Research Article
Solvability of Second-Order m-Point Boundary Value Problems with Impulses

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By Leray-Schauder continuation theorem and the nonlinear alternative of Leray-Schauder type, the existence of a solution for an m-point boundary value problem with impulses is proved.

1. Introduction

The main purpose of this paper is to get results on the solvability of the following boundary value problem (BVP):

$$x''(t) = f(t, x(t), x'(t)),$$

$$\Delta x'(t_k) = b_k x'(t_k), \qquad \Delta x(t_k) = c_k x(t_k),$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),$$

(1.1)

where $\xi_i \in (0,1)$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in R$, $i = 1, 2, \ldots, m-2$, $\sum_{i=1}^{m-2} a_i \neq 1$, $0 = t_0 < t_1 < t_2 < \cdots < t_T < t_{T+1} = 1$.

Such problems without impulses effects have been solved before, for example, in [1–3]. But as far as we know the publication on the solvability of m-point problems with impulses is fewer [4]. Our main goal is to find condition for $f, b_k, c_k, 1 \le k \le T$, which guarantees the existence of at least one solution of problem (1.1). The proofs are based on the Leray-Schauder continuation theorem [5] and the nonlinear alternative of Leray-Schauder type [6].

In order to define the concept of solution for BVP (1.1), we introduce the following spaces of functions:

(i) $PC[0,1] = \{u : [0,1] \to R, u$ is continuous at $t \neq t_k, u(t_k^+), u(t_k^-)$ exist, and $u(t_k^-) = u(t_k)\}$;

(ii) $PC^1[0,1] = \{u \in PC[0,1] : u$ is continuously differentiable at $t \neq t_k, u'(0^+), u'(t_k^+), u'(t_k^-)$ exist and $u'(t_k^-) = u'(t_k)\}$;

(iii) $PC^2[0,1] = \{u \in PC^1[0,1] : u$ is twice continuously differentiable at $t \neq t_k\}$.

Note that $PC[0,1]$ and $PC^1[0,1]$ are Banach spaces with the norms

$$\|u\|_\infty = \sup\{|u(t)| : t \in [0,1]\}, \qquad \|u\|_1 = \max\{\|u\|_\infty, \|u'\|_\infty\}, \qquad (1.2)$$

respectively.

Definition 1.1. The set $\mathcal{F}$ is said to be quasiequicontinuous in $[0,c]$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathcal{F}, k \in Z, t^*, t^{**} \in (t_{k-1}, t_k] \cap [0,c]$, and $|t^* - t^{**}| < \delta$, then $|x(t^*) - x(t^{**})| < \varepsilon$.

LEMMA 1.2 (compactness criterion [7]). *The set $\mathcal{F} \subset PC([0,c], R^n)$ is relatively compact if and only if one has the following:*

(1) *$\mathcal{F}$ is bounded;*

(2) *$\mathcal{F}$ is quasiequicontinuous in $[0,c]$.*

LEMMA 1.3 [7]. *Let $s \in [0,T), c_k \geq 0, \alpha_k, k = 1,\dots,p$, are constants and let $p,q \in PC(J,R), x \in PC^1(J,R)$. If*

$$x'(t) \leq p(t)x(t) + q(t), \quad t \in [s,T), t \neq t_k,$$
$$x(t_k^+) \leq c_k x(t_k) + \alpha_k, \quad t_k \in [s,T), \qquad (1.3)$$

then for $t \in [s,T]$,

$$x(t) \leq x(s^+)\left(\prod_{s<t_k<t} c_k\right)\exp\left(\int_s^t p(u)du\right)$$

$$+ \int_s^t \left(\prod_{u<t_k<t} c_k\right)\exp\left(\int_u^t p(\tau)d\tau\right)q(u)du \qquad (1.4)$$

$$+ \sum_{s<t_k<t}\left(\prod_{t_k<t_i<t} c_i\right)\exp\left(\int_{t_k}^t p(\tau)d\tau\right)\alpha_k.$$

The result also holds if the above inequalities are reversed.

2. Main results

THEOREM 2.1. *Let $f : [0,1] \times R^2 \to R$ be a continuous function. Assume that there exist $p(t), q(t),$ and $r(t) : [0,1] \to [0,\infty)$ such that*

$$|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t) \qquad (2.1)$$

for $t \in [0,1]$ and all $(u,v) \in R^2$. Then the BVP (1.1) has at least one solution in $PC^1[0,1]$ provided

$$Q + B < 1, \tag{2.2}$$

$$\left(1 + \frac{\sum_{i=1}^{m-2}|a_i|}{|1 - \sum_{i=1}^{m-2}a_i|}\right)\left(\frac{P}{1 - Q - B} + C\right) < 1, \tag{2.3}$$

where $P = \int_0^1 p(t)dt$, $Q = \int_0^1 q(t)dt$, $B = \sum_{k=1}^{T}|b_k|$, $C = \sum_{k=1}^{T}|c_k|$.

Proof. Let $Y = X = PC^1[0,1]$. Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \left\{x \in PC^2[0,1], \ x'(0) = 0, \ x(1) = \sum_{i=1}^{m-2}a_i x(\xi_i)\right\}, \tag{2.4}$$

and for $x \in D(L) : Lx = (x'', \Delta x'(t_k), \Delta x(t_k))$. We also define a nonlinear mapping $F : X \to Y$ by setting

$$(Fx)(t) = (f(t,x(t),x'(t)), b_k x'(t_k), c_k x(t_k)). \tag{2.5}$$

From the assumption on f, we see that F is a bounded mapping from X to Y. Next, it is easy to see that $L : D(L) \to Y$ is one-to-one mapping. Moreover, it follows easily using Lemma 1.2 that $L^{-1}F : X \to X$ is a compact mapping.

We note that $x \in PC^1[0,1]$ is a solution of (1.1) if and only if x is a fixed point of the equation

$$x = L^{-1}Fx. \tag{2.6}$$

We apply the Leray-Schauder continuation theorem to obtain the existence of a solution for $x = L^{-1}Fx$.

To do this, it suffices to verify that the set of all possible solutions of the family of equations:

$$x''(t) = \lambda f(t,x(t),x'(t)),$$

$$\Delta x'(t_k^+) = \lambda b_k x'(t_k), \qquad \Delta x(t_k) = \lambda c_k x(t_k),$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2}a_i x(\xi_i). \tag{2.7}$$

Integrate (2.7) from 0 to t to obtain

$$x'(t) = \lambda \int_0^t f(s,x(s),x'(s))ds + \lambda \sum_{0 < t_k < t} b_k x'(t_k). \tag{2.8}$$

By condition (2.1), we have

$$|x'(t)| \leq \int_0^t [p(s)\|x\| + q(s)\|x'\| + r(s)]ds + \sum_{k=1}^T |b_k|\|x'\|$$

$$\leq (Q+B)\|x'\| + P\|x\| + R_1,$$

(2.9)

where $R_1 = \int_0^1 r(t)dt$. Thus,

$$\|x'\| \leq \frac{1}{1-Q-B}(P\|x\| + R_1).$$

(2.10)

Integrate (2.8) from t to 1 to obtain

$$-x(t)$$

$$= \lambda\left\{\int_0^1 H(t,s)f(s,x(s),x'(s))ds + \int_t^1 \sum_{0<t_k<s} b_k x'(t_k)ds + \sum_{t<t_k<1} c_k x(t_k)\right.$$

$$\left. + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\left[\int_0^1 H(\xi_i,s)f(s,x(s),x'(s))ds\int_{\xi_i}^1 \sum_{0<t_k<s} b_k x'(t_k)ds + \sum_{\xi_i<t_k<1} c_k x(t_k)\right]\right\},$$

(2.11)

where

$$H(t,s) = \begin{cases} 1-t, & 0 \leq s \leq t \leq 1, \\ 1-s, & 0 \leq t \leq s \leq 1. \end{cases}$$

(2.12)

So

$$\|x\| \leq \left(1 + \frac{\sum_{i=1}^{m-2}|a_i|}{|1-\sum_{i=1}^{m-2} a_i|}\right)[(P+C)\|x\| + (Q+B)\|x'\| + R_1].$$

(2.13)

Equations (2.10) and (2.13) imply

$$\|x\| \leq \left(1 + \frac{\sum_{i=1}^{m-2}|a_i|}{|1-\sum_{i=1}^{m-2} a_i|}\right)\left[\left(\frac{P}{1-Q-B} + C\right)\|x\| + R_1\right].$$

(2.14)

It follows from the assumption (2.3) that there is a constant M_1 in dependent of $\lambda \in [0,1]$ such that $\|x\| \leq M_1$. Furthermore, by (2.10), there is a constant M_2 such that $\|x'\| \leq M_2$. It is now immediate that the set of solutions of the family of equations (2.7) is, a priori, bounded in $PC^1[0,1]$ by a constant independent of $\lambda \in [0,1]$. This completes the proof of the theorem.

THEOREM 2.2. *Let* $f : [0,1] \times R^2 \to R$. *Assume that the following conditions hold:*

(H_1) $|f(t,u,v)| \leq q(t)w(\max\{|u|,|v|\})$ *on* $[0,1] \times R^2$ *with* $w > 0$ *continuous and non-decreasing on* $[0,\infty)$, $q(t) : [0,1] \to [0,\infty)$ *is continuous;*

(H$_2$) $b_k \geq 0$, and

$$C \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) < 1,$$

$$\sup_{r \geq 0} \frac{r}{w(r)} > M_3 = \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \left[1 - C \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \right]^{-1} Q,$$

(2.15)

where $Q = \int_0^1 \prod_{0 < t_k < 1} (1 + b_k) q(s) ds$.

Then (1.1) has at least one solution.

Choose $\widetilde{M} > 0$ such that

$$\frac{\widetilde{M}}{w(\widetilde{M})} > M_3. \tag{2.16}$$

To show that (1.1)) has at least one solution, we consider the operator

$$x = \lambda L^{-1} F x, \quad \lambda \in [0, 1], \tag{2.17}$$

which is equivalent to (2.7). Let $x \in PC^1[0, 1]$ be any solution of (2.7), from (H$_1$), we have

$$-q(t) w(\|x\|_1) \leq x''(t) \leq q(t) w(\|x\|_1). \tag{2.18}$$

Consider the inequalities

$$
\begin{aligned}
x''(t) &\leq q(t) w(\|x\|_1), \\
x'(t_k) &= (1 + b_k) x(t_k), \\
x'(0) &= 0, \\
x''(t) &\geq -q(t) w(\|x\|_1), \\
x'(t_k) &= (1 + b_k) x(t_k), \\
x'(0) &= 0.
\end{aligned}
\tag{2.19}
$$

By Lemma 1.3, we have

$$
\begin{aligned}
x'(t) &\leq w(\|x\|_1) \int_0^t \prod_{0 < t_k < t} (1 + b_k) q(s) ds \\
&\leq Q w(\|x\|_1), \\
x'(t) &\geq -w(\|x\|_1) \int_0^t \prod_{0 < t_k < t} (1 + b_k) q(s) ds \\
&\geq -Q w(\|x\|_1).
\end{aligned}
\tag{2.20}
$$

From (2.20), we can deduce

$$|x'(t)| \leq Q w(\|x\|_1), \tag{2.21}$$

and so

$$\|x'\| \le Qw(\|x\|_1). \tag{2.22}$$

Using $x(t) = x(1) - \int_t^1 x'(s)ds - \sum_{t<t_k<1} c_k x(t_k)$ and $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$, we have

$$x(t) = -\frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \left[\int_{\xi_i}^1 x'(s)ds + \sum_{\xi_i < t_k < 1} c_k x(t_k) \right] - \int_t^1 x'(s)ds - \sum_{t<t_k<1} c_k x(t_k), \tag{2.23}$$

which implies

$$|x(t)| \le \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) (\|x'\| + C\|x\|), \tag{2.24}$$

and so

$$\begin{aligned}
\|x\| &\le \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \left[1 - C \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \right]^{-1} \|x'\| \\
&\le \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \left[1 - C \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|} \right) \right]^{-1} Qw(\|x\|_1).
\end{aligned} \tag{2.25}$$

Now, (2.22) together with (2.25) imply $\|x\|_1 \ne \widetilde{M}$. Set

$$U = \{u \in PC^1[0,1] : \|u\|_1 < \widetilde{M}\}, \qquad K = E = PC^1[0,1], \tag{2.26}$$

then the nonlinear alternative of Leray-Schauder type [6] guarantees that $L^{-1}F$ has a fixed point, that is, (1.1) has a solution $x \in PC^1[0,1]$, which completes the proof. $\qquad\square$

3. Examples

Example 3.1. Consider the boundary value problem

$$\begin{aligned}
x'' &= f(t,x,x'), \quad t \in [0,1], \ t \ne \frac{1}{2}, \\
\Delta x'(t_k) &= \frac{1}{6} x'(t_k), \quad \Delta x(t_k) = \frac{1}{4} x(t_k), \quad t_k = \frac{1}{2}, \\
x'(0) &= 0, \ x(1) = \frac{1}{2} x\left(\frac{1}{3}\right) - \frac{1}{3} x\left(\frac{2}{3}\right),
\end{aligned} \tag{3.1}$$

where

$$f(t,u,v) = t^5 u + \frac{1}{2} t^3 v + t^2 [1 + \cos(u^{200} + v^{30})]. \tag{3.2}$$

It is easy to see that

$$|f(t,u,v)| \le p(t)|u| + q(t)|v| + r(t) \tag{3.3}$$

with $p(t) = t^5$, $q(t) = (1/2)t^3$, $r(t) = 2t^2$. Clearly, $P = 1/6$, $Q = 1/8$, $B = 1/6$, $C = 1/4$, and

$$Q + B = \frac{7}{24} < 1, \qquad \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|}\right)\left(\frac{P}{1 - Q - B} + C\right) = \frac{33}{34} < 1. \tag{3.4}$$

By Theorem 2.1, (3.1) has at least one solution.

Example 3.2. Consider the boundary value problem

$$x'' = f(t, x, x'), \quad t \in [0, 1], \; t \neq \frac{1}{2},$$

$$\Delta x'(t_k) = x'(t_k), \quad \Delta x(t_k) = \frac{1}{3}x(t_k), \quad t_k = \frac{1}{2}, \tag{3.5}$$

$$x'(0) = 0, \quad x(1) = \frac{1}{2}x\left(\frac{1}{3}\right) - \frac{1}{2}x\left(\frac{2}{3}\right),$$

where

$$f(t, u, v) = e^{-t}(u^\alpha + v^\beta) + \mu e^{-t} \tag{3.6}$$

with $\alpha \in [0, 1]$, $\beta \in [0, 1]$, $\mu > 0$. It is easy to see that

$$|f(t, u, v)| \leq q(t)w(\max\{|u|, |v|\}) \tag{3.7}$$

with $q(t) = e^{-t}$, $w(s) = s^\alpha + s^\beta + \mu$. Clearly

$$C\left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|}\right) = \frac{2}{3} < 1,$$

$$\sup_{r \geq 0} \frac{r}{w(r)} = \sup_{r \geq 0} \frac{r}{r^\alpha + r^\beta + \mu} = \infty, \tag{3.8}$$

so (H_2) is true. Theorem 2.2 shows that (3.5) has at least one solution.

Acknowledgments

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