

## Research Article

# On the Sets of Regularity of Solutions for a Class of Degenerate Nonlinear Elliptic Fourth-Order Equations with $L^1$ Data

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Received 24 January 2007; Accepted 29 January 2007

Recommended by V. Lakshmikantham

We establish Hölder continuity of generalized solutions of the Dirichlet problem, associated to a degenerate nonlinear fourth-order equation in an open bounded set  $\Omega \subset \mathbb{R}^n$ , with  $L^1$  data, on the subsets of  $\Omega$  where the behavior of weights and of the data is regular enough.

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## 1. Introduction

In this paper, we will deal with equations involving an operator  $A : \overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega) \rightarrow (\overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega))^*$  of the form

$$Au = \sum_{|\alpha|=1,2} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \nabla_2 u), \quad (1.1)$$

where  $\Omega$  is a bounded open set of  $\mathbb{R}^n$ ,  $n > 4$ ,  $2 < p < n/2$ ,  $\max(2p, \sqrt{n}) < q < n$ ,  $\nu$  and  $\mu$  are positive functions in  $\Omega$  with properties precised later,  $\overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$  is the Banach space of all functions  $u : \Omega \rightarrow \mathbb{R}$  with the properties  $|u|^q, \nu |D^\alpha u|^q, \mu |D^\beta u|^p \in L^1(\Omega)$ ,  $|\alpha| = 1$ ,  $|\beta| = 2$ , and “zero” boundary values;  $\nabla_2 u = \{D^\alpha u : |\alpha| \leq 2\}$ .

The functions  $A_\alpha$  satisfy growth and monotonicity conditions, and in particular, the following strengthened ellipticity condition (for a.e.  $x \in \Omega$  and  $\xi = \{\xi_\alpha : |\alpha| = 1, 2\}$ ):

$$\sum_{|\alpha|=1,2} A_\alpha(x, \xi) \xi_\alpha \geq c_2 \left\{ \sum_{|\alpha|=1} \nu(x) |\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x) |\xi_\alpha|^p \right\} - g_2(x), \quad (1.2)$$

where  $c_2 > 0$ ,  $g_2(x) \in L^1(\Omega)$ .

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We will assume that the right-hand sides of our equations, depending on unknown function, belong to  $L^1(\Omega)$ .

A model representative of the given class of equations is the following:

$$\begin{aligned} - \sum_{|\alpha|=1} D^\alpha \left[ \nu \left( \sum_{|\beta|=1} |D^\beta u|^2 \right)^{(q-2)/2} D^\alpha u \right] + \sum_{|\alpha|=2} D^\alpha \left[ \mu \left( \sum_{|\beta|=2} |D^\beta u|^2 \right)^{(p-2)/2} D^\alpha u \right] \\ = -|u|^{\sigma-1} u + f \quad \text{in } \Omega, \end{aligned} \quad (1.3)$$

where  $\sigma > 1$  and  $f \in L^1(\Omega)$ .

The assumed conditions and known results of the theory of monotone operators allow us to prove existence of generalized solutions of the Dirichlet problem associated to our operator (see, e.g., [1]), bounded on the sets  $G \subset \Omega$  where the behavior of weights and of the data of the problem is regular enough (see [2]).

In our paper, following the approach of [3], we establish on such sets a result on Hölder continuity of generalized solutions of the same Dirichlet problem.

We note that for one high-order equation with degenerate nonlinear operator satisfying a strengthened ellipticity condition, regularity of solutions was studied in [4, 5] (non-degenerate case) and in [6, 7] (degenerate case). However, it has been made for equations with right-hand sides in  $L^t$  with  $t > 1$ .

### 2. Hypotheses

Let  $n \in \mathbb{N}$ ,  $n > 4$ , and let  $\Omega$  be a bounded open set of  $\mathbb{R}^n$ . Let  $p, q$  be two real numbers such that  $2 < p < n/2$ ,  $\max(2p, \sqrt{n}) < q < n$ .

Let  $\nu : \Omega \rightarrow \mathbb{R}^+$  be a measurable function such that

$$\nu \in L^1_{\text{loc}}(\Omega), \quad \left( \frac{1}{\nu} \right)^{1/(q-1)} \in L^1_{\text{loc}}(\Omega). \quad (2.1)$$

$W^{1,q}(\nu, \Omega)$  is the space of all functions  $u \in L^q(\Omega)$  such that their derivatives, in the sense of distribution,  $D^\alpha u$ ,  $|\alpha| = 1$ , are functions for which the following properties hold:  $\nu^{1/q} D^\alpha u \in L^q(\Omega)$  if  $|\alpha| = 1$ ;  $W^{1,q}(\nu, \Omega)$  is a Banach space with respect to the norm

$$\|u\|_{1,q,\nu} = \left( \int_{\Omega} |u|^q dx + \sum_{|\alpha|=1} \int_{\Omega} \nu |D^\alpha u|^q dx \right)^{1/q}. \quad (2.2)$$

$\overset{\circ}{W}^{1,q}(\nu, \Omega)$  is the closure of  $C_0^\infty(\Omega)$  in  $W^{1,q}(\nu, \Omega)$ .

Let  $\mu(x) : \Omega \rightarrow \mathbb{R}^+$  be a measurable function such that

$$\mu \in L^1_{\text{loc}}(\Omega), \quad \left( \frac{1}{\mu} \right)^{1/(p-1)} \in L^1_{\text{loc}}(\Omega). \quad (2.3)$$

$W^{1,q}_{2,p}(\nu, \mu, \Omega)$  is the space of all functions  $u \in W^{1,q}(\nu, \Omega)$ , such that their derivatives, in the sense of distribution,  $D^\alpha u$ ,  $|\alpha| = 2$ , are functions with the following properties:

$\mu^{1/p} D^\alpha u \in L^p(\Omega)$ ,  $|\alpha| = 2$ ;  $W_{2,p}^{1,q}(\nu, \mu, \Omega)$  is a Banach space with respect to the norm

$$\|u\| = \|u\|_{1,q,\nu} + \left( \sum_{|\alpha|=2} \int_{\Omega} \mu |D^\alpha u|^p dx \right)^{1/p}. \quad (2.4)$$

$\overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$  is the closure of  $C_0^\infty(\Omega)$  in  $W_{2,p}^{1,q}(\nu, \mu, \Omega)$ .

*Hypothesis 2.1.* Let  $\nu(x)$  be a measurable positive function:

$$\begin{aligned} \frac{1}{\nu} &\in L^t(\Omega) \quad \text{with } t > \frac{nq}{q^2 - n}, \\ \nu &\in L^{\bar{t}}(\Omega) \quad \text{with } \bar{t} > \frac{nt}{qt - n}. \end{aligned} \quad (2.5)$$

We put  $\tilde{q} = nqt/(n(1+t) - qt)$ . We can easily prove that a constant  $c_0 > 0$  exists such that if  $u \in W^{1,q}(\nu, \Omega)$ , the following inequality holds:

$$\int_{\Omega} |u|^{\tilde{q}} dx \leq c_0 \left\{ \int_{\text{supp } u} \left( \frac{1}{\nu} \right)^t dx \right\}^{\tilde{q}/qt} \left\{ \sum_{|\alpha|=1} \int_{\Omega} \nu |D^\alpha u|^q dx \right\}^{\tilde{q}/q}. \quad (2.6)$$

We set  $\tilde{\nu} = \mu^{q/(q-2p)} (1/\nu)^{2p/(q-2p)}$ .

*Hypothesis 2.2.*  $\tilde{\nu} \in L^1(\Omega)$ .

*Hypothesis 2.3.* There exists a real number  $r > \tilde{q}(q-1)/(\tilde{q}(q-1)(p-1) - q)$  such that

$$\frac{1}{\mu} \in L^r(\Omega). \quad (2.7)$$

For more details about weight functions, see [8, 9].

Let  $\Omega_1$  be a nonempty open set of  $\mathbb{R}^n$  such that  $\Omega_1 \subset \Omega$ .

*Definition 2.4.* It is said that  $G$  closed set of  $\mathbb{R}^n$  is a “regular set” if  $\overset{\circ}{G}$  is nonempty and  $G \subset \Omega_1$ .

Denote by  $\mathbb{R}^{n,2}$  the space of all sets  $\xi = \{\xi_\alpha \in \mathbb{R} : |\alpha| = 1, 2\}$  of real numbers; if a function  $u \in L_{\text{loc}}^1(\Omega)$  has the weak derivatives  $D^\alpha u$ ,  $|\alpha| = 1, 2$  then  $\nabla_2 u = \{D^\alpha u : |\alpha| = 1, 2\}$ . Suppose that  $A_\alpha : \Omega \times \mathbb{R}^{n,2} \rightarrow \mathbb{R}$  are Carathéodory functions.

*Hypothesis 2.5.* There exist  $c_1, c_2 > 0$  and  $g_1(x), g_2(x)$  nonnegative functions such that  $g_1, g_2 \in L^1(\Omega)$  and, for almost every  $x \in \Omega$ , for every  $\xi \in \mathbb{R}^{n,2}$ , the following inequalities

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hold:

$$\begin{aligned} & \sum_{|\alpha|=1} [\nu(x)]^{-1/(q-1)} |A_\alpha(x, \xi)|^{q/(q-1)} + \sum_{|\alpha|=2} [\mu(x)]^{-1/(p-1)} |A_\alpha(x, \xi)|^{p/(p-1)} \\ & \leq c_1 \left\{ \sum_{|\alpha|=1} \nu(x) |\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x) |\xi_\alpha|^p \right\} + g_1(x), \end{aligned} \quad (2.8)$$

$$\sum_{|\alpha|=1,2} A_\alpha(x, \xi) \xi_\alpha \geq c_2 \left\{ \sum_{|\alpha|=1} \nu(x) |\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x) |\xi_\alpha|^p \right\} - g_2(x). \quad (2.9)$$

Moreover, we will assume that for almost every  $x \in \Omega$  and every  $\xi, \xi' \in \mathbb{R}^{m,2}$ ,  $\xi \neq \xi'$ ,

$$\sum_{|\alpha|=1,2} [A_\alpha(x, \xi) - A_\alpha(x, \xi')] (\xi_\alpha - \xi'_\alpha) > 0. \quad (2.10)$$

Let  $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function such that

- (a) for almost every  $x \in \Omega$ , the function  $F(x, \cdot)$  is nonincreasing in  $\mathbb{R}$ ;
- (b) for every  $x \in \Omega$ , the function  $F(\cdot, s)$  belongs to  $L^1(\Omega)$ .

Let  $A : \overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega) \rightarrow (\overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega))^*$  be the operator such that for every  $u, v \in \overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$ ,

$$\langle Au, v \rangle = \int_{\Omega} \left\{ \sum_{|\alpha|=1,2} A_\alpha(x, \nabla_2 u) D^\alpha v \right\} dx. \quad (2.11)$$

We consider the following Dirichlet problem:

$$(P) = \begin{cases} Au = F(x, u) & \text{in } \Omega \\ D^\alpha u = 0, \quad |\alpha| = 0, 1, & \text{on } \partial\Omega. \end{cases} \quad (2.12)$$

**Definition 2.6.** A  $W$ -solution of problem (P) is a function  $u \in \overset{\circ}{W}^{2,1}(\Omega)$  such that

- (i)  $F(x, u) \in L^1(\Omega)$ ;
- (ii)  $A_\alpha(x, \nabla_2 u) \in L^1(\Omega)$ , for every  $\alpha : |\alpha| = 1, 2$ ;
- (iii)  $\langle Au, \phi \rangle = \langle F(x, u), \phi \rangle$  in distributional sense.

It is well known that Hypotheses 2.1–2.3, 2.5, and assumptions on  $F(x, s)$  imply the existence of a  $W$ -solution of problem (P) (see [1]). Moreover, a boundedness local result for such solution has been established in [2] under more restrictive hypotheses on data and weight functions.

More precisely, the following holds (see [2, Theorem 5.1]).

**THEOREM 2.7.** Suppose that Hypotheses 2.1–2.3 and 2.5 are satisfied. Let  $q_1 \in (q, \tilde{q}(q-1)/q)$ ,  $\tau > \tilde{q}/(\tilde{q}-q_1)$ . Assume that restrictions of the functions  $\nu^{q_1/(q_1-q)}$ ,  $\tilde{\nu}$ ,  $g_1$ ,  $g_2$ , and  $|F(\cdot, 0)|^{q_1/(q_1-1)}$  on  $G$  belong to  $L^\tau(G)$ , for every “regular set”  $G$ .

Then there exists  $\bar{u}$   $W$ -solution of problem (P) such that for every  $G$ ,  $\text{ess}_G \sup |\bar{u}| \leq M_G < +\infty$ , with  $M_G$  positive constant depending only on known values.

### 3. Main result

In the sequel of paper,  $G$  will be a “regular set.” In order to obtain our regularity result on  $G$ , we need the following further hypotheses.

*Hypothesis 3.1.* There exists a constant  $c' > 0$  such that for all  $y \in \overset{\circ}{G}$  and for all  $\rho > 0$ , with  $\overline{B(y, \rho)} \subset G$ , we have

$$\left\{ \rho^{-n} \int_{B(y, \rho)} \left( \frac{1}{\nu} \right)^t dx \right\}^{1/t} \left\{ \rho^{-n} \int_{B(y, \rho)} \nu^\tau dx \right\}^{1/\tau} \leq c'. \quad (3.1)$$

With regard to this assumption, see [3].

*Hypothesis 3.2.* There exist a real positive number  $\sigma$  and two real functions  $h(x) (\geq 0)$ ,  $f(x) (> 0)$  defined on  $G$ , such that

$$|F(x, s)| \leq h(x)|s|^\sigma + f(x), \quad \text{for almost every } x \in G \text{ and every } s \in \mathbb{R}. \quad (3.2)$$

Moreover, we assume that

$$h(x), f(x) \in L^\tau(G), \quad (3.3)$$

with  $\tau$  defined as above.

Using considerations stated in [1], following the approach of [3], we establish the following result.

**THEOREM 3.3.** *Let all above-stated hypotheses hold and let conditions of Theorem 2.7 be satisfied. Then, the  $W$ -solution  $\bar{u}$  of Dirichlet problem (P), essentially bounded on  $G$ , is also locally Hölderian on  $G$ .*

*More precisely, there exist positive constant  $C$  and  $\lambda$  ( $0 < \lambda < 1$ ) such that for every open set  $\Omega', \bar{\Omega}' \subset G$ , and every  $x, y \in \Omega'$*

$$|\bar{u}(x) - \bar{u}(y)| \leq C[d(\Omega', \partial G)]^{-\lambda} |x - y|^\lambda, \quad (3.4)$$

where  $C$  and  $\lambda$  depend only on  $c_1, c_2, c_0, c', n, q, p, t, \tau, \sigma, M_G, \text{diam } G, \text{meas } G, \|f\|_{L^\tau(G)}, \|h\|_{L^\tau(G)}, \|g_1\|_{L^\tau(G)}, \|g_2\|_{L^\tau(G)}, \|\tilde{\nu}\|_{L^\tau(G)}$ , and  $\|1/\nu\|_{L^t(\Omega)}$ .

*Proof.* For every  $l \in \mathbb{N}$ , we define the function  $F_l : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$F_l(x, s) = \begin{cases} -l & \text{if } F(x, 0) - F(x, s) < -l, \\ F(x, 0) - F(x, s) & \text{if } |F(x, 0) - F(x, s)| \leq l, \\ l & \text{if } F(x, 0) - F(x, s) > l, \end{cases} \quad (3.5)$$

and the function  $f_l : \Omega \rightarrow \mathbb{R}$  by

$$f_l(x) = \begin{cases} F(x, 0) & \text{if } |F(x, 0)| \leq l, \\ 0 & \text{if } |F(x, 0)| > l. \end{cases} \quad (3.6)$$

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By Lebesgue's theorem and property (b) of  $F(x, s)$ , we have that  $f_l(x)$  goes to  $F(x, 0)$  in  $L^1(\Omega)$ .

Next, inequalities (2.6), (2.8)–(2.10), property (a) of  $F(x, s)$ , and known results of the theory of monotone operators (see, e.g., [10]) imply that for any  $l \in \mathbb{N}$ , there exists  $u_l \in W_{2,p}^{1,q}(\nu, \mu, \Omega)$  such that

$$\int_{\Omega} \left\{ \sum_{|\alpha|=1,2} A_{\alpha}(x, \nabla_2 u_l) D^{\alpha} v + F_l(x, u_l) v \right\} dx = \int_{\Omega} f_l v dx, \quad (3.7)$$

for every  $v \in \mathring{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$ .

From considerations stated in [1, Section 3], we deduce that there exists a  $W$ -solution  $\bar{u}$  of problem (P) such that

$$u_l \longrightarrow \bar{u} \quad \text{a.e. in } \Omega. \quad (3.8)$$

Moreover, see proof of Theorem 2.7,

$$\operatorname{ess\,sup}_G |u_l| \leq M_G, \quad \text{for every } l \in \mathbb{N}. \quad (3.9)$$

We set  $\bar{n} = q^2/(q-2p)$ ,  $a = (1/\bar{n})(q - n/t - n/\tau)$ .

Let us fix  $y \in G$ ,  $\rho > 0$  and  $\overline{B}(y, 2\rho) \subset G$ . Let us put

$$\begin{aligned} \omega_{1,l} &= \operatorname{ess\,inf}_{B(y, 2\rho)} u_l, & \omega_{2,l} &= \operatorname{ess\,sup}_{B(y, 2\rho)} u_l, \\ \omega_l &= \omega_{2,l} - \omega_{1,l}. \end{aligned} \quad (3.10)$$

We will show that

$$\operatorname{osc} \{u_l, B(y, \rho)\} \leq \tilde{c} \omega_l + \rho^a, \quad (3.11)$$

with  $\tilde{c} \in ]0, 1[$  independent of  $l \in \mathbb{N}$ .

To this aim, we fix  $l \in \mathbb{N}$  and we set

$$\begin{aligned} \Phi_l &= \sum_{|\alpha|=1} \nu |D^{\alpha} u_l|^q + \sum_{|\alpha|=2} \mu |D^{\alpha} u_l|^p, \\ \psi(x) &= \rho^{-a\bar{n}} (1 + f(x) + h(x) + g_1(x) + g_2(x) + \tilde{v}(x)) + \rho^{-q} \nu. \end{aligned} \quad (3.12)$$

Obviously, we will assume that

$$\omega_l \geq \rho^a \quad (\text{otherwise, it is clear that (3.11) is true}). \quad (3.13)$$

We introduce now the following functions:

$$F_{1,l}(x) = \begin{cases} \frac{2e\omega_l}{u_l(x) - \omega_{1,l} + \rho^a} & \text{if } x \in B(y, 2\rho), \\ e & \text{if } x \in \Omega \setminus B(y, 2\rho); \end{cases} \quad (3.14)$$

$\varphi \in C_0^\infty(\Omega)$ :  $0 \leq \varphi \leq 1$  in  $\Omega$ ,  $\varphi = 0$  in  $\Omega \setminus B(y, 2\rho)$  and satisfying

$$|D^\alpha \varphi| \leq \bar{c} \rho^{-|\alpha|}, \quad |\alpha| = 1, 2, \quad (3.15)$$

where the positive constant  $\bar{c}$  depends only on  $n$ .

Let us fix  $s > q$  and  $r \geq 0$  and define

$$\begin{aligned} v_l &= (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^s, \\ z_l &= -\frac{1}{2e\omega_l} [r(\lg F_{1,l})^{r-1} + (q-1)(\lg F_{1,l})^r] F_{1,l}^q \varphi^s. \end{aligned} \quad (3.16)$$

From Hypothesis 2.2 and (3.15), we have that  $v_l \in \mathring{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$  and the next inequalities are true:

$$|D^\alpha v_l - z_l D^\alpha u_l| \leq \bar{c} s \varphi^{s-1} (\lg F_{1,l})^r F_{1,l}^{q-1} \rho^{-1} \quad \text{if } |\alpha| = 1 \text{ a.e. in } B(y, 2\rho), \quad (3.17)$$

$$\begin{aligned} |D^\alpha v_l - z_l D^\alpha u_l| &\leq 5q^2 s(r+1)^2 (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^s \left\{ \sum_{|\beta|=1} \frac{|D^\beta u_l|^2}{(u_l - \omega_{1,l} + \rho^a)^2} \right\} \\ &\quad + 2nq s^2 \bar{c}^2 \rho^{-2} (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^{s-2} \quad \text{if } |\alpha| = 2 \text{ a.e. in } B(y, 2\rho). \end{aligned} \quad (3.18)$$

Since  $u_l(x)$  satisfies (3.7), for  $v = v_l$ , we obtain

$$\int_\Omega \left\{ \sum_{|\alpha|=1,2} A_\alpha(x, \nabla_2 u_l) D^\alpha v_l + F_l(x, u_l) v_l \right\} dx = \int_\Omega f_l v_l dx. \quad (3.19)$$

From this, taking into account (3.9) and Hypothesis 3.2, we have

$$\int_\Omega \sum_{|\alpha|=1,2} A_\alpha(x, \nabla_2 u_l) D^\alpha v_l dx \leq (3 + M_G^G) \int_\Omega \{1 + f(x) + h(x)\} v_l dx. \quad (3.20)$$

Hence

$$\int_\Omega \sum_{|\alpha|=1,2} \{A_\alpha(x, \nabla_2 u_l) D^\alpha u_l\} (-z_l) dx \leq (3 + M_G^G) \int_\Omega \{1 + f(x) + h(x)\} v_l dx + I_1 + I_2, \quad (3.21)$$

where

$$I_i = \int_\Omega \sum_{|\alpha|=i} |A_\alpha(x, \nabla_2 u_l)| |D^\alpha v_l - z_l D^\alpha u_l| dx, \quad i = 1, 2. \quad (3.22)$$

Using Hypothesis 2.5 and definition of  $z_l$ , we have

$$\begin{aligned} \frac{(q-1)c_2}{2e\omega_l} \int_\Omega \Phi_l (\lg F_{1,l})^r F_{1,l}^q \varphi^s dx &\leq (3 + M_G^G) \int_\Omega \{1 + f(x) + h(x)\} (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^s dx \\ &\quad + \int_\Omega g_2(x) (-z_l) dx + I_1 + I_2. \end{aligned} \quad (3.23)$$

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Note that

$$\begin{aligned} F_{1,l}^{q-1} &\leq (\text{diam } G)^a (2e\omega_l)^{q-1} \rho^{-aq}, \\ -z_l &\leq (q-1)(r+1)(2e\omega_l)^{q-1} \rho^{-aq} \varphi^s (\lg F_{1,l})^r \quad \text{a.e. in } B(y, 2\rho), \end{aligned} \quad (3.24)$$

consequently, from (3.23), we obtain

$$\begin{aligned} &\frac{c_2}{2e\omega_l} \int_{B(y, 2\rho)} \Phi_l (\lg F_{1,l})^r F_{1,l}^q \varphi^s dx \\ &\leq c_3 (r+1) (2e\omega_l)^{q-1} \int_{B(y, 2\rho)} \rho^{-aq} \{1 + f(x) + h(x) + g_2(x)\} (\lg F_{1,l})^r \varphi^s dx + I_1 + I_2, \end{aligned} \quad (3.25)$$

where  $c_3 = (q-1)(3 + M_G^\sigma)(\text{diam } G + 1)$ .

Let us fix  $|\alpha| = 1$ . Let  $\epsilon > 0$ , then, applying Young's inequality and using (2.8) and (3.17), we establish

$$\begin{aligned} I_1 &\leq \frac{c_1 \epsilon}{2e\omega_l} \int_{B(y, 2\rho)} \Phi_l F_{1,l}^q (\lg F_{1,l})^r \varphi^s dx \\ &\quad + c_1 \epsilon (2e\omega_l)^{q-1} \int_{B(y, 2\rho)} \rho^{-aq} g_1(x) (\lg F_{1,l})^r \varphi^s dx \\ &\quad + \epsilon^{1-q} (2e\omega_l)^{q-1} n(\bar{c}s)^q \int_{B(y, 2\rho)} \rho^{-q} \nu (\lg F_{1,l})^r \varphi^{s-q} dx. \end{aligned} \quad (3.26)$$

Let us fix  $|\alpha| = 2$  and estimate  $I_2$ . To this aim, it will be useful to observe that the following equalities are true:

$$\frac{p-1}{p} + \frac{2}{q} + \frac{q-2p}{qp} = 1, \quad q-1 = \frac{p-1}{p} q + \left(\frac{q}{p} - 1\right). \quad (3.27)$$

Moreover,

$$\rho^{-aq-2p} \mu \leq \rho^{-a\bar{n}} \tilde{\nu} + \rho^{-q} \nu \quad \text{in } \Omega. \quad (3.28)$$

Furthermore, due to (2.8), (3.18), and Young's inequality, we have

$$\begin{aligned} I_2 &\leq \frac{c_4 \epsilon}{2e\omega_l} \int_{B(y, 2\rho)} \Phi_l F_{1,l}^q (\lg F_{1,l})^r \varphi^s dx \\ &\quad + c_5 (2e\omega_l)^{q-1} \epsilon \left(1 + \frac{1}{\epsilon}\right)^{\bar{n}} s^{\bar{n}} (r+1)^{\bar{n}} \int_{B(y, 2\rho)} \{\rho^{-a\bar{n}} (g_1(x) + \tilde{\nu}(x)) + \rho^{-q} \nu\} (\lg F_{1,l})^r \varphi^{s-q} dx, \end{aligned} \quad (3.29)$$

where  $c_4$  depends only on  $c_1, n, q$ ; and  $c_5$  depends only on  $c_1, n, q, p, \bar{c}$ , and  $\text{diam } G$ .



From (3.25), (3.26), and (3.29), we get

$$\begin{aligned}
& \frac{c_2}{2e\omega_l} \int_{B(y,2\rho)} \Phi_l(\lg F_{1,l})^r F_{1,l}^q \varphi^s dx \\
& \leq \frac{(c_1 + c_4)\epsilon}{2e\omega_l} \int_{B(y,2\rho)} \Phi_l F_{1,l}^q (\lg F_{1,l})^r \varphi^s dx \\
& \quad + (2e\omega_l)^{q-1} c_6 (r+1)^{\bar{n}} s^{\bar{n}} \left(1 + \epsilon + \frac{1}{\epsilon}\right)^{\bar{n}+1} \int_{B(y,2\rho)} \psi (\lg F_{1,l})^r \varphi^{s-q} dx,
\end{aligned} \tag{3.30}$$

where the constant  $c_6$  depends only on  $c_1, \bar{c}, n, q, p, M_G, \sigma$ , and  $\text{diam } G$ .

Setting

$$\epsilon = \frac{c_2}{2(c_1 + c_4)}, \tag{3.31}$$

from the last inequality, we deduce

$$\int_{B(y,2\rho)} \Phi_l(\lg F_{1,l})^r F_{1,l}^q \varphi^s dx \leq c_7 (2e\omega_l)^q (r+1)^{\bar{n}} s^{\bar{n}} \int_{B(y,2\rho)} \psi (\lg F_{1,l})^r \varphi^{s-q} dx, \tag{3.32}$$

where the constant  $c_7$  depends only on  $c_1, c_2, \bar{c}, n, q, p, M_G, \sigma$ , and  $\text{diam } G$ .

Now, if we choose  $\varphi$  such that  $\varphi = 1$  in  $B(y, (4/3)\rho)$ , from (3.32), with  $r = 0$  and  $s = q + 1$ , we get

$$\int_{B(y, (4/3)\rho)} \left\{ \sum_{|\alpha|=1} \nu |D^\alpha u_l|^q \right\} F_{1,l}^q dx \leq c_7 (2e\omega_l)^q (q+1)^{\bar{n}} \int_{B(y,2\rho)} \psi dx. \tag{3.33}$$

Moreover, if we take in (3.32) instead of  $\varphi$  the function  $\varphi_1 \in C_0^\infty(\Omega)$  with the properties  $0 \leq \varphi_1 \leq 1$  in  $\Omega$ ,  $\varphi_1 = 0$  in  $\Omega \setminus B(y, (4/3)\rho)$ ,  $\varphi_1 = 1$  in  $B(y, \rho)$ , and  $|D^\alpha \varphi| \leq \bar{c} \rho^{-|\alpha|}$  in  $\Omega$ ,  $|\alpha| = 1, 2$ , we obtain that for every  $r > 0$  and  $s > q$ ,

$$\int_{B(y,2\rho)} \left\{ \sum_{|\alpha|=1} \nu |D^\alpha u_l|^q \right\} (\lg F_{1,l})^r F_{1,l}^q dx \leq c_7 (2e\omega_l)^q s^{\bar{n}} (r+1)^{\bar{n}} \int_{B(y,2\rho)} \psi (\lg F_{1,l})^r \varphi_1^{s-q} dx. \tag{3.34}$$

We fix arbitrary  $r > 0$  and  $s > \tilde{q}$ , and let

$$z_l = (\lg F_{1,l})^{r/\tilde{q}} \varphi_1^{s/\tilde{q}}. \tag{3.35}$$

By means of Hypothesis 2.1, we establish that  $z_l \in \mathring{W}^{1,q}(\nu, \Omega)$  and for  $|\alpha| = 1$ ,

$$\begin{aligned}
\nu |D^\alpha z_l|^q & \leq 2^{q-1} \left( \frac{r}{\tilde{q}} \right)^q (\lg F_{1,l})^{(r/\tilde{q}-1)q} (F_{1,l})^q \frac{1}{(2e\omega_l)^q} |D^\alpha u_l|^q \nu \varphi_1^{sq/\tilde{q}} \\
& \quad + 2^{q-1} \left( \frac{s}{\tilde{q}} \right)^q (\lg F_{1,l})^{rq/\tilde{q}} \varphi_1^{(s/\tilde{q}-1)q} \bar{c}^q \rho^{-q} \nu.
\end{aligned} \tag{3.36}$$

## 10 Boundary Value Problems

Now, it is convenient to observe that  $\tilde{q}/(\tilde{q} - q_1) > nt/(qt - n)$ , then  $\tau > nt/(qt - n)$ ; moreover,  $\psi(x) \in L^\tau(G)$ . From (3.34) and (3.36), we deduce

$$\begin{aligned} & \int_{\Omega} \nu |D^\alpha z_l|^q dx \\ & \leq c_8 s^{\bar{n}} (r+1)^{\bar{n}+q} \left( \int_{B(y, 2\rho)} \psi^\tau dx \right)^{1/\tau} \left( \int_{B(y, 2\rho)} (\lg F_{1,l})^{r(q/\tilde{q})(\tau/(\tau-1))} \varphi_1^{(s/\tilde{q}-1)q(\tau/(\tau-1))} dx \right)^{(\tau-1)/\tau}, \end{aligned} \quad (3.37)$$

where the constant  $c_8$  depends only on  $c_1, c_2, \bar{c}, n, q, p, M_G, \sigma$ , and  $\text{diam } G$ .

We set

$$\theta = \frac{\tilde{q}(\tau-1)}{q\tau}, \quad m = \frac{q\tau}{\tau-1}, \quad (3.38)$$

and for every  $r, s > 0$ , we define

$$I(r, s) = \int_{B(y, 2\rho)} (\lg F_{1,l})^r \varphi_1^s dx. \quad (3.39)$$

Consequently, last inequality can be rewritten in this manner:

$$\int_{\Omega} \nu |D^\alpha z_l|^q dx \leq c_8 s^{\bar{n}} (r+1)^{\bar{n}+q} \left( \int_{B(y, 2\rho)} \psi^\tau dx \right)^{1/\tau} \left[ I\left(\frac{r}{\theta}, \frac{s}{\theta} - m\right) \right]^{(\tau-1)/\tau}. \quad (3.40)$$

Due to Hypothesis 2.1,

$$I(r, s) = \int_{B(y, 2\rho)} z_l^{\tilde{q}} dx \leq c_0 \left[ \int_{B(y, 2\rho)} \left( \frac{1}{\nu} \right)^t dx \right]^{\tilde{q}/qt} \left[ \sum_{|\alpha|=1} \int_{\Omega} \nu |D^\alpha z_l|^q dx \right]^{\tilde{q}/q}. \quad (3.41)$$

Let us denote by  $\prod_G$  the norm of  $(1 + f(x) + h(x) + g_1(x) + g_2(x) + \tilde{\nu}(x))$  in  $L^\tau(G)$ . By simple computation, we have

$$\left( \int_{B(y, 2\rho)} \psi^\tau dx \right)^{1/\tau} \leq \rho^{-q} \left( \int_{B(y, 2\rho)} \nu^\tau dx \right)^{1/\tau} + \prod_G \rho^{-a\bar{n}}. \quad (3.42)$$

Now, it is convenient to observe that  $(q - n/t - n/\tau)(\tilde{q}/q) = n(\theta - 1)$ .

Then, from (3.40)–(3.42), using Hypothesis 3.1, we get

$$I(r, s) \leq M(r+s)^{\bar{m}} \rho^{n(1-\theta)} \left[ I\left(\frac{r}{\theta}, \frac{s}{\theta} - m\right) \right]^\theta, \quad \text{for every } r > 0, s > \tilde{q}, \quad (3.43)$$

where  $\bar{m} = 2(q + \bar{n})\tilde{q}$  and the positive constant  $M$  depends only on  $c_1, c_2, \bar{c}, c_0, c', n, q, p, t, \|1/\nu\|_{L^t(\Omega)}, M_G, \sigma, \text{meas } G, \text{diam } G$ , and  $\prod_G$ .

We set for  $i = 0, 1, 2, \dots$  that

$$r_i = \frac{tq}{t+1} \theta^i, \quad s_i = \frac{m\theta}{\theta-1} (\theta^{i+1} - 1). \quad (3.44)$$

Then by (3.43), it is trivial to establish the following iterative relation:

$$I(r_i, s_i) \leq M c_9 \rho^{n(1-\theta)} \theta^{i\overline{m}} [I(r_{i-1}, s_{i-1})]^\theta \quad \text{for every } i \in \mathbb{N}, \quad (3.45)$$

where  $c_9$  depends only on  $n, q, p, t$ , and  $\tau$ .

Using this recurrent relation, we obtain that for every  $i \in \mathbb{N}$ ,

$$I(r_i, s_i) \leq \left[ (M c_9 + 1)^{1/(1-\theta)} \theta^{S\overline{m}} (\text{diam } G + 1)^n \rho^{-n} I(r_0, s_0) \right]^{\theta^i}, \quad (3.46)$$

where  $S$  is a positive constant depending only on  $n, q, t$ , and  $\tau$ .

Now, we assume that

$$\text{meas} \left\{ x \in B\left(y, \frac{4}{3}\rho\right) : u_l(x) \geq \frac{\omega_{1,l} + \omega_{2,l}}{2} \right\} \geq \frac{1}{2} \text{meas} B\left(y, \frac{4}{3}\rho\right). \quad (3.47)$$

We observe that if  $x \in B(y, (4/3)\rho)$  satisfies  $u_l(x) \geq (\omega_{1,l} + \omega_{2,l})/2$ , then  $F_{1,l}(x) \leq 4e$ , so by [11, Lemma 4], we deduce

$$\int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx \leq c \rho^n + \frac{c \rho r_0}{2e \omega_l} \int_{B(y, (4/3)\rho)} \left\{ \sum_{|\alpha|=1} |D^\alpha u_l| (\lg F_{1,l})^{r_0-1} F_{1,l} \right\} dx, \quad (3.48)$$

where  $c$  depends only on  $n$ .

Then, using Young's inequality, we get

$$\int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx \leq c r_0 \rho^n + r_0 \left( \frac{c r_0 \rho}{2e \omega_l} \right)^{r_0} \int_{B(y, (4/3)\rho)} \left\{ \sum_{|\alpha|=1} |D^\alpha u_l| \right\}^{r_0} F_{1,l}^{r_0} dx. \quad (3.49)$$

Last inequality, using Hölder's inequality and (3.33), gives

$$\begin{aligned} \int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx &\leq c r_0 \rho^n + r_0 [c r_0]^{r_0} 2^{r_0-1} [c_7 (q+1)^{\overline{n}}]^{t/(t+1)} \rho^{r_0} \\ &\quad \times \left( \int_{B(y, 2\rho)} \psi dx \right)^{t/(t+1)} \left( \int_{B(y, 2\rho)} \left( \frac{1}{\gamma} \right)^t dx \right)^{1/(t+1)}. \end{aligned} \quad (3.50)$$

Observe that due to (3.42) and Hypothesis 3.1,

$$\left( \int_{B(y, 2\rho)} \psi dx \right)^{t/(t+1)} \left( \int_{B(y, 2\rho)} \left( \frac{1}{\gamma} \right)^t dx \right)^{1/(t+1)} \leq c_{10} (1+M) \rho^{n-r_0}, \quad (3.51)$$

where  $c_{10}$  depends only on measure of the unit ball in  $\mathbb{R}^n$ .

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Consequently, from (3.50), we obtain

$$\int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx \leq \left( c_{10}(1+M)r_0 [cr_0]^{r_0} 2^{r_0-1} [c_7(q+1)^{\bar{n}}]^{t/(t+1)} + cr_0 \right) \rho^n. \quad (3.52)$$

Taking into account that

$$I(r_0, s_0) \leq \int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx, \quad (3.53)$$

from (3.46) we get

$$I(r_i, s_i) \leq [c_{11}]^{\theta^i}, \quad \text{for every } i \in \mathbb{N}. \quad (3.54)$$

Last inequality allow us to conclude that

$$\operatorname{ess\,sup}_{B(y,\rho)} F_{1,l}(x) \leq (1 + c_{11}), \quad (3.55)$$

and so

$$\operatorname{osc} \{u_l, B(y, \rho)\} \leq (1 - 2e^{-1-c_{11}}) \omega_l + \rho^a. \quad (3.56)$$

Recall that we proved (3.11) under assumption (3.47). If (3.47) is not true, we take instead of  $F_{1,l}$  the function  $F_{2,l} : \Omega \rightarrow \mathbb{R}^n$  such that  $F_{2,l} = 2e\omega_l(\omega_{2,l} - u_l + \rho^a)^{-1}$  in  $B(y, 2\rho)$ , and arguing as above, we establish (3.11) again.

It is important to observe that the positive constant  $c_{11}$  depends only on  $c_1, c_2, c, \bar{c}, c_0, c', n, q, p, t, \|1/\nu\|_{L^t(\Omega)}, M_G, \sigma, \operatorname{diam} G$ , and  $\prod_G$ , and is independent of  $l \in \mathbb{N}$ .

Now from (3.11), taking into account [12, Chapter 2, Lemma 4.8], we deduce that there exist positive constant  $C$  and  $\lambda(< 1)$  depending on  $c_{11}$  and  $a$  but independent of  $l \in \mathbb{N}$  such that

$$\operatorname{osc} \{u_l, B(y, \rho)\} \leq C[d(y, \partial^\circ G)]^{-\lambda} \rho^\lambda, \quad \text{for every } \rho \in ]0, d(y, \partial^\circ G)]. \quad (3.57)$$

This and (3.8) imply that

$$\operatorname{osc} \{\bar{u}, B(y, \rho)\} \leq C[d(y, \partial^\circ G)]^{-\lambda} \rho^\lambda, \quad \text{for every } \rho \in ]0, d(y, \partial^\circ G)]. \quad (3.58)$$

The proof is complete.  $\square$

### 4. An example

Let  $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ ,  $0 < \gamma < \min(q - n/q, q/2)$ , and let  $\nu, \mu$  be the restriction in  $\Omega \setminus \{0\}$  of real functions

$$|x|^\gamma, \quad |x|^{2p\gamma/q}. \quad (4.1)$$

According to considerations stated in [3, Section 7], we have that functions  $\nu, \mu$  satisfy Hypotheses 2.1 and 2.3.

Now, we will verify that  $\nu(x)$  satisfies Hypothesis 3.1, for all  $t: nq/(q^2 - n) < t < n/\gamma$ . To this aim, let  $G \subset \Omega \setminus \{0\}$  be a “regular set,” and fix  $y \in G, \rho > 0: \overline{B(y, \rho)} \subset G$ .

If  $|y| < 2\rho$ , it follows that  $B(y, \rho) \subset B(0, 3\rho)$ . Hence, we have

$$\begin{aligned} \int_{B(y, \rho)} \frac{1}{|x|^{\gamma t}} dx &\leq \int_{B(0, 3\rho)} \frac{1}{|x|^{\gamma t}} dx = n\chi_n \int_0^{3\rho} r^{n-1-\gamma t} dr = n\chi_n \frac{3^{n-\gamma t}}{n-\gamma t} \rho^{n-\gamma t}, \\ \int_{B(y, \rho)} |x|^{\gamma \tau} dx &\leq \int_{B(0, 3\rho)} |x|^{\gamma \tau} dx = n\chi_n \frac{3^{n+\gamma \tau}}{n+\gamma \tau} \rho^{n+\gamma \tau}. \end{aligned} \quad (4.2)$$

From (4.2), taking into account that  $\tau > nt/(qt - n)$ , we get

$$\left( \rho^{-n} \int_{B(y, \rho)} \frac{1}{|x|^{\gamma t}} dx \right)^{1/t} \left( \rho^{-n} \int_{B(y, \rho)} |x|^{\gamma \tau} dx \right)^{1/\tau} \leq (n\chi_n + 1) 3^n \left( \frac{1}{n-\gamma t} + 1 \right) \quad \text{if } |y| < 2\rho. \quad (4.3)$$

Instead if  $|y| \geq 2\rho$ , we denote by  $\Xi$  that

$$\Xi = \left\{ k \in \mathbb{N} : \frac{|y|}{\rho} \geq k \right\}. \quad (4.4)$$

Note that  $\Xi \neq \emptyset$  and is bounded from above. Consequently, if we denote  $\bar{k} = \max \Xi$ , we obtain

$$\bar{k}\rho \leq |y| < \rho(\bar{k} + 1). \quad (4.5)$$

Last inequality implies that for every  $x \in B(y, \rho)$ , it results that

$$(\bar{k} - 1)\rho \leq |x| \leq (\bar{k} + 2)\rho. \quad (4.6)$$

From (4.6), we obtain

$$\begin{aligned} \int_{B(y, \rho)} \frac{1}{|x|^{\gamma t}} dx &\leq \frac{\chi_n}{(\bar{k} - 1)^{\gamma t}} \rho^{n-\gamma t}, \\ \int_{B(y, \rho)} |x|^{\gamma \tau} dx &\leq \chi_n (\bar{k} + 2)^{\gamma \tau} \rho^{n+\gamma \tau}, \end{aligned} \quad (4.7)$$

where  $\chi_n$  is the measure of the unit ball in  $\mathbb{R}^n$ .

Therefore, we get

$$\left(\rho^{-n} \int_{B(y,\rho)} \frac{1}{|x|^{\gamma t}}\right)^{1/t} \left(\rho^{-n} \int_{B(y,\rho)} |x|^{\gamma t} dx\right)^{1/\tau} \leq 4^n (\chi_n + 1) \quad \text{if } |y| \geq 2\rho. \quad (4.8)$$

We can conclude that (3.1) holds with  $c' = 4^n(n\chi_n + 1)(1/(n - \gamma t) + 1)$ .

Next, let  $f : \Omega \rightarrow \mathbb{R}$  be the function such that for every  $x \in \Omega \setminus \{0\}$ ,

$$f(x) = \frac{|x|^{-n}}{(1 - \lg |x|)^2} + \frac{1}{\sqrt{1 - |x|}}. \quad (4.9)$$

Observe that  $f(x) \in L^1(\Omega)$  but  $f(x)$  does not belong to  $L^\gamma(\Omega)$ , for every  $\gamma > 1$ .

Let  $\sigma > 1$ , we consider the following Dirichlet problem:

$$\begin{aligned} - \sum_{|\alpha|=1} D^\alpha \left[ \nu \left( \sum_{|\beta|=1} |D^\beta u|^2 \right)^{(q-2)/2} D^\alpha u \right] + \sum_{|\alpha|=2} D^\alpha \left[ \mu \left( \sum_{|\beta|=2} |D^\beta u|^2 \right)^{(p-2)/2} D^\alpha u \right] \\ = -|u|^{\sigma-1} u + f \quad \text{in } \Omega, \\ D^\alpha u = 0, \quad |\alpha| = 0, 1, \text{ on } \partial\Omega. \end{aligned} \quad (4.10)$$

By Theorem 2.7, we establish that there exists a  $W$ -solution  $\bar{u}$  of problem (4.10), bounded in every “regular set”  $G \subset \Omega \setminus \{0\}$ , and moreover, applying our result, Hölderian in every open set  $A : \bar{A} \subset \Omega \setminus \{0\}$ .

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