

# MULTIPLE POSITIVE SOLUTIONS OF SINGULAR $p$ -LAPLACIAN PROBLEMS BY VARIATIONAL METHODS

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We obtain multiple positive solutions of singular  $p$ -Laplacian problems using variational methods. The techniques are applicable to other types of singular problems as well.

## 1. Introduction

We consider the singular quasilinear elliptic boundary value problem

$$\begin{aligned} -\Delta_p u &= a(x)u^{-\gamma} + \lambda f(x, u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where  $\Omega$  is a bounded  $C^2$  domain in  $\mathbb{R}^n$ ,  $n \geq 1$ ,  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplacian,  $1 < p < \infty$ ,  $a \geq 0$  is a nontrivial measurable function,  $\gamma > 0$  is a constant,  $\lambda > 0$  is a parameter, and  $f$  is a Carathéodory function on  $\Omega \times [0, \infty)$  satisfying

$$\sup_{(x,t) \in \Omega \times [0,T]} |f(x,t)| < \infty \quad \forall T > 0. \tag{1.2}$$

The semilinear case  $p = 2$  with  $\gamma < 1$  and  $f = 0$  has been studied extensively in both bounded and unbounded domains (see [5, 6, 7, 10, 11, 12, 14, 20] and their references). In particular, Lair and Shaker [11] showed the existence of a unique (weak) solution when  $\Omega$  is bounded and  $a \in L^2(\Omega)$ . Their result was extended to the sublinear case  $f(t) = t^\beta$ ,  $0 < \beta \leq 1$  by Shi and Yao [15] and Wiegner [18]. In the superlinear case  $1 < \beta < 2^* - 1$  and for small  $\lambda$ , Coclite and Palmieri [4] obtained a solution when  $a = 1$  and Sun et al. [16] obtained two solutions using the Ekeland's variational principle for more general  $a$ 's. Zhang [19] extended their multiplicity result to more general superlinear terms  $f(t) \geq 0$  using critical point theory on closed convex sets. The ODE case  $n = 1$  was studied by Agarwal and O'Regan [1] using fixed point theory and by Agarwal et al. [2] using variational methods. The purpose of the present paper is to treat the general quasilinear case  $p \in (1, \infty)$ ,  $\gamma \in (0, \infty)$ , and  $f$  is allowed to change sign. We use a simple cutoff argument and only the basic critical point theory. Our results seem to be new even for  $p = 2$ .

First we assume

$(H_1)$   $\exists \varphi \geq 0$  in  $C_0^1(\overline{\Omega})$  and  $q > n$  such that  $a\varphi^{-\gamma} \in L^q(\Omega)$ .

This does not require  $\gamma < 1$  as usually assumed in the literature. For example, when  $\Omega$  is the unit ball,  $a(x) = (1 - |x|^2)^\sigma$ ,  $\sigma \geq 0$ , and  $\gamma < \sigma + 1/n$ , we can take  $\varphi(x) = 1 - |x|^2$  and  $q < 1/(\gamma - \sigma)$  (resp.,  $q$  with no additional restrictions) if  $\gamma > \sigma$  (resp.,  $\gamma \leq \sigma$ ).

**THEOREM 1.1.** *If  $(H_1)$  and (1.2) hold and  $f \geq 0$ , then  $\exists \lambda_0 > 0$  such that problem (1.1) has a solution  $\forall \lambda \in (0, \lambda_0)$ .*

**COROLLARY 1.2.** *Problem (1.1) with  $f = 0$  has a solution if  $(H_1)$  holds.*

Next we allow  $f$  to change sign, but strengthen  $(H_1)$  to

$(H_2)$   $a \in L^\infty(\Omega)$  with  $a_0 := \inf_\Omega a > 0$  and  $\gamma < 1/n$ .

This implies that  $a\varphi^{-\gamma} \in L^q(\Omega)$  for any  $\varphi$  whose interior normal derivative  $\partial\varphi/\partial\nu > 0$  on  $\partial\Omega$  and  $q < 1/\gamma$ .

**THEOREM 1.3.** *If  $(H_2)$  and (1.2) hold, then  $\exists \lambda_0 > 0$  such that problem (1.1) has a solution  $\forall \lambda \in (0, \lambda_0)$ .*

Finally we assume that  $f$  is  $C^1$  in  $t$ , satisfies

$$|f_t(x, t)| \leq C(t^{r-2} + 1) \quad (1.3)$$

for some  $2 \leq r < p^*$ , and  $p$ -superlinear:

$$0 < \theta F(x, t) \leq t f(x, t), \quad t \text{ large} \quad (1.4)$$

for some  $\theta > p$ . Here  $p^* = np/(n - p)$  (resp.,  $\infty$ ) if  $p < n$  (resp.,  $p \geq n$ ) is the critical Sobolev exponent and  $C$  denotes a generic positive constant.

**THEOREM 1.4.** *If  $p \geq 2$ ,  $(H_1)$ , (1.3), and (1.4) hold, and  $f \geq 0$ , then  $\exists \lambda_0 > 0$  such that problem (1.1) has two solutions  $\forall \lambda \in (0, \lambda_0)$ .*

**THEOREM 1.5.** *If  $p \geq 2$  and  $(H_2)$ , (1.3), and (1.4) hold, then  $\exists \lambda_0 > 0$  such that problem (1.1) has two solutions  $\forall \lambda \in (0, \lambda_0)$ .*

## 2. Preliminaries on the $p$ -Laplacian

Consider the problem

$$\begin{aligned} -\Delta_p u &= g(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (2.1)$$

**PROPOSITION 2.1.** *If  $g \in L^q(\Omega)$  for some  $q > n$ , then (2.1) has a unique weak solution  $u \in C_0^1(\overline{\Omega})$ . If, in addition,  $g \geq 0$  is nontrivial, then*

$$u > 0 \quad \text{in } \Omega, \quad \partial u / \partial \nu > 0 \quad \text{on } \partial\Omega. \quad (2.2)$$

*Proof.* The existence of a unique solution  $u \in W_0^{1,p}(\Omega)$  is well-known. The problem

$$\begin{aligned} -\Delta v &= g(x) \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (2.3)$$

has a unique solution  $v \in W^{2,q}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$ ,  $\alpha = 1 - n/q$ . Then  $u$  satisfies

$$\begin{aligned} \operatorname{div}(|\nabla u|^{p-2} \nabla u - G(x)) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (2.4)$$

where  $G = \nabla v \in C^\alpha(\overline{\Omega})$ , and  $u$  is bounded by Guedda and Véron [8] since  $q > n/p$  if  $p \leq n$ , so  $u \in C_0^1(\overline{\Omega})$  by Lieberman [13]. The rest now follows from Vázquez [17].  $\square$

### 3. Proofs of Theorems 1.1 and 1.3

*Proof of Theorem 1.1.* Since  $a \in L^q(\Omega)$  by  $(H_1)$ , the problem

$$\begin{aligned} -\Delta_p v &= a(x) \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (3.1)$$

has a unique positive solution  $v \in C_0^1(\overline{\Omega})$  with  $\partial v / \partial \nu > 0$  on  $\partial\Omega$  by Proposition 2.1. Then  $\inf_\Omega (v/\varphi) > 0$  and hence  $av^{-\gamma} \in L^q(\Omega)$ . Fix  $0 < \varepsilon \leq 1$  so small that  $\underline{u} := \varepsilon^{1/(p-1)} v \leq 1$ . Then

$$-\Delta_p \underline{u} - a(x) \underline{u}^{-\gamma} - \lambda f(x, \underline{u}) \leq -(1 - \varepsilon)a(x) \leq 0, \quad (3.2)$$

so  $\underline{u}$  is a subsolution of (1.1).

Since  $a\underline{u}^{-\gamma} \in L^q(\Omega)$ , the problem

$$\begin{aligned} -\Delta_p u &= a(x) \underline{u}(x)^{-\gamma} + 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (3.3)$$

has a unique solution  $\overline{u} \in C_0^1(\overline{\Omega})$  by Proposition 2.1, and  $\overline{u} \geq \underline{u}$  since

$$-\Delta_p \overline{u} \geq a(x) \geq \varepsilon a(x) = -\Delta_p \underline{u}. \quad (3.4)$$

Then

$$-\Delta_p \overline{u} - a(x) \overline{u}^{-\gamma} - \lambda f(x, \overline{u}) \geq 1 - \lambda \sup_{x \in \Omega, t \leq \max_\Omega \overline{u}} f(x, t), \quad (3.5)$$

so  $\exists \lambda_0 > 0$  such that  $\overline{u}$  is a supersolution of (1.1)  $\forall \lambda \in (0, \lambda_0)$  by (1.2).

Let

$$\begin{aligned}
 g_{\lambda, \bar{u}}(x, t) &= \begin{cases} a(x)\bar{u}(x)^{-\gamma} + \lambda f(x, \bar{u}(x)), & t > \bar{u}(x) \\ a(x)t^{-\gamma} + \lambda f(x, t), & \underline{u}(x) \leq t \leq \bar{u}(x) \\ a(x)\underline{u}(x)^{-\gamma} + \lambda f(x, \underline{u}(x)), & t < \underline{u}(x), \end{cases} \\
 G_{\lambda, \bar{u}}(x, t) &= \int_0^t g_{\lambda, \bar{u}}(x, s) ds, \\
 \Phi_{\lambda, \bar{u}}(u) &= \int_{\Omega} |\nabla u|^p - p G_{\lambda, \bar{u}}(x, u), \quad u \in W_0^{1,p}(\Omega).
 \end{aligned} \tag{3.6}$$

Since

$$0 \leq g_{\lambda, \bar{u}}(x, t) \leq a(x)\underline{u}(x)^{-\gamma} + \lambda \sup_{x \in \Omega, t \leq \max_{\Omega} \bar{u}} f(x, t), \quad \forall (x, t) \in \Omega \times \mathbb{R}, \tag{3.7}$$

and  $a\underline{u}^{-\gamma} \in L^q(\Omega)$ ,  $\Phi_{\lambda, \bar{u}}$  is bounded from below and has a global minimizer  $u_0$ , which then is a solution of (1.1) in the order interval  $[\underline{u}, \bar{u}]$ .  $\square$

*Proof of Theorem 1.3.* The problem

$$\begin{aligned}
 -\Delta_p v &= a_0 \quad \text{in } \Omega, \\
 v &= 0 \quad \text{on } \partial\Omega
 \end{aligned} \tag{3.8}$$

has a unique positive solution  $v \in C_0^1(\bar{\Omega})$  with  $\partial v / \partial \nu > 0$  on  $\partial\Omega$ . Fix  $0 < \varepsilon < 1$  so small that  $\underline{u} := \varepsilon^{1/(p-1)} v \leq 1$ . Then

$$-\Delta_p \underline{u} - a(x)\underline{u}^{-\gamma} - \lambda f(x, \underline{u}) \leq -(1 - \varepsilon)a_0 + \lambda \sup_{x \in \Omega, t \leq \max_{\Omega} \underline{u}} |f(x, t)|, \tag{3.9}$$

so  $\exists \lambda_0 > 0$  such that  $\underline{u}$  is a subsolution of (1.1)  $\forall \lambda \in (0, \lambda_0)$ . The rest of the proof now proceeds as above.  $\square$

#### 4. Proofs of Theorems 1.4 and 1.5

*Proof of Theorem 1.4.* Define a Carathéodory function on  $\Omega \times \mathbb{R}$  by

$$g_{\lambda}(x, t) = \begin{cases} a(x)t^{-\gamma} + \lambda f(x, t), & t \geq \underline{u}(x) \\ a(x)\underline{u}(x)^{-\gamma} + \lambda f(x, \underline{u}(x)), & t < \underline{u}(x) \end{cases} \tag{4.1}$$

and consider the problem

$$\begin{aligned}
 -\Delta_p u &= g_{\lambda}(x, u) \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial\Omega.
 \end{aligned} \tag{4.2}$$

Every solution of (4.2) is  $\geq \underline{u}$  and hence also a solution of (1.1). By (1.3),

$$0 \leq g_{\lambda}(x, t) \leq a(x)\underline{u}(x)^{-\gamma} + \lambda C \left( (t^+)^{r-1} + 1 \right), \quad \forall (x, t) \in \Omega \times \mathbb{R} \tag{4.3}$$

so solutions of (4.2) are the critical points of the  $C^1$  functional

$$\Phi_\lambda(u) = \int_\Omega |\nabla u|^p - pG_\lambda(x, u), \quad u \in W_0^{1,p}(\Omega), \quad (4.4)$$

where  $G_\lambda(x, t) = \int_0^t g_\lambda(x, s) ds$ .

Since  $u_0$  solves

$$\begin{aligned} -\Delta_p u &= g_{\lambda, \bar{u}}(x, u_0(x)) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (4.5)$$

and  $g_{\lambda, \bar{u}}(\cdot, u_0(\cdot)) \in L^q(\Omega)$  by (3.7),  $u_0 \in C_0^1(\bar{\Omega})$  by Proposition 2.1. Note that, with a possibly smaller  $\lambda_0$ ,  $2\bar{u}$  is also a supersolution of (1.1)  $\forall \lambda \in (0, \lambda_0)$ . We assume that  $u_0$  is the global minimizer of the corresponding functional  $\Phi_{\lambda, 2\bar{u}}$  also, for otherwise we are done. Since

$$u_0 \leq \bar{u} < 2\bar{u} \quad \text{in } \Omega, \quad \partial u_0 / \partial \nu \leq \partial \bar{u} / \partial \nu < \partial(2\bar{u}) / \partial \nu \quad \text{on } \partial\Omega, \quad (4.6)$$

$\Phi_{\lambda, 2\bar{u}} = \Phi_\lambda$  in a  $C_0^1(\bar{\Omega})$ -neighborhood of  $u_0$ , so  $u_0$  is a local minimizer of  $\Phi_\lambda|_{C_0^1(\bar{\Omega})}$ , and hence also of  $\Phi_\lambda$  by Brezis and Nirenberg [3] for  $p = 2$  and by Guo and Zhang [9] for  $p > 2$ . The mountain pass lemma now gives a second critical point as (1.4) implies that  $\Phi_\lambda$  satisfies the (PS) condition and  $\Phi_\lambda(tu) \rightarrow -\infty$  as  $t \rightarrow \infty$ .  $\square$

Proof of Theorem 1.5 is similar and therefore omitted.

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