

## Research Article

# Extremal Solutions of Periodic Boundary Value Problems for First-Order Impulsive Integrodifferential Equations of Mixed-Type on Time Scales

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We consider the existence of minimal and maximal solutions of periodic boundary value problems for first-order impulsive integrodifferential equations of mixed-type on time scales by establishing a comparison result and using the monotone iterative technique.

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## 1. Introduction

The theory of calculus on time scales (see [1, 2] and references cited therein) was initiated by Stefan Hilger in his Ph.D. thesis in 1990 [3] in order to unify continuous and discrete analyses, and it has a tremendous potential for applications and has recently received much attention since his foundational work. In this paper, we will study the periodic boundary value problem for the first-order impulsive integrodifferential equations of mixed-type (PBVP):

$$\begin{aligned} u^\Delta(t) &= f(t, u(t), [Tu](t), [Su](t)), \quad t \neq t_k, \quad t \in J_{\mathbb{T}}, \\ u(t_k^+) - u(t_k^-) &= I_k(u(t_k^-)), \quad k = 1, 2, \dots, p, \\ u(0) &= u(T), \end{aligned} \tag{1.1}$$

where  $\mathbb{T}$  is a time scale which has the subspace topology inherited from the standard topology on  $\mathbb{R}$ . For each interval  $J$  of  $\mathbb{R}$ , we denote by  $J_{\mathbb{T}} = J \cap \mathbb{T}$ ,  $f \in C[J_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ ,  $J = [0, T]$ ,  $I_k \in C[\mathbb{R}, \mathbb{R}]$ , where  $u(t_k^+)$  and  $u(t_k^-)$  represent right and left limits of  $u(t)$  at  $t = t_k$  ( $k = 1, 2, \dots, p$ ) in the sense of time scales, and in addition, if  $t_k$  is right scattered, then  $\gamma(t_k^+) = \gamma(t_k)$ , whereas if  $t_k$  is left scattered, then  $\gamma(t_k^-) = \gamma(t_k)$ ,

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$$0 < t_1 < t_2 < \cdots < t_k < \cdots < t_p < T,$$

$$[Tu](t) = \int_0^t k(t,s)u(s)\Delta s, \quad [Su](t) = \int_0^T h(t,s)u(s)\Delta s, \quad (1.2)$$

$k(t,s) \in C[D, \mathbb{R}^+]$ ,  $D = \{(t,s) \in J_{\mathbb{T}} \times J_{\mathbb{T}} : t \geq s\}$ ,  $h(t,s) \in C[J_{\mathbb{T}} \times J_{\mathbb{T}}, \mathbb{R}^+]$ ,  $\mathbb{R}^+ = [0, +\infty)$ ,  $k_0 = \max\{k(t,s) : (t,s) \in D\}$ ,  $h_0 = \max\{h(t,s) : (t,s) \in J_{\mathbb{T}} \times J_{\mathbb{T}}\}$ .

The study of impulsive dynamic equations on time scales has been initiated by Henderson [4], Benchohra et al. [5], and Atici and Biles [6]. Extremal solutions of PBVP for impulsive differential equations and difference equations has been studied by some authors (see [7, 8]). In this paper, we will obtain an inequality on time scales. And then, using this inequality, a comparison result is obtained. At last, we obtain an existence theorem of minimal and maximal solutions of PBVP (1.1) by using monotone iterative technique (see [7–9]).

### 2. Preliminaries and comparison principle

In this section, we will first recall some basic definitions and lemmas, which are used in what follows.

Let  $\mathbb{T}$  be a nonempty closed subset (time scale) of  $\mathbb{R}$ . The forward and backward jump operators  $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ , and the graininess  $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$  are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t. \quad (2.1)$$

A point  $t \in \mathbb{T}$  is called left dense if  $t > \inf \mathbb{T}$  and  $\rho(t) = t$ , left scattered if  $\rho(t) < t$ , right dense if  $t < \sup \mathbb{T}$  and  $\sigma(t) = t$ , and right scattered if  $\sigma(t) > t$ . If  $\mathbb{T}$  has a left-scattered maximum  $m$ , then  $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$ ; otherwise  $\mathbb{T}^k = \mathbb{T}$ . If  $\mathbb{T}$  has a right-scattered minimum  $m$ , then  $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$ ; otherwise  $\mathbb{T}_k = \mathbb{T}$ .

A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is right-dense continuous provided it is continuous at right-dense point in  $\mathbb{T}$  and its left-side limits exist at left-dense points in  $\mathbb{T}$ . If  $f$  is continuous at each right-dense point and each left-dense point, then  $f$  is said to be continuous function on  $\mathbb{T}$ .

For  $y : \mathbb{T} \rightarrow \mathbb{R}$  and  $t \in \mathbb{T}^k$ , we define the delta derivative of  $y(t)$ ,  $y^\Delta(t)$  to be the number (if it exists) with the property that for a given  $\varepsilon > 0$ , there exists a neighborhood  $U$  of  $t$  such that

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| < \varepsilon |\sigma(t) - s| \quad (2.2)$$

for all  $s \in U$ .

If  $y$  is continuous, then  $y$  is right-dense continuous, and if  $y$  is delta differentiable at  $t$ , then  $y$  is continuous at  $t$ .

LEMMA 2.1 (see [1]). Assume that  $f, g : \mathbb{T} \rightarrow \mathbb{R}$  are delta differentiable at  $t \in \mathbb{T}^k$ . Then,

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)). \quad (2.3)$$

Let  $y$  be right-dense continuous. If  $Y^\Delta(t) = y(t)$ , then we define the delta integral by

$$\int_a^t y(s) \Delta s = Y(t) - Y(a). \quad (2.4)$$

A function  $r : \mathbb{T} \rightarrow \mathbb{R}$  is called regressive if

$$1 + \mu(t)r(t) \neq 0 \quad (2.5)$$

for all  $t \in \mathbb{T}^k$ .

If  $r$  is regressive function, then the generalized exponential function  $e_r$  is defined by

$$e_r(t, s) = \exp \left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau)) \Delta \tau \right\} \quad \text{for } s, t \in \mathbb{T} \quad (2.6)$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\text{Log}(1 + hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases} \quad (2.7)$$

Let  $p, q : \mathbb{T} \rightarrow \mathbb{R}$  be two regressive functions, we define

$$p \oplus q := p + q + \mu p q, \quad \ominus p := -\frac{p}{1 + \mu p}, \quad p \ominus q := p \oplus (\ominus q). \quad (2.8)$$

Then, the generalized exponential function has the following properties.

LEMMA 2.2 (see [1]). Assume that  $p, q : \mathbb{T} \rightarrow \mathbb{R}$  are two regressive functions, then

- (i)  $e_0(t, s) \equiv 1$  and  $e_p(t, t) \equiv 1$ ;
- (ii)  $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$ ;
- (iii)  $e_p(t, \sigma(s)) = e_p(t, s)/(1 + \mu(s)p(s))$ ;
- (iv)  $1/e_p(t, s) = e_{\ominus p}(t, s)$ ;
- (v)  $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$ ;
- (vi)  $e_p(t, s)e_p(s, r) = e_p(t, r)$ ;
- (vii)  $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$ ;
- (viii)  $e_p(t, s)/e_q(t, s) = e_{p \ominus q}(t, s)$ .

LEMMA 2.3 [1]. Let  $r : \mathbb{T} \rightarrow \mathbb{R}$  be right-dense continuous and regressive,  $a \in \mathbb{T}$ , and  $y_a \in \mathbb{R}$ . The unique solution of the initial value problem

$$y^\Delta(t) = r(t)y(t) + h(t), \quad y(a) = y_a, \quad (2.9)$$

is given by

$$y(t) = e_r(t, a)y_a + \int_a^t e_r(t, \sigma(s))h(s) \Delta s. \quad (2.10)$$

Throughout this paper, we assume that, for each  $k = 1, \dots, p$ , the points of impulse  $t_k$  are right dense. For convenience, we introduce the notation  $PC[J_{\mathbb{T}}, \mathbb{R}] = \{u : J_{\mathbb{T}} \rightarrow \mathbb{R}, u(t)$

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is continuous everywhere except some  $t_k$  at which  $u(t_k^-)$  and  $u(t_k^+)$  exist and  $u(t_k^-) = u(t_k)$ . Evidently,  $PC[J_{\mathbb{T}}, \mathbb{R}]$  is a Banach space with norm  $\|u\|_{PC} = \sup_{t \in J_{\mathbb{T}}} |u(t)|$ . Let  $J'_{\mathbb{T}} = J_{\mathbb{T}} \setminus \{t_1, t_2, \dots, t_p\}$ ,  $C^1[J'_{\mathbb{T}}, \mathbb{R}] = \{u^\Delta(t) \text{ is continuous on } J'_{\mathbb{T}}\}$ ,  $\Omega = PC[J_{\mathbb{T}}, \mathbb{R}] \cap C^1[J'_{\mathbb{T}}, \mathbb{R}]$ ,  $\mathbb{T}^+ = \mathbb{T} \cap \mathbb{R}^+$ ,  $PC^1[\mathbb{T}^+, \mathbb{R}] = PC[\mathbb{T}^+, \mathbb{R}] \cap C^1[\mathbb{T}^+, \mathbb{R}]$ . A function  $u \in \Omega$  is called a solution of PBVP (1.1) if it satisfies (1.1).

Next, we combine [10, 11] to obtain an inequality as follows.

LEMMA 2.4. Assume that

(A<sub>0</sub>) the sequence  $\{t_k\}$  satisfies  $0 \leq t_0 < t_1 < t_2 < \dots < t_k < \dots$  with  $\lim_{k \rightarrow +\infty} t_k = +\infty$ ,

(A<sub>1</sub>)  $m \in PC^1[\mathbb{T}^+, \mathbb{R}]$  is right-dense continuous at  $t_k$  for  $k = 1, 2, \dots$ ,

(A<sub>2</sub>)  $\inf_{t \in J_{\mathbb{T}}} \{\mu(t)p(t)\} > -1$ . For  $k = 1, 2, \dots, t \geq t_0$ ,

$$m^\Delta(t) \geq p(t)m(t) + q(t), \quad t \neq t_k, \quad m(t_k^+) \geq d_k m(t_k) + b_k, \quad (2.11)$$

where  $p, q \in C(\mathbb{T}^+, \mathbb{R})$ ,  $d_k \geq 0$ , and  $b_k$  are real constants. Then,

$$\begin{aligned} m(t) &\geq m(t_0) \prod_{t_0 < t_k < t} d_k e_p(t, t_0) + \int_{t_0}^t \prod_{s < t_k < t} d_k e_p(t, \sigma(s)) q(s) \Delta s \\ &\quad + \sum_{t_0 < t_k < t} \prod_{t_k < t_j < t} d_j e_p(t, t_k) b_k. \end{aligned} \quad (2.12)$$

*Proof.* By condition (A<sub>2</sub>), we know that  $e_{\ominus p}(\sigma(t), t_0) \geq 0$  for  $t \in [t_0, +\infty)_{\mathbb{T}}$ . For the following inequality:

$$m^\Delta(t) \geq p(t)m(t) + q(t), \quad (2.13)$$

on multiplying  $e_{\ominus p}(\sigma(t), t_0)$  and arranging the terms, we obtain

$$e_{\ominus p}(\sigma(t), t_0) m^\Delta(t) - p(t)m(t)e_{\ominus p}(\sigma(t), t_0) \geq e_{\ominus p}(\sigma(t), t_0) q(t), \quad (2.14)$$

which is the same as

$$(e_{\ominus p}(t, t_0) m(t))^\Delta \geq e_{\ominus p}(\sigma(t), t_0) q(t). \quad (2.15)$$

Integrating (2.15) from  $t_0$  to  $t_1$ , then

$$e_{\ominus p}(t_1, t_0) m(t_1) \geq m(t_0) + \int_{t_0}^{t_1} e_{\ominus p}(\sigma(s), t_0) q(s) \Delta s. \quad (2.16)$$

Again integrating (2.15) from  $t_1$  to  $t$ , where  $t \in (t_1, t_2]$ , then

$$\begin{aligned} e_{\ominus p}(t, t_0) m(t) &\geq e_{\ominus p}(t_1, t_0) m(t_1^+) + \int_{t_1}^t e_{\ominus p}(\sigma(s), t_0) q(s) \Delta s \\ &\geq e_{\ominus p}(t_1, t_0) (d_1 m(t_1) + b_1) + \int_{t_1}^t e_{\ominus p}(\sigma(s), t_0) q(s) \Delta s \\ &\geq d_1 \left( m(t_0) + \int_{t_0}^{t_1} e_{\ominus p}(\sigma(s), t_0) q(s) \Delta s \right) + b_1 e_{\ominus p}(t_1, t_0) \\ &\quad + \int_{t_1}^t e_{\ominus p}(\sigma(s), t_0) q(s) \Delta s, \end{aligned} \quad (2.17)$$

that is,

$$m(t) \geq m(t_0)d_1e_p(t, t_0) + \int_{t_0}^t \prod_{s < t_k < t} d_ke_p(t, \sigma(s))q(s)\Delta s + b_1e_p(t, t_1). \quad (2.18)$$

Repeating the above procession for  $t \in [t_0, +\infty)_{\mathbb{T}}$ , we have

$$\begin{aligned} m(t) &\geq m(t_0) \prod_{t_0 < t_k < t} d_ke_p(t, t_0) + \int_{t_0}^t \prod_{s < t_k < t} d_ke_p(t, \sigma(s))q(s)\Delta s \\ &\quad + \sum_{t_0 < t_k < t} \prod_{t_k < t_j < t} d_je_p(t, t_k)b_k. \end{aligned} \quad (2.19)$$

Thus the proof of Lemma 2.4 is complete.  $\square$

The following comparison result plays an important role in this paper.

LEMMA 2.5. *Let  $t_0 = 0$ ,  $t_{p+1} = T$ . Assume that  $u \in \Omega$  satisfies*

$$\begin{aligned} u^\Delta(t) &\geq -a(t)u(t) - b(t)[Tu](t) - c(t)[Su](t), \quad t \neq t_k, t \in J_{\mathbb{T}}, \\ u(t_k^+) - u(t_k) &\geq -L_k u(t_k), \quad k = 1, 2, \dots, p, \\ u(0) &\geq u(T), \end{aligned} \quad (2.20)$$

where  $a, b, c \in C[J_{\mathbb{T}}, \mathbb{R}^+]$ ,  $a$  is not identically vanishing, and  $\sup_{t \in J_{\mathbb{T}}} \{\mu(t)a(t)\} < 1$ ,  $0 \leq L_k < 1$  ( $k = 1, 2, \dots, p$ ). If

$$(Bk_0 + Ch_0)e_{\ominus(-a)}(T, 0) \leq \frac{\left\{ \prod_{0 < t_k < T} (1 - L_k) \right\}^2}{\int_0^T \prod_{s < t_k < T} (1 - L_k) \Delta s} \quad (2.21)$$

with  $B = \sup_{t \in J_{\mathbb{T}}} \{b(t) \int_0^t e_{\ominus(-a)}(\sigma(t), s) \Delta s\}$  and  $C = \sup_{t \in J_{\mathbb{T}}} \{c(t) \int_0^T e_{\ominus(-a)}(\sigma(t), s) \Delta s\}$ , then  $u(t) \geq 0$  for  $t \in J_{\mathbb{T}}$ .

*Proof.* Let  $p(t) = u(t)e_{\ominus(-a)}(t, 0)$  for  $t \in J_{\mathbb{T}}$ . Then  $p \in \Omega$  satisfies

$$\begin{aligned} p^\Delta(t) &\geq -b(t) \int_0^t e_{\ominus(-a)}(\sigma(t), s) k(t, s) p(s) \Delta s \\ &\quad - c(t) \int_0^T e_{\ominus(-a)}(\sigma(t), s) h(t, s) p(s) \Delta s, \quad t \neq t_k, t \in J_{\mathbb{T}}, \\ p(t_k^+) - p(t_k) &\geq -L_k p(t_k), \quad k = 1, 2, \dots, p, \\ p(0) &\geq e_{\ominus(-a)}(T, 0) p(T). \end{aligned} \quad (2.22)$$

We now prove

$$p(t) \geq 0 \quad \text{for } t \in J_{\mathbb{T}}. \quad (2.23)$$

Assume that (2.23) is not true. Then, there are two cases:

- (a) there exists  $t_1^* \in J_{\mathbb{T}}$  such that  $p(t_1^*) < 0$  and  $p(t) \leq 0$  for  $t \in J_{\mathbb{T}}$ ;
- (b) there exists  $t_1^*, t_2^* \in J_{\mathbb{T}}$  such that  $p(t_1^*) < 0$  and  $p(t_2^*) > 0$ .

In case (a), (2.22) implies that

$$\begin{aligned} p^\Delta(t) &\geq 0, \quad t \neq t_k, \quad t \in J_{\mathbb{T}}, \\ p(t_k^+) - p(t_k) &\geq 0, \quad k = 1, 2, \dots, p. \end{aligned} \quad (2.24)$$

This means that  $p(t)$  is nondecreasing in  $J_{\mathbb{T}}$ ; therefore,

$$\begin{aligned} p(0) &\leq p(t_1^*) < 0, \\ p(0) &\leq p(T) \leq 0, \end{aligned} \quad (2.25)$$

which contradicts  $p(T) \leq e_{\ominus(-a)}(T, 0)p(0) < 0$ .

In case (b) let  $\sup_{t \in J_{\mathbb{T}}} p(t) = \lambda$ . Then,  $\lambda > 0$  and there exists  $t_i < t_0^* \leq t_{i+1}$  for some  $i$  such that  $p(t_0^*) = \lambda$  or  $p(t_i^+) = \lambda$ . We may assume that  $p(t_0^*) = \lambda$  (since, in case of  $p(t_i^+) = \lambda$ , the proof is similar). From (2.22), we have

$$\begin{aligned} p^\Delta(t) &\geq -\lambda k_0 b(t) \int_0^t e_{\ominus(-a)}(\sigma(t), s) \Delta s - \lambda h_0 c(t) \int_0^T e_{\ominus(-a)}(\sigma(t), s) \Delta s \\ &\geq -\lambda(Bk_0 + Ch_0), \quad t \neq t_k, \quad t \in J_{\mathbb{T}}. \end{aligned} \quad (2.26)$$

For  $t \in [t_0^*, T]_{\mathbb{T}}$ ,  $k = i+1, i+2, \dots, p$ ,

$$p^\Delta(t) \geq -\lambda(Bk_0 + Ch_0), \quad t \neq t_k, \quad p(t_k^+) \geq (1 - L_k)p(t_k). \quad (2.27)$$

By Lemma 2.4, we have

$$p(t) \geq p(t_0^*) \prod_{t_0^* < t_k < t} (1 - L_k) + \int_{t_0^*}^t \prod_{s < t_k < t} (1 - L_k) (-\lambda(Bk_0 + Ch_0)) \Delta s. \quad (2.28)$$

Let  $t = T$  in (2.28), then

$$p(T) \geq \lambda \prod_{t_0^* < t_k < T} (1 - L_k) - \lambda(Bk_0 + Ch_0) \int_{t_0^*}^T \prod_{s < t_k < T} (1 - L_k) \Delta s. \quad (2.29)$$

If  $p(T) < 0$ , then (2.29) gives

$$(Bk_0 + Ch_0) > \frac{\prod_{t_0^* < t_k < T} (1 - L_k)}{\int_{t_0^*}^T \prod_{s < t_k < T} (1 - L_k) \Delta s} \geq \frac{\prod_{0 < t_k < T} (1 - L_k)}{\int_0^T \prod_{s < t_k < T} (1 - L_k) \Delta s}, \quad (2.30)$$

which contradicts (2.21), so, we have  $p(T) \geq 0$ , and by (2.22),  $p(0) \geq p(T)e_{-a}(T, 0) \geq 0$ . Hence,  $0 < t_1^* < T$ . Let  $t_j < t_1^* \leq t_{j+1}$  for some  $j$ . We first assume that  $t_0^* < t_1^*$ , so  $i \leq j$ . Let  $t = t_1^*$  in (2.28), we have

$$0 > p(t_1^*) \geq \lambda \prod_{t_0^* < t_k < t_1^*} (1 - L_k) + \int_{t_0^*}^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k) [-\lambda(Bk_0 + Ch_0)] \Delta s, \quad (2.31)$$

which gives

$$(Bk_0 + Ch_0) > \frac{\prod_{t_0^* < t_k < t_1^*} (1 - L_k)}{\int_{t_0^*}^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k) \Delta s} \geq \frac{\prod_{0 < t_k < T} (1 - L_k)}{\int_0^T \prod_{s < t_k < T} (1 - L_k) \Delta s}, \quad (2.32)$$

which contradicts (2.21).

Next we assume that  $t_1^* < t_0^*$ . So  $j \leq i$ . For  $t \in J_T$ ,  $k = 1, 2, \dots, p$ ,

$$p^\Delta(t) \geq -\lambda(Bk_0 + Ch_0), \quad t \neq t_k, \quad p(t_k^+) \geq (1 - L_k)p(t_k). \quad (2.33)$$

By Lemma 2.4, we have

$$p(t) \geq p(0) \prod_{0 < t_k < t} (1 - L_k) + \int_0^t \prod_{s < t_k < t} (1 - L_k) (-\lambda(Bk_0 + Ch_0)) \Delta s. \quad (2.34)$$

Let  $t = t_1^*$  in (2.34), then

$$0 > p(t_1^*) \geq p(0) \prod_{0 < t_k < t_1^*} (1 - L_k) - \lambda(Bk_0 + Ch_0) \int_0^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k) \Delta s, \quad (2.35)$$

which implies

$$p(0) \prod_{0 < t_k < t_1^*} (1 - L_k) < \lambda(Bk_0 + Ch_0) \int_0^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k) \Delta s. \quad (2.36)$$

By (2.22), we obtain

$$\lambda(Bk_0 + Ch_0) \int_0^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k) \Delta s > e_{(-a)}(T, 0)p(T) \prod_{0 < t_k < t_1^*} (1 - L_k). \quad (2.37)$$

From (2.29), (2.37), we have

$$\begin{aligned} & \lambda(Bk_0 + Ch_0) \int_0^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k) \Delta s \\ & > e_{(-a)}(T, 0) \prod_{0 < t_k < t_1^*} (1 - L_k) \left\{ \lambda \prod_{t_0^* < t_k < T} (1 - L_k) - \lambda(Bk_0 + Ch_0) \int_{t_0^*}^T \prod_{s < t_k < T} (1 - L_k) \Delta s \right\} \end{aligned} \quad (2.38)$$

or

$$\begin{aligned} & \prod_{0 < t_k < t_1^*} (1 - L_k) \prod_{t_0^* < t_k < T} (1 - L_k) < (Bk_0 + Ch_0) \prod_{0 < t_k < t_1^*} (1 - L_k) \int_{t_0^*}^T \prod_{s < t_k < T} (1 - L_k) \Delta s \\ & \quad + (Bk_0 + Ch_0) e_{\ominus(-a)}(T, 0) \int_0^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k) \Delta s. \end{aligned} \quad (2.39)$$

Hence

$$\begin{aligned}
\left\{ \prod_{0 < t_k < T} (1 - L_k) \right\}^2 &\leq \prod_{0 < t_k < t_1^*} (1 - L_k) \prod_{t_0^* < t_k < T} (1 - L_k) \prod_{0 < t_k < T} (1 - L_k) \\
&< (Bk_0 + Ch_0) \prod_{0 < t_k < t_1^*} (1 - L_k) \prod_{0 < t_k < T} (1 - L_k) \int_{t_0^*}^T \prod_{s < t_k < T} (1 - L_k) \Delta s \\
&\quad + (Bk_0 + Ch_0) e_{\ominus(-a)}(T, 0) \prod_{0 < t_k < T} (1 - L_k) \int_0^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k) \Delta s \\
&< (Bk_0 + Ch_0) e_{\ominus(-a)}(T, 0) \int_0^T \prod_{s < t_k < T} (1 - L_k) \Delta s,
\end{aligned} \tag{2.40}$$

which contradicts (2.21).

Thus the proof of Lemma 2.5 is complete.  $\square$

For any  $\delta(t) \in PC[J_{\mathbb{T}}, \mathbb{R}]$  and  $\eta \in \Omega$ ,  $a, b, c \in C[J_{\mathbb{T}}, \mathbb{R}^+]$ ,  $a$  is not identically vanishing, and  $0 \leq L_k < 1$  ( $k = 1, 2, \dots, p$ ),  $I_k \in C[\mathbb{R}, \mathbb{R}]$  ( $k = 1, 2, \dots, p$ ), we consider the linear periodic boundary value problem for a linear impulsive integrodifferential equation (PBVP):

$$\begin{aligned}
u^\Delta(t) + a(t)u(t) &= -b(t)[Tu](t) - c(t)[Su](t) + \delta(t), \quad t \neq t_k, \quad t \in J_{\mathbb{T}}, \\
u(t_k^+) - u(t_k) &= -L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k), \quad k = 1, 2, \dots, p, \\
u(0) &= u(T).
\end{aligned} \tag{2.41}$$

LEMMA 2.6.  $u \in \Omega$  is a solution of PBVP (2.41) if and only if  $u \in PC[J_{\mathbb{T}}, \mathbb{R}]$  is a solution of the following impulsive integral equation:

$$\begin{aligned}
u(t) &= \int_0^T G(t, s) \{ \delta(s) - b(s)[Tu](s) - c(s)[Su](s) \} \Delta s \\
&\quad + \sum_{0 < t_k < T} G(t, t_k) e_{(-a)}(\sigma(t_k), t_k) ( -L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k) ), \quad t \in J_{\mathbb{T}},
\end{aligned} \tag{2.42}$$

where

$$G(t, s) = \frac{1}{1 - e_{(-a)}(T, 0)} \begin{cases} e_{(-a)}(t, \sigma(s)), & 0 \leq s < t \leq T, \\ e_{(-a)}(T, 0) e_{(-a)}(t, \sigma(s)), & 0 \leq t \leq s \leq T. \end{cases} \tag{2.43}$$

*Proof.* Assume that  $u \in \Omega$  is a solution of (2.41). For the first equation of (2.41), using Lemma 2.3 on  $t \in [0, t_1]_{\mathbb{T}}$ , we have

$$u(t) = e_{(-a)}(t, 0)u(0) + \int_0^t e_{(-a)}(t, \sigma(s)) \{ \delta(s) - b(s)[Tu](s) - c(s)[Su](s) \} \Delta s. \tag{2.44}$$

Then

$$u(t_1) = e_{(-a)}(t_1, 0)u(0) + \int_0^{t_1} e_{(-a)}(t_1, \sigma(s)) \{ \delta(s) - b(s)[Tu](s) - c(s)[Su](s) \} \Delta s. \quad (2.45)$$

Again using Lemma 2.3 on  $t \in (t_1, t_2]_{\mathbb{T}}$ , then

$$\begin{aligned} u(t) &= u(t_1^+)e_{(-a)}(t, t_1) + \int_{t_1}^t e_{(-a)}(t, \sigma(s)) \{ \delta(s) - b(s)[Tu](s) - c(s)[Su](s) \} \Delta s \\ &= u(t_1)e_{(-a)}(t, t_1) + \int_{t_1}^t e_{(-a)}(t, \sigma(s)) \{ \delta(s) - b(s)[Tu](s) - c(s)[Su](s) \} \Delta s \\ &\quad + e_{(-a)}(t, t_1) (-L_1 u(t_1) + I_1(\eta(t_1)) + L_1 \eta(t_1)) \\ &= e_{(-a)}(t, 0)u(0) + \int_0^t e_{(-a)}(t, \sigma(s)) \{ \delta(s) - b(s)[Tu](s) - c(s)[Su](s) \} \Delta s \\ &\quad + e_{(-a)}(t, t_1) (-L_1 u(t_1) + I_1(\eta(t_1)) + L_1 \eta(t_1)). \end{aligned} \quad (2.46)$$

Repeating the above procession for  $t \in J_{\mathbb{T}}$ , we have

$$\begin{aligned} u(t) &= u(0)e_{(-a)}(t, 0) + \int_0^t e_{(-a)}(t, \sigma(s)) \{ \delta(s) - b(s)[Tu](s) - c(s)[Su](s) \} \Delta s \\ &\quad + \sum_{0 < t_k < t} e_{(-a)}(t, t_k) (-L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k)). \end{aligned} \quad (2.47)$$

Setting  $t = T$  in (2.47) and using the boundary condition  $u(0) = u(T)$ , we obtain

$$\begin{aligned} u(0) &= \frac{1}{1 - e_{(-a)}(T, 0)} \left\{ \int_0^T e_{(-a)}(T, \sigma(s)) (\delta(s) - b(s)[Tu](s) - c(s)[Su](s)) \Delta s \right. \\ &\quad \left. + \sum_{0 < t_k < T} e_{(-a)}(T, t_k) (-L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k)) \right\}. \end{aligned} \quad (2.48)$$

Substituting (2.48) into (2.47), we see that  $u \in PC[J_{\mathbb{T}}, \mathbb{R}]$  satisfies (2.42).

If  $u \in PC[J_{\mathbb{T}}, \mathbb{R}]$  is a solution of (2.42), then  $u \in C^1(J_{\mathbb{T}}^+, R)$  and

$$\begin{aligned} u^\Delta(t) + a(t)u(t) &= -b(t)[Tu](t) - c(t)[Su](t) + \delta(t), \quad t \neq t_k, \quad t \in J_{\mathbb{T}}, \\ u(t_k^+) - u(t_k) &= -L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k), \quad k = 1, 2, \dots, p. \end{aligned} \quad (2.49)$$

Setting  $t = 0, T$  in (2.42), respectively, we have

$$\begin{aligned} u(T) &= \frac{1}{1 - e_{(-a)}(T, 0)} \left\{ \int_0^T e_{(-a)}(T, \sigma(s)) (\delta(s) - b(s)[Tu](s) - c(s)[Su](s)) \Delta s \right. \\ &\quad \left. + \sum_{0 < t_k < T} e_{(-a)}(T, t_k) (-L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k)) \right\} = u(0). \end{aligned} \quad (2.50)$$

Therefore,  $u \in \Omega$  is a solution of (2.41). Thus Lemma 2.6 is proved.  $\square$

## 10 Boundary Value Problems

LEMMA 2.7. Assume that  $a, b, c \in C[J_{\mathbb{T}}, \mathbb{R}^+]$  and  $0 \leq L_k < 1$  ( $k = 1, 2, \dots, p$ ),  $I_k \in C[\mathbb{R}, \mathbb{R}]$  ( $k = 1, 2, \dots, p$ ),  $\delta \in PC[J_{\mathbb{T}}, \mathbb{R}]$ ,  $\eta \in \Omega$ , and the following inequality holds:

$$\frac{1}{1 - e_{(-a)}(T, 0)} \left( \int_0^T (k_0 s b(s) + Th_0 c(s)) \Delta s + \sum_{k=1}^p L_k \right) < 1. \quad (2.51)$$

Then PBVP (2.41) possesses a unique solution in  $\Omega$ .

*Proof.* For any  $u \in \Omega$ , consider the operator  $F$  defined by the formula

$$\begin{aligned} (Fu)(t) = & \int_0^T G(t, s) \{ \delta(s) - b(s) [Tu](s) - c(s) [Su](s) \} \Delta s \\ & + \sum_{0 < t_k < T} G(t, t_k) e_{(-a)}(\sigma(t_k), t_k) (-L_k u(t_k) + I_k(\eta(t_k)) + L_k \eta(t_k)), \quad t \in J_{\mathbb{T}}. \end{aligned} \quad (2.52)$$

Then  $Fu \in \Omega$ , that is,  $F\Omega \subset \Omega$ .

For every  $u, v \in \Omega, t \in J_{\mathbb{T}}$ , we have

$$\begin{aligned} |(Fu)(t) - (Fv)(t)| \leq & \int_0^T G(t, s) \{ |b(s)| |[Tu](s) - [Tv](s)| + |c(s)| |[Su](s) - [Sv](s)| \} \Delta s \\ & + \sum_{0 < t_k < T} G(t, t_k) e_{(-a)}(\sigma(t_k), t_k) L_k |u(t_k) - v(t_k)| \\ & < \frac{1}{1 - e_{(-a)}(T, 0)} \left( \int_0^T (k_0 s b(s) + Th_0 c(s)) \Delta s + \sum_{k=1}^p L_k \right) \|u - v\|_{PC}. \end{aligned} \quad (2.53)$$

Hence

$$\|Fu - Fv\|_{PC} = \sup_{t \in J_{\mathbb{T}}} |(Fu)(t) - (Fv)(t)| \leq \alpha \|u - v\|_{PC}, \quad (2.54)$$

where

$$\alpha = \frac{1}{1 - e_{(-a)}(T, 0)} \left( \int_0^T (k_0 s b(s) + Th_0 c(s)) \Delta s + \sum_{k=1}^p L_k \right) < 1. \quad (2.55)$$

Thus the operator  $F$  is a contraction on  $\Omega$ . That is, there is a unique element  $u \in \Omega$  such that  $u = Fu$ . Therefore,  $u$  is the unique solution of PBVP (2.41). The proof of Lemma 2.7 is complete.  $\square$

LEMMA 2.8.  $u \in \Omega$  is a solution of PBVP (1.1) if and only if  $u \in PC[J_{\mathbb{T}}, \mathbb{R}]$  is solution of the following integral equation:

$$\begin{aligned} u(t) = & \int_0^T G(t, s) [f(s, u(s), [Tu](s), [Su](s)) + a(s)u(s)] \Delta s \\ & + \sum_{0 < t_k < 1} G(t, t_k) e_{(-a)}(\sigma(t_k), t_k) I_k(u(t_k)), \end{aligned} \quad (2.56)$$

where

$$G(t, s) = \frac{1}{1 - e_{(-a)}(T, 0)} \begin{cases} e_{(-a)}(t, \sigma(s)), & 0 \leq s < t \leq T, \\ e_{(-a)}(T, 0)e_{(-a)}(t, \sigma(s)), & 0 \leq t \leq s \leq T. \end{cases} \quad (2.57)$$

The proof of Lemma 2.8 is similar to that of Lemma 2.6 and we will omit it here.

### 3. Main results

In this section, we will use the monotone iterative technique to prove the existence of minimal and maximal solutions of the PBVP (1.1).

**THEOREM 3.1.** *Assume that the following conditions hold.*

(H<sub>1</sub>) *There exist functions  $u_0, v_0 \in \Omega$ ,  $u_0(t) \leq v_0(t)$  for all  $t \in J_{\mathbb{T}}$  such that*

$$\begin{aligned} u_0^\Delta(t) &\leq f(t, u_0(t), [Tu_0](t), [Su_0](t)), \quad t \neq t_k, t \in J_{\mathbb{T}}, \\ u_0(t_k^+) - u_0(t_k) &\leq I_k(u_0(t_k)), \quad k = 1, 2, \dots, p, \\ u_0(0) &\leq u_0(T), \\ v_0^\Delta(t) &\geq f(t, v_0(t), [Tv_0](t), [Sv_0](t)), \quad t \neq t_k, t \in J_{\mathbb{T}}, \\ v_0(t_k^+) - v_0(t_k) &\geq I_k(v_0(t_k)), \quad k = 1, 2, \dots, p, \\ v_0(0) &\geq v_0(T). \end{aligned} \quad (3.1)$$

(H<sub>2</sub>) *The function  $f \in C[J_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$  satisfies*

$$f(t, u_2, v_2, w_2) - f(t, u_1, v_1, w_1) \geq -a(t)(u_2 - u_1) - b(t)(v_2 - v_1) - c(t)(w_2 - w_1), \quad (3.2)$$

*whenever  $u_0(t) \leq u_1 \leq u_2 \leq v_0(t)$ ,  $[Tu_0](t) \leq v_1 \leq v_2 \leq [Tv_0](t)$ ,  $[Su_0](t) \leq w_1 \leq w_2 \leq [Sv_0](t)$ ,  $t \in J_{\mathbb{T}}$ , where for  $a, b, c \in C[J_{\mathbb{T}}, \mathbb{R}^+]$ ,  $\sup_{t \in J_{\mathbb{T}}} \{\mu(t)a(t)\} < 1$ ,  $a$  is not identically vanishing.*

(H<sub>3</sub>) *The function  $I_k \in C[\mathbb{R}, \mathbb{R}]$  satisfies*

$$I_k(x) - I_k(y) \geq -L_k(x - y), \quad (3.3)$$

*whenever  $u_0(t_k) \leq y \leq x \leq v_0(t_k)$  ( $k = 1, 2, \dots, p$ ), and  $0 \leq L_k < 1$  ( $k = 1, 2, \dots, p$ ).*

*Further, assume that the inequalities (2.21) and (2.51) hold. Then PBVP (1.1) has the minimal solution  $u^*$  and maximal  $v^*$  in  $[u_0, v_0]$ . Moreover, there exist monotone iteration sequences  $\{u_n(t)\}, \{v_n(t)\} \subset [u_0, v_0]$  such that*

$$u_n(t) \longrightarrow u^*(t), v_n(t) \longrightarrow v^*(t) \quad \text{as } n \longrightarrow \infty \text{ uniformly on } t \in J_{\mathbb{T}}, \quad (3.4)$$

where  $\{u_n(t)\}, \{v_n(t)\}$  satisfy

$$\begin{aligned} u_n^\Delta(t) &= f(t, u_{n-1}(t), [Tu_{n-1}](t), [Su_{n-1}](t)) - a(t)(u_n - u_{n-1})(t) \\ &\quad - b(t)[T(u_n - u_{n-1})](t) - c(t)[S(u_n - u_{n-1})](t), \quad t \neq t_k, t \in J_{\mathbb{T}}, \\ u_n(t_k^+) - u_n(t_k) &= -L_k u_n(t_k) + I_k(u_{n-1}(t_k)) + L_k u_{n-1}(t_k), \quad k = 1, 2, \dots, p, \\ u_n(0) &= u_n(T) \quad (n = 1, 2, 3, \dots), \end{aligned} \tag{3.5}$$

$$\begin{aligned} v_n^\Delta(t) &= f(t, v_{n-1}(t), [Tv_{n-1}](t), [Sv_{n-1}](t)) - a(t)(v_n - v_{n-1})(t) \\ &\quad - b(t)[T(v_n - v_{n-1})](t) - c(t)[S(v_n - v_{n-1})](t), \quad t \neq t_k, t \in J_{\mathbb{T}}, \\ v_n(t_k^+) - v_n(t_k) &= -L_k v_n(t_k) + I_k(v_{n-1}(t_k)) + L_k v_{n-1}(t_k), \quad k = 1, 2, \dots, p, \\ v_n(0) &= v_n(T) \quad (n = 1, 2, 3, \dots), \\ u_0 &\leq u_1 \leq \dots \leq u_n \leq \dots \leq u^* \leq v^* \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0. \end{aligned} \tag{3.6}$$

*Proof.* For any  $u_{n-1}, v_{n-1} \in \Omega$ , by Lemma 2.7, we know that (3.5) has unique solution  $u_n$  and  $v_n$  in  $\Omega$ , respectively.

In the following, we will show by induction that

$$u_{n-1} \leq u_n \leq v_n \leq v_{n-1}, \quad n = 1, 2, 3, \dots \tag{3.7}$$

By (3.5) and the conditions  $(H_1)$ ,  $(H_2)$ , and  $(H_3)$ , we have

$$\begin{aligned} (u_1 - u_0)^\Delta(t) &\geq -a(t)(u_1 - u_0)(t) - b(t)[T(u_1 - u_0)](t) \\ &\quad - c(t)[S(u_1 - u_0)](t), \quad t \neq t_k, t \in J_{\mathbb{T}}, \\ (u_1 - u_0)(t_k^+) - (u_1 - u_0)(t_k) &\geq -L_k(u_1 - u_0)(t_k), \quad k = 1, 2, \dots, p, \\ (u_1 - u_0)(0) &\geq (u_1 - u_0)(T), \\ (v_0 - v_1)^\Delta(t) &\geq -a(t)(v_0 - v_1)(t) - b(t)[T(v_0 - v_1)](t) \\ &\quad - c(t)[S(v_0 - v_1)](t), \quad t \neq t_k, t \in J_{\mathbb{T}}, \\ (v_0 - v_1)(t_k^+) - (v_0 - v_1)(t_k) &\geq -L_k(v_0 - v_1)(t_k), \quad k = 1, 2, \dots, p, \\ (v_0 - v_1)(0) &\geq (v_0 - v_1)(T), \\ (v_1 - u_1)^\Delta(t) &\geq -a(t)(v_1 - u_1)(t) - b(t)[T(v_1 - u_1)](t) \\ &\quad - c(t)[S(v_1 - u_1)](t), \quad t \neq t_k, t \in J_{\mathbb{T}}, \\ (v_1 - u_1)(t_k^+) - (v_1 - u_1)(t_k) &\geq -L_k(v_1 - u_1)(t_k), \quad k = 1, 2, \dots, p, \\ (v_1 - u_1)(0) &= (v_1 - u_1)(T). \end{aligned} \tag{3.8}$$

Thus, by Lemma 2.5, we have  $u_0 \leq u_1 \leq v_1 \leq v_0$ .

Now we assume that (3.7) is true for  $i > 1$ , that is,  $u_{i-1} \leq u_i \leq v_i \leq v_{i-1}$ , and we prove that (3.7) is true for  $i + 1$  too. In fact, by (3.5), and the conditions  $H_2$  and  $H_3$ , we have that

$$\begin{aligned}
(u_{i+1} - u_i)^\Delta(t) &\geq -a(t)(u_{i+1} - u_i)(t) - b(t)[T(u_{i+1} - u_i)](t) \\
&\quad - c(t)[S(u_{i+1} - u_i)](t), \quad t \neq t_k, t \in J_{\mathbb{T}}, \\
(u_{i+1} - u_i)(t_k^+) - (u_{i+1} - u_i)(t_k) &\geq -L_k(u_{i+1} - u_i)(t_k), \quad k = 1, 2, \dots, p, \\
(u_{i+1} - u_i)(0) &= (u_{i+1} - u_i)(T), \\
(v_{i+1} - v_i)^\Delta(t) &\geq -a(t)(v_{i+1} - v_i)(t) - b(t)[T(v_{i+1} - v_i)](t) \\
&\quad - c(t)[S(v_{i+1} - v_i)](t), \quad t \neq t_k, t \in J_{\mathbb{T}}, \\
(v_{i+1} - v_i)(t_k^+) - (v_{i+1} - v_i)(t_k) &\geq -L_k(v_{i+1} - v_i)(t_k), \quad k = 1, 2, \dots, p, \\
(v_{i+1} - v_i)(0) &= (v_{i+1} - v_i)(T), \\
(v_{i+1} - u_{i+1})^\Delta(t) &\geq -a(t)(v_{i+1} - u_{i+1})(t) - b(t)[T(v_{i+1} - u_{i+1})](t) \\
&\quad - c(t)[S(v_{i+1} - u_{i+1})](t), \quad t \neq t_k, t \in J_{\mathbb{T}}, \\
(v_{i+1} - u_{i+1})(t_k^+) - (v_{i+1} - u_{i+1})(t_k) &\geq -L_k(v_{i+1} - u_{i+1})(t_k), \quad k = 1, 2, \dots, p, \\
(v_{i+1} - u_{i+1})(0) &= (v_{i+1} - u_{i+1})(T).
\end{aligned} \tag{3.9}$$

Thus, by Lemma 2.5, we have that  $u_i \leq u_{i+1} \leq v_{i+1} \leq v_i$ . So, by induction, (3.7) holds for any positive integer  $n$ .

It is easy to know by (3.7) that

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0. \tag{3.10}$$

Furthermore, by (3.5), and Lemma 2.6, we have

$$\begin{aligned}
u_n(t) &= \int_0^T G(t, s) \{ f(s, u_{n-1}(s), [Tu_{n-1}](s), [Su_{n-1}](s)) + a(s)u_{n-1}(s) \\
&\quad - b(s)[T(u_n - u_{n-1})](s) - c(s)[S(u_n - u_{n-1})](s) \} \Delta s \\
&\quad + \sum_{0 < t_k < T} G(t, t_k) e_{(-a)}(\sigma(t_k), t_k) (-L_k u_n(t_k) + I_k(u_{n-1}(t_k)) + L_k u_{n-1}(t_k)), \quad t \in J_{\mathbb{T}}, \\
v_n(t) &= \int_0^T G(t, s) \{ f(s, v_{n-1}(s), [Tv_{n-1}](s), [Sv_{n-1}](s)) + a(s)v_{n-1}(s) \\
&\quad - b(s)[T(v_n - v_{n-1})](s) - c(s)[S(v_n - v_{n-1})](s) \} \Delta s \\
&\quad + \sum_{0 < t_k < T} G(t, t_k) e_{(-a)}(\sigma(t_k), t_k) (-L_k v_n(t_k) + I_k(v_{n-1}(t_k)) + L_k v_{n-1}(t_k)), \quad t \in J_{\mathbb{T}}.
\end{aligned} \tag{3.11}$$

By (3.5) and the condition  $(H_2)$ , we have

$$\begin{aligned}
 & f(t, u_0(t), T[u_0](t), S[u_0](t)) - a(t)(v_0 - u_0)(t) \\
 & \quad - b(t)T[(v_0 - u_0)](t) - c(t)S[(v_0 - u_0)](t) \\
 & \leq u_n^\Delta(t) \leq f(t, v_0(t), T[v_0](t), S[v_0](t)) \\
 & \quad + a(t)(v_0 - u_0)(t) + b(t)T[(v_0 - u_0)](t) + c(t)S[(v_0 - u_0)](t).
 \end{aligned} \tag{3.12}$$

Thus,  $\{u_n^\Delta(t)\}$  is uniformly bounded. Also, similarly to the above we can show that  $\{v_n^\Delta(t)\}$  is uniformly bounded. Using Lemma 2.4 [12], we know that there exist  $u^*, v^*$  such that  $\lim_{n \rightarrow \infty} u_n(t) = u^*(t)$ ,  $\lim_{n \rightarrow \infty} v_n(t) = v^*(t)$  uniformly on  $J_{\mathbb{T}}$ .

Taking limits as  $n \rightarrow \infty$ , by (3.11), we have that

$$\begin{aligned}
 u^*(t) &= \int_0^T G(t, s)[f(s, u^*(s), [Tu^*](s), [Su^*](s)) + a(s)u^*(s)]\Delta s \\
 & \quad + \sum_{0 < t_k < 1} G(t, t_k)e_{(-a)}(\sigma(t_k), t_k)I_k(u^*(t_k)), \\
 v^*(t) &= \int_0^T G(t, s)[f(s, v^*(s), [Tv^*](s), [Sv^*](s)) + a(s)v^*(s)]\Delta s \\
 & \quad + \sum_{0 < t_k < 1} G(t, t_k)e_{(-a)}(\sigma(t_k), t_k)I_k(v^*(t_k)).
 \end{aligned} \tag{3.13}$$

From the above, by Lemma 2.8, we know that  $u^*$  and  $v^*$  are solutions of PBVP (1.1) in  $[u_0, v_0]$ .

Next we prove that  $u^*$  and  $v^*$  are the minimal and maximal solutions of PBVP (1.1) in  $[u_0, v_0]$ .

In fact, let  $w \in [u_0, v_0]$  be a solution of PBVP(1.1), that is,

$$\begin{aligned}
 w^\Delta(t) &= f(t, w(t), [Tw](t), [Sw](t)), \quad t \neq t_k, \quad t \in J_{\mathbb{T}}, \\
 w(t_k^+) - w(t_k) &= I_k(w(t_k)), \quad k = 1, 2, \dots, p, \\
 w(0) &= w(T).
 \end{aligned} \tag{3.14}$$

Using induction, suppose that there exists a positive integer  $n$  such that  $u_n(t) \leq w(t) \leq v_n(t)$  on  $J_{\mathbb{T}}$ . Then,

$$\begin{aligned}
 (w - u_{n+1})^\Delta(t) &= f(t, w(t), [Tw](t), [Sw](t)) \\
 & \quad - \{f(t, u_n(t), [Tu_n](t), [Su_n](t)) - a(t)(u_n - u_{n+1})(t) \\
 & \quad - b(t)[T(u_n - u_{n+1})](t) - c(t)[S(u_n - u_{n+1})](t)\} \\
 &\geq -a(t)(w(t) - u_{n+1}(t)) - b(t)[T(w - u_{n+1})](t) \\
 & \quad - c(t)[S(w - u_{n+1})](t), \quad t \neq t_k, \quad t \in J_{\mathbb{T}},
 \end{aligned}$$

$$\begin{aligned}
(w - u_{n+1})(t_k^+) &= (w - u_{n+1})(t_k) + I_k(w(t_k)) - [-L_k u_{n+1}(t_k) + I_k(u_n(t_k)) + L_k u_n(t_k)] \\
&\geq (1 - L_k)(w - u_{n+1})(t_k), \quad k = 1, 2, \dots, p, \\
(w - u_{n+1})(0) &= (w - u_{n+1})(T).
\end{aligned}
\tag{3.15}$$

By Lemma 2.5, it follows that  $w(t) \geq u_{n+1}(t)$  on  $J_{\mathbb{T}}$ . Similarly, we obtain  $v_{n+1}(t) \geq w(t)$  on  $J_{\mathbb{T}}$ . Since  $u_0(t) \leq w(t) \leq v_0(t)$  on  $J_{\mathbb{T}}$ , by induction we get

$$u_{n+1}(t) \leq w(t) \leq v_{n+1}(t), \quad n = 1, 2, 3, \dots \tag{3.16}$$

Thus, letting  $n \rightarrow \infty$  in (3.16), we have that

$$u^* \leq w \leq v^*, \tag{3.17}$$

that is,  $u^*$  and  $v^*$  are the minimal and maximal solutions of the PBVP (1.1) in the interval  $[u_0, v_0]$ .

The proof of Theorem 3.1 is complete.  $\square$

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