F. Nicolosi, “Existence theorems for boundary value problems associated with quasilinear elliptic equations,” *Ricerche di Matematica*, vol. 37, no. 1, pp. 157–176, 1988.

- [10] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Paris, France, 1969.
- [11] I. V. Skrypnik, *Nonlinear Elliptic Equations of Higher Order*, Naukova Dumka, Kiev, Ukraine, 1973.
- [12] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, NY, USA, 1968.

S. Bonafede: Dipartimento di Economia dei Sistemi Agro-Forestali, Università delgi Studi di Palermo, Viale delle Scienze, 90128 Palermo, Italy

Email address: bonafedes@unipa.it

F. Nicolosi: Dipartimento di Matematica e Informatica, Università delgi Studi di Catania, Viale A. Doria 6, 95125 Catania, Italy

Email address: fnicolosi@dmi.unict.it

Special Issue on Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at <http://www.hindawi.com/journals/mpe/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	March 1, 2009
First Round of Reviews	June 1, 2009
Publication Date	September 1, 2009

Guest Editors

Edson Denis Leonel, Department of Statistics, Applied Mathematics and Computing, Institute of Geosciences and Exact Sciences, State University of São Paulo at Rio Claro, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru