SECOND-ORDER ESTIMATES FOR BOUNDARY BLOWUP SOLUTIONS OF SPECIAL ELLIPTIC EQUATIONS

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We find a second-order approximation of the boundary blowup solution of the equation $\Delta u = e^{u|u|^{\beta-1}}$, with $\beta > 0$, in a bounded smooth domain $\Omega \subset R^N$. Furthermore, we consider the equation $\Delta u = e^{u+e^u}$. In both cases, we underline the effect of the geometry of the domain in the asymptotic expansion of the solutions near the boundary $\partial\Omega$.

1. Introduction

Let $\Omega \subset R^N$ be a bounded smooth domain. In 1916, Bieberbach [10] has investigated the problem

$$\Delta u = e^u \quad \text{in } \Omega, \qquad u(x) \longrightarrow \infty \quad \text{as } x \longrightarrow \partial\Omega, \tag{1.1}$$

and has proved the existence of a classical solution called a boundary blowup (explosive, large) solution. Moreover, if $\delta = \delta(x)$ denotes the distance from x to $\partial\Omega$, we have [10] $u(x) - \log(2/\delta^2(x)) \to 0$ as $x \to \partial\Omega$. Recently, Bandle [4] has improved the previous estimate finding the expansion

$$u(x) = \log \frac{2}{\delta^2(x)} + (N-1)K(\overline{x})\delta(x) + o(\delta(x)), \tag{1.2}$$

where $K(\overline{x})$ denotes the mean curvature of $\partial\Omega$ at the point $\overline{x}$ nearest to x, and $o(\delta)$ has the usual meaning. Boundary estimates for various nonlinearities have been discussed in several papers, see for example [1, 3, 5, 8, 13–16].

In Section 2 of the present paper we investigate boundary blowup solutions of the equation $\Delta u = e^{u|u|^{\beta-1}}$, with $\beta > 0$, $\beta \neq 1$. We prove the estimate

$$u(x) = \Phi(\delta) + \beta^{-1}(N-1)K(x)\delta(\Phi(\delta))^{1-\beta} + O(1)\delta(\Phi(\delta))^{1-2\beta}, \tag{1.3}$$

where $\Phi(\delta)$ is defined by the equation

$$\int_{\Phi(s)}^{\infty} (2F(t))^{-1/2} = s, \quad F(t) = \int_{-\infty}^{t} e^{\tau |\tau|^{\beta-1}} d\tau, \tag{1.4}$$

$K(x)$ is the mean curvature of the surface $\{x \in \Omega : \delta(x) = \text{constant}\}$, and $O(1)$ denotes a bounded quantity.

In Section 3 we consider boundary blowup solutions of the equation $\Delta u = e^{u+e^u}$. We find the estimate

$$u(x) = \Psi(\delta) + (N-1)K(x)e^{-\Psi(\delta)}\delta + O(1)e^{-2\Psi(\delta)}\delta, \tag{1.5}$$

where Ψ is defined by the equation

$$\int_{\Psi(s)}^{\infty} (2e^{e^t} - 2)^{-1/2} dt = s. \tag{1.6}$$

In this paper, the distance function $\delta = \delta(x)$ plays an important role. Recall that if Ω is smooth then also $\delta(x)$ is smooth for x near to $\partial\Omega$, and [12]

$$\sum_{i=1}^{N} \delta_{x_i}\delta_{x_i} = 1, \quad -\sum_{i=1}^{N} \delta_{x_ix_i} = (N-1)K = H, \tag{1.7}$$

where $K = K(x)$ is the mean curvature of the surface $\{x \in \Omega : \delta(x) = \text{constant}\}$.

The effect of the geometry of the domain in the behaviour of boundary blowup solutions for special equations has been observed in various papers, see for example, [2, 7, 9, 11].

2. The equation $\Delta u = e^{u|u|^{\beta-1}}$

In what follows we denote with $O(1)$ a bounded quantity.

LEMMA 2.1. *Let $\beta > 0$, $f(s) = e^{s|s|^{\beta-1}}$, $F(s) = \int_{-\infty}^{s} f(t)dt$. Then*

$$F(s)f'(s)(f(s))^{-2} = 1 + O(1)s^{-\beta}. \tag{2.1}$$

Proof. For $s > 0$ we have

$$\begin{aligned}
F(s)f'(s)(f(s))^{-2} &= f'(s)(f(s))^{-2}F(0) + f'(s)(f(s))^{-2}\int_{0}^{s} f(t)dt \\
&= \beta e^{-s^{\beta}}s^{\beta-1}F(0) + e^{-s^{\beta}}\int_{0}^{s} e^{t^{\beta}}\beta t^{\beta-1}dt + \beta e^{-s^{\beta}}\int_{0}^{s} e^{t^{\beta}}(s^{\beta-1} - t^{\beta-1})dt \\
&= \beta e^{-s^{\beta}}s^{\beta-1}F(0) + 1 - e^{-s^{\beta}} + \beta e^{-s^{\beta}}\int_{0}^{s} e^{t^{\beta}}(s^{\beta-1} - t^{\beta-1})dt.
\end{aligned} \tag{2.2}$$

We have

$$\lim_{s \to \infty} s^{\beta}\beta e^{-s^{\beta}}s^{\beta-1}F(0) = 0,$$

$$\lim_{s \to \infty} s^{\beta}e^{-s^{\beta}} = 0. \tag{2.3}$$

Moreover, using de l'Hôpital's rule we find

$$
\begin{aligned}
\lim_{s \to \infty} \frac{\beta \int_0^s e^{t^\beta} \left(s^{2\beta-1} - s^\beta t^{\beta-1}\right) dt}{e^{s^\beta}} &= \lim_{s \to \infty} \frac{\int_0^s e^{t^\beta} \left((2\beta-1)s^{\beta-1} - \beta t^{\beta-1}\right) dt}{e^{s^\beta}} \\
&= \lim_{s \to \infty} \frac{(\beta-1)e^{s^\beta} s^{\beta-1} + \int_0^s e^{t^\beta} (2\beta-1)(\beta-1)s^{\beta-2} dt}{\beta e^{s^\beta} s^{\beta-1}} \\
&= \frac{\beta-1}{\beta} + (2\beta-1)(\beta-1) \lim_{s \to \infty} \frac{\int_0^s e^{t^\beta} dt}{\beta e^{s^\beta} s} \\
&= \frac{\beta-1}{\beta} + (2\beta-1)(\beta-1) \lim_{s \to \infty} \frac{1}{\beta(1 + \beta s^\beta)} = \frac{\beta-1}{\beta}.
\end{aligned}
\tag{2.4}
$$

The lemma follows. □

Remark 2.2. If $\beta = 1$, we have $F(s)f'(s)(f(s))^{-2} = 1$. We do not care of this special case because it has been discussed in [2].

LEMMA 2.3. *Let $\Phi = \Phi(\delta)$ be defined by*

$$
\int_{\Phi(\delta)}^\infty (2F(t))^{-1/2} dt = \delta, \quad F(t) = \int_{-\infty}^t f(\tau) d\tau, \quad f(\tau) = e^{\tau|\tau|^{\beta-1}}.
\tag{2.5}
$$

Then

$$
-\Phi'(\delta) = \left[1 + O(1)(\Phi(\delta))^{-\beta}\right] \delta f(\Phi(\delta)).
\tag{2.6}
$$

Proof. By the (trivial) relation

$$
-1 + 2(1 + O(1)s^{-\beta}) = 1 + O(1)s^{-\beta},
\tag{2.7}
$$

using (2.1) we have

$$
-1 + 2F(s)f'(s)(f(s))^{-2} = 1 + O(1)s^{-\beta}.
\tag{2.8}
$$

Multiplying by $(2F(s))^{-1/2}$ we find

$$
\begin{aligned}
-(2F(s))^{-1/2} + (2F(s))^{1/2} f'(s)(f(s))^{-2} &= (2F(s))^{-1/2} + O(1)(2F(s))^{-1/2} s^{-\beta}, \\
-\left((2F(s))^{1/2}(f(s))^{-1}\right)' &= (2F(s))^{-1/2} + O(1)(2F(s))^{-1/2} s^{-\beta}.
\end{aligned}
\tag{2.9}
$$

Integrating on (s, ∞) we get

$$
(2F(s))^{1/2}(f(s))^{-1} = \int_s^\infty (2F(t))^{-1/2} dt + O(1) \int_s^\infty (2F(t))^{-1/2} t^{-\beta} dt.
\tag{2.10}
$$

4   Second-order estimates

Using de l'Hôpital's rule we find

$$\lim_{s\to\infty} \frac{s^{-\beta}\int_s^\infty (2F(t))^{-1/2}dt}{\int_s^\infty (2F(t))^{-1/2}t^{-\beta}dt} = \lim_{s\to\infty} \frac{(2F(s))^{-1/2}s^{-\beta} + \beta s^{-\beta-1}\int_s^\infty (2F(t))^{-1/2}dt}{(2F(s))^{-1/2}s^{-\beta}}$$

$$= 1 + \lim_{s\to\infty} \frac{\beta \int_s^\infty (2F(t))^{-1/2}dt}{s(2F(s))^{-1/2}} \qquad (2.11)$$

$$= 1 + \lim_{s\to\infty} \frac{-\beta}{1 - s(2F(s))^{-1}f(s)} = 1.$$

In the last step we have used the limit

$$\lim_{s\to\infty} \frac{sf(s)}{F(s)} = \infty, \qquad (2.12)$$

which can be proved easily with de l'Hôpital's rule. Using (2.11), (2.10) can be rewritten as

$$(2F(s))^{1/2}(f(s))^{-1} = \int_s^\infty (2F(t))^{-1/2}dt + O(1)s^{-\beta}\int_s^\infty (2F(t))^{-1/2}dt. \qquad (2.13)$$

Putting $s = \Phi(\delta)$ and using the equation $-\Phi'(\delta) = (2F(\Phi(\delta)))^{1/2}$, the lemma follows. $\qquad \square$

THEOREM 2.4. *Let Ω be a bounded smooth domain in R^N, $N \geq 2$, and let $\beta > 0$, $\beta \neq 1$. If $u(x)$ is a boundary blowup solution of $\Delta u = e^{u|u|^{\beta-1}}$ in Ω, then*

$$u(x) = \Phi(\delta) + \beta^{-1}H\delta(\Phi(\delta))^{1-\beta} + O(1)\delta(\Phi(\delta))^{1-2\beta}, \qquad (2.14)$$

where $\Phi(\delta)$ is defined as in (2.5), $\delta = \delta(x)$ is the distance from x to $\partial\Omega$ and H is defined by (1.7).

Proof. We look for a super-solution of the form

$$w(x) = \Phi(\delta) + \beta^{-1}H\delta(\Phi(\delta))^{1-\beta} + \alpha\delta(\Phi(\delta))^{1-2\beta}, \qquad (2.15)$$

where α is a positive constant to be determined. Denoting by $'$ differentiation with respect to δ, we have

$$w_{x_i} = \Phi'(\delta)\delta_{x_i} + \beta^{-1}H_{x_i}\delta(\Phi(\delta))^{1-\beta} + \beta^{-1}H(\delta(\Phi(\delta))^{1-\beta})'\delta_{x_i} + \alpha(\delta(\Phi(\delta))^{1-2\beta})'\delta_{x_i}. \qquad (2.16)$$

Using (1.7) we find

$$\Delta w = \Phi''(\delta) - \Phi'(\delta)H + \beta^{-1}\Delta H\delta(\Phi(\delta))^{1-\beta} + 2\beta^{-1}\nabla H \cdot \nabla\delta(\delta(\Phi(\delta))^{1-\beta})'$$

$$+ \beta^{-1}H(\delta(\Phi(\delta))^{1-\beta})'' - \beta^{-1}H^2(\delta(\Phi(\delta))^{1-\beta})' \qquad (2.17)$$

$$+ \alpha(\delta(\Phi(\delta))^{1-2\beta})'' - \alpha(\delta(\Phi(\delta))^{1-2\beta})'H.$$

With $f(\tau) = e^{\tau|\tau|^{\beta-1}}$, by (2.5) we have $\Phi''(\delta) = f(\Phi)$. Often we write Φ instead of $\Phi(\delta)$ and Φ' instead of $\Phi'(\delta)$. Lemma 2.3 yields

$$-\Phi' = [1 + O(1)\Phi^{-\beta}]\delta f(\Phi). \tag{2.18}$$

Using (2.18) and the equation $\Phi' = -(2F(\Phi))^{1/2}$ we find

$$\lim_{\delta \to 0} \frac{(\Phi(\delta))^{1-\beta}}{\delta(\Phi(\delta))^{-\beta} f(\Phi)} = \lim_{\delta \to 0} \frac{\Phi}{-\Phi'} = \lim_{\delta \to 0} \frac{\Phi}{(2F(\Phi))^{1/2}}$$
$$= \lim_{s \to \infty} \left(\frac{s^2}{2F(s)}\right)^{1/2} = \lim_{s \to \infty} \left(\frac{s}{f(s)}\right)^{1/2} = 0. \tag{2.19}$$

Let us write the last result as

$$(\Phi(\delta))^{1-\beta} = o(1)\delta(\Phi(\delta))^{-\beta} f(\Phi), \tag{2.20}$$

where $o(1)$ denotes a quantity which tends to zero as $\delta \to 0$. Using (2.18) again we find

$$\lim_{\delta \to 0} \frac{(\Phi(\delta))^{-\beta}\Phi'}{\delta(\Phi(\delta))^{-\beta} f(\Phi)} = -1. \tag{2.21}$$

Therefore,

$$\left(\delta(\Phi(\delta))^{1-\beta}\right)' = (\Phi(\delta))^{1-\beta} + (1-\beta)\delta(\Phi(\delta))^{-\beta}\Phi'$$
$$= o(1)\delta(\Phi(\delta))^{-\beta} f(\Phi). \tag{2.22}$$

Further differentiation yields

$$\left(\delta(\Phi(\delta))^{1-\beta}\right)'' = 2(1-\beta)(\Phi(\delta))^{-\beta}\Phi' - \beta(1-\beta)\delta(\Phi(\delta))^{-\beta-1}(\Phi')^2$$
$$+ (1-\beta)\delta(\Phi(\delta))^{-\beta} f(\Phi). \tag{2.23}$$

Moreover, recalling (2.12) we find

$$\lim_{\delta \to 0} \frac{\delta(\Phi(\delta))^{-\beta-1}(\Phi')^2}{\delta(\Phi(\delta))^{-\beta} f(\Phi)} = \lim_{\delta \to 0} \frac{2F(\Phi)}{\Phi f(\Phi)} = \lim_{s \to \infty} \frac{2F(s)}{s f(s)} = 0. \tag{2.24}$$

Using the last result and (2.21), from (2.23) we find

$$\left(\delta(\Phi(\delta))^{1-\beta}\right)'' = O(1)\delta(\Phi(\delta))^{-\beta} f(\Phi). \tag{2.25}$$

Similarly, we find

$$\left(\delta(\Phi(\delta))^{1-2\beta}\right)' = o(1)\delta(\Phi(\delta))^{-2\beta} f(\Phi),$$
$$\left(\delta(\Phi(\delta))^{1-2\beta}\right)'' = O(1)\delta(\Phi(\delta))^{-2\beta} f(\Phi). \tag{2.26}$$

Denoting by M_1 a nonnegative constant independent of α and using (2.18), (2.20), (2.22), (2.25), (2.26), by (2.17) we get

$$\Delta w < f(\Phi)[1 + H\delta + M_1\delta\Phi^{-\beta} + \alpha M_1\delta\Phi^{-2\beta}]. \tag{2.27}$$

On the other side, we have

$$\begin{aligned} f(w) &= e^{(\Phi + \beta^{-1}H\delta\Phi^{1-\beta} + \alpha\delta\Phi^{1-2\beta})^\beta} \\ &= e^{\Phi^\beta(1 + \beta^{-1}H\delta\Phi^{-\beta} + \alpha\delta\Phi^{-2\beta})^\beta}. \end{aligned} \tag{2.28}$$

Let us take $\delta_0 > 0$ and α such that for $\{x \in \Omega : \delta(x) < \delta_0\}$ we have

$$-\frac{1}{2} < \beta^{-1}H\delta(\Phi(\delta))^{-\beta} + \alpha\delta(\Phi(\delta))^{-2\beta} < 1. \tag{2.29}$$

Then, denoting by M_2 a nonnegative constant independent of α we find

$$\begin{aligned} f(w) &> e^{\Phi^\beta(1 + H\delta\Phi^{-\beta} + \alpha\beta\delta\Phi^{-2\beta} - M_2(\delta\Phi^{-\beta})^2 - M_2(\alpha\delta\Phi^{-2\beta})^2)} \\ &= f(\Phi)e^{H\delta + \alpha\beta\delta\Phi^{-\beta} - M_2\delta^2\Phi^{-\beta} - M_2(\alpha\delta)^2\Phi^{-3\beta}} \\ &> f(\Phi)[1 + H\delta + \alpha\beta\delta\Phi^{-\beta} - M_2\delta^2\Phi^{-\beta} - M_2(\alpha\delta)^2\Phi^{-3\beta}]. \end{aligned} \tag{2.30}$$

By (2.27) and (2.30) we find that

$$\Delta w < f(w) \tag{2.31}$$

when

$$1 + H\delta + M_1\delta\Phi^{-\beta} + \alpha M_1\delta\Phi^{-2\beta} < 1 + H\delta + \alpha\beta\delta\Phi^{-\beta} - M_2\delta^2\Phi^{-\beta} - M_2(\alpha\delta)^2\Phi^{-3\beta}. \tag{2.32}$$

Rearranging we find

$$M_1 + M_2\delta < \alpha[\beta - M_2\alpha\delta\Phi^{-2\beta} - M_1\Phi^{-\beta}]. \tag{2.33}$$

We can take δ_0 small and α large so that (2.33) and (2.29) hold for $\delta(x) < \delta_0$.

Our function $f(t) = e^{t|t|^{\beta-1}}$ is positive and increasing for all t, and $F(t)t^{-2}$ is increasing for large t. Moreover, if $G(t) = \int_0^t \sqrt{F(s)}ds$, for a and b such that $1 < a < 2 < b$, we have

$$a\frac{F(t)}{f(t)} \leq \frac{G(t)}{G'(t)} \leq b\frac{F(t)}{f(t)} \quad \text{for large } t. \tag{2.34}$$

Therefore, by [7, Theorem 4(ii)] we have, for some constant $C > 0$,

$$C\delta^2\Phi'(\delta) + \Phi(\delta) \leq u(x) \leq \Phi(\delta) + C\delta\Phi(\delta). \tag{2.35}$$

Using the right-hand side of (2.35) we find

$$w(x) - u(x) \geq \Phi(\delta)\left[\beta^{-1}H\delta(\Phi(\delta))^{-\beta} + \alpha\delta(\Phi(\delta))^{-2\beta} - C\delta\right]. \tag{2.36}$$

Take α and δ_0 such that (2.33) holds and put $\alpha\delta_0(\Phi(\delta_0))^{-2\beta} = q$. Decrease δ_0 and increase α so that $\alpha\delta_0(\Phi(\delta_0))^{-\beta} = q$ and

$$\beta^{-1}H\delta(\Phi(\delta))^{-\beta} + q - C\delta > 0 \qquad (2.37)$$

for $\delta(x) = \delta_0$. Then, $w(x) \geq u(x)$ on $\{x \in \Omega : \delta(x) = \delta_0\}$. When α is fixed, by (2.36) we get $\liminf_{x \to \partial\Omega}[w(x) - u(x)] \geq 0$. Hence, using (2.31) we find $w(x) \geq u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$.

We look for a subsolution of the form

$$v(x) = \Phi(\delta) + \beta^{-1}H\delta(\Phi(\delta))^{1-\beta} - \alpha\delta(\Phi(\delta))^{1-2\beta}, \qquad (2.38)$$

where α is a positive constant to be determined. Instead of (2.27), now we find

$$\Delta v > f(\Phi)[1 + H\delta - M_1\delta\Phi^{-\beta} - \alpha M_1\delta\Phi^{-2\beta}]. \qquad (2.39)$$

Of course, the constant M_1 in (2.39) and the constants M_i in what follows are not necessarily the same as in the previous case.

Now we have

$$f(v) = e^{\Phi^\beta(1+\beta^{-1}H\delta\Phi^{-\beta}-\alpha\delta\Phi^{-2\beta})^\beta}. \qquad (2.40)$$

Let us take $\delta_0 > 0$ and α such that, for $\{x \in \Omega : \delta(x) < \delta_0\}$ we have

$$-\frac{1}{2} < \beta^{-1}H\delta(\Phi(\delta))^{-\beta} - \alpha\delta(\Phi(\delta))^{-2\beta} < 1. \qquad (2.41)$$

Then,

$$\begin{aligned}
f(v) &< e^{\Phi^\beta(1+H\delta\Phi^{-\beta}-\alpha\beta\delta\Phi^{-2\beta}+M_2(\delta\Phi^{-\beta})^2+M_2(\alpha\delta\Phi^{-2\beta})^2)} \\
&= f(\Phi)e^{H\delta-\alpha\beta\delta\Phi^{-\beta}+M_2\delta^2\Phi^{-\beta}+M_2(\alpha\delta)^2\Phi^{-3\beta}}.
\end{aligned} \qquad (2.42)$$

In our next step, we take δ and α such that

$$\alpha\delta\Phi^{-\beta} < 1, \qquad H\delta - \alpha\beta\delta\Phi^{-\beta} + M_2\delta^2\Phi^{-\beta} + M_2(\alpha\delta)^2\Phi^{-3\beta} < 1. \qquad (2.43)$$

Then we find

$$f(v) < f(\Phi)[1 + H\delta - \alpha\beta\delta\Phi^{-\beta} + M_3\delta^2 + M_3(\alpha\delta)^2\Phi^{-2\beta}]. \qquad (2.44)$$

By (2.39) and (2.44) we find that $\Delta v > f(v)$ provided

$$1 + H\delta - M_1\delta\Phi^{-\beta} - \alpha M_1\delta\Phi^{-2\beta} > 1 + H\delta - \alpha\beta\delta\Phi^{-\beta} + M_3\delta^2 + M_3(\alpha\delta)^2\Phi^{-2\beta}. \qquad (2.45)$$

Rearranging we have

$$\alpha[\beta - M_1\Phi^{-\beta} - M_3\alpha\delta\Phi^{-\beta}] > M_1 + M_3\delta\Phi^\beta. \qquad (2.46)$$

Since $\delta\Phi^\beta \to 0$ as $\delta \to 0$, inequality (2.46) (in addition to (2.41) and (2.43)) holds for $\delta(x) < \delta_0$ with suitable δ_0 and α.

Using the left-hand side of (2.35) we find

$$\begin{aligned} v(x) - u(x) &\leq \beta^{-1} H \delta \left(\Phi(\delta)\right)^{1-\beta} - \alpha \delta \left(\Phi(\delta)\right)^{1-2\beta} - C\delta^2 \Phi'(\delta) \\ &= \left(\Phi(\delta)\right)^{1-\beta} \left[\beta^{-1} H \delta - \alpha \delta \left(\Phi(\delta)\right)^{-\beta} - C\delta^2 \Phi'(\delta) \left(\Phi(\delta)\right)^{\beta-1}\right]. \end{aligned} \tag{2.47}$$

Take α and δ_0 such that (2.46) holds, and put $\alpha\delta_0(\Phi(\delta_0))^{-\beta} = q$. Decrease δ_0 and increase α so that $\alpha\delta_0(\Phi(\delta_0))^{-\beta} = q$ and

$$\beta^{-1} H \delta - q - C\delta^2 \Phi'(\delta) \left(\Phi(\delta)\right)^{\beta-1} < 0 \tag{2.48}$$

for $\delta(x) = \delta_0$. Note that the previous inequality holds for δ small because

$$\lim_{\delta \to 0} \frac{\delta^2 \Phi'(\delta)}{\left(\Phi(\delta)\right)^{1-\beta}} = 0, \tag{2.49}$$

as one can prove using Lemma 2.3 and de l'Hôpital's rule. It follows from (2.47) that $v(x) \leq u(x)$ on $\{x \in \Omega : \delta(x) = \delta_0\}$. By (2.47) we also find that $v(x) - u(x) \leq 0$ on $\partial\Omega$. Hence $v(x) \leq u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$. The theorem follows. □

3. The equation $\Delta u = e^{u+e^u}$

LEMMA 3.1. *Let* $f(t) = e^{t+e^t}$, $F(s) = \int_{-\infty}^{s} f(t)dt$. *Then*

$$F(s)f'(s)\left(f(s)\right)^{-2} = 1 + O(1)e^{-s}, \tag{3.1}$$

where $O(1)$ *is a bounded quantity.*

Proof. By computation we find

$$F(s)f'(s)\left(f(s)\right)^{-2} = 1 + e^{-s} - e^{-e^s} - e^{-s-e^s}. \tag{3.2}$$

The lemma follows. □

LEMMA 3.2. *Let* $f(t)$ *and* $F(s)$ *be as in Lemma 3.1. If*

$$\int_{\Psi(\delta)}^{\infty} \left(2F(s)\right)^{-1/2} ds = \delta \tag{3.3}$$

we have

$$-\Psi'(\delta) = \left[1 + O(1)e^{-\Psi(\delta)}\right]\delta f\left(\Psi(\delta)\right). \tag{3.4}$$

Proof. By the (trivial) relation

$$-1 + 2\left(1 + O(1)e^{-s}\right) = 1 + O(1)e^{-s}, \tag{3.5}$$

using (3.1) we have

$$-1 + 2F(s)f'(s)\left(f(s)\right)^{-2} = 1 + O(1)e^{-s}. \tag{3.6}$$

Multiplying by $(2F(s))^{-1/2}$ we find

$$-(2F(s))^{-1/2} + (2F(s))^{1/2} f'(s) (f(s))^{-2} = (2F(s))^{-1/2} + O(1)(2F(s))^{-1/2} e^{-s},$$
$$-\left((2F(s))^{1/2} (f(s))^{-1}\right)' = (2F(s))^{-1/2} + O(1)(2F(s))^{-1/2} e^{-s}. \tag{3.7}$$

Integrating on (s, ∞) we get

$$(2F(s))^{1/2} (f(s))^{-1} = \int_s^\infty (2F(t))^{-1/2} dt + O(1) \int_s^\infty (2F(t))^{-1/2} e^{-t} dt. \tag{3.8}$$

Using de l'Hôpital's rule we find

$$\lim_{s \to \infty} \frac{e^{-s} \int_s^\infty (2F(t))^{-1/2} dt}{\int_s^\infty (2F(t))^{-1/2} e^{-t} dt} = 1 + \lim_{s \to \infty} \frac{\int_s^\infty (2F(t))^{-1/2} dt}{(2F(s))^{-1/2}} = 1. \tag{3.9}$$

Using (3.9), (3.8) can be rewritten as

$$(2F(s))^{1/2} (f(s))^{-1} = \int_s^\infty (2F(t))^{-1/2} dt + O(1) e^{-s} \int_s^\infty (2F(t))^{-1/2} dt. \tag{3.10}$$

Putting $s = \Psi(\delta)$ and recalling that $-\Psi'(\delta) = (2F(\Psi(\delta)))^{1/2}$, the lemma follows. □

THEOREM 3.3. *Let Ω be a bounded smooth domain in R^N, $N \geq 2$, and let $f(t) = e^{t+e^t}$. If $u(x)$ is a boundary blowup solution of $\Delta u = f(u)$ in Ω, then we have*

$$u(x) = \Psi + He^{-\Psi} \delta + O(1) e^{-2\Psi} \delta, \tag{3.11}$$

where $\Psi = \Psi(\delta)$ is defined as in Lemma 3.2 and $H = H(x)$ is defined by (1.7).

Proof. We look for a super-solution of the form

$$w(x) = \Psi + He^{-\Psi} \delta + \alpha e^{-2\Psi} \delta, \tag{3.12}$$

where α is a positive constant to be determined. Denoting by $'$ differentiation with respect to δ, we have

$$w_{x_i} = \Psi' \delta_{x_i} + H_{x_i} e^{-\Psi} \delta + H \left(e^{-\Psi} \delta\right)' \delta_{x_i} + \alpha \left(e^{-2\Psi} \delta\right)' \delta_{x_i}. \tag{3.13}$$

Using (1.7) we find

$$\Delta w = \Psi'' - \Psi' H + \Delta H e^{-\Psi} \delta + \left(2\nabla H \cdot \nabla \delta - H^2\right) \left(e^{-\Psi} \delta\right)' + H \left(e^{-\Psi} \delta\right)''$$
$$- \alpha H \left(e^{-2\Psi} \delta\right)' + \alpha \left(e^{-2\Psi} \delta\right)''. \tag{3.14}$$

By Lemma 3.2 we have $-\Psi' = [1 + O(1) e^{-\Psi}] \delta f(\Psi)$, and $\Psi'' = f(\Psi)$. Moreover, since $\Psi' \delta \to 0$ as $\delta \to 0$, for δ small we also find

$$0 < \left(e^{-\Psi} \delta\right)' = e^{-\Psi} - e^{-\Psi} \Psi' \delta < C_1 e^{-\Psi}. \tag{3.15}$$

We denote with C_i positive constants (independent of α). Since $f(\Psi)\delta^2 \to 0$ and $f(\Psi)\delta \to \infty$ as $\delta \to 0$, we get

$$0 < \left(e^{-\Psi}\delta\right)'' = -2e^{-\Psi}\Psi' - e^{-\Psi}f(\Psi)\delta + e^{-\Psi}(\Psi')^2\delta < C_2 e^{-\Psi}f(\Psi)\delta. \tag{3.16}$$

Similarly, we find

$$0 < \left(e^{-2\Psi}\delta\right)' < C_3 e^{-2\Psi},$$
$$0 < \left(e^{-2\Psi}\delta\right)'' < C_4 e^{-2\Psi}f(\Psi)\delta. \tag{3.17}$$

Therefore, by (3.14) we infer

$$\Delta w < f(\Psi)\left[1 + H\delta + M_1 e^{-\Psi}\delta + \alpha M_2 e^{-2\Psi}\delta\right]. \tag{3.18}$$

On the other side, since

$$e^w = e^{\Psi + He^{-\Psi}\delta + \alpha e^{-2\Psi}\delta} > e^{\Psi}\left[1 + He^{-\Psi}\delta + \alpha e^{-2\Psi}\delta\right], \tag{3.19}$$

we find

$$\begin{aligned}
f(w) = e^{w+e^w} &> e^{\Psi + He^{-\Psi}\delta + \alpha e^{-2\Psi}\delta + e^{\Psi}[1 + He^{-\Psi}\delta + \alpha e^{-2\Psi}\delta]} \\
&= e^{\Psi + e^{\Psi}} e^{[He^{-\Psi}\delta + \alpha e^{-2\Psi}\delta + H\delta + \alpha e^{-\Psi}\delta]} \\
&> f(\Psi)\left[1 - M_3 e^{-\Psi}\delta + H\delta + \alpha e^{-\Psi}\delta\right].
\end{aligned} \tag{3.20}$$

By (3.18) and (3.20) we have

$$\Delta w < f(w) \tag{3.21}$$

provided

$$1 + H\delta + M_1 e^{-\Psi}\delta + \alpha M_2 e^{-2\Psi}\delta < 1 - M_3 e^{-\Psi}\delta + H\delta + \alpha e^{-\Psi}\delta. \tag{3.22}$$

Rearranging we find

$$M_1 + M_3 < \alpha\left[1 - M_2 e^{-\Psi(\delta)}\right]. \tag{3.23}$$

Inequality (3.23) holds provided δ is small and α is large enough.

The function $f(t) = e^{t+e^t}$ is positive and increasing for all t. If $F(t)$ is defined as in Lemma 3.1, the function $F(t)t^{-2}$ is increasing for large t. Moreover, if $G(t) = \int_0^t \sqrt{F(s)}ds$, for $1 < a < 2 < b$ we have

$$a\frac{F(t)}{f(t)} \le \frac{G(t)}{G'(t)} \le b\frac{F(t)}{f(t)} \quad \text{for large } t. \tag{3.24}$$

Therefore, by [7, Theorem 4(ii)] we have, for some constant $C > 0$,

$$C\delta^2\Psi'(\delta) + \Psi(\delta) \le u(x) \le \Psi(\delta) + C\delta\Psi(\delta). \tag{3.25}$$

Using the right-hand side of (3.25) we find

$$w(x) - u(x) \geq He^{-\Psi}\delta + \alpha e^{-2\Psi}\delta - C\delta\Psi(\delta). \tag{3.26}$$

Take α and δ_0 so that (3.23) holds for $\delta(x) = \delta_0$ and put $q = \alpha e^{-2\Psi(\delta_0)}\delta_0$. Decrease δ_0 and increase α so that $\alpha e^{-2\Psi(\delta_0)}\delta_0 = q$ and $He^{-\Psi}\delta + q - C\delta\Psi(\delta) > 0$ for $\delta(x) = \delta_0$. Recall that $\delta\Psi(\delta) \to 0$ as $\delta \to 0$. Then, $w(x) \geq u(x)$ on $\{x \in \Omega : \delta(x) = \delta_0\}$. Moreover, by (3.26) we have $w(x) - u(x) \geq 0$ on $\partial\Omega$. Hence, using (3.21) we find $w(x) \geq u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$.

Let us prove that

$$v = \Psi + He^{-\Psi}\delta - \alpha e^{-2\Psi}\delta \tag{3.27}$$

is a subsolution provided α is a suitable positive constant. By computation, instead of (3.18), now we find

$$\Delta v > f(\Psi)[1 + H\delta - M_4 e^{-\Psi}\delta - \alpha M_5 e^{-2\Psi}\delta]. \tag{3.28}$$

The next step is slightly delicate. Take α and δ such that

$$e\alpha e^{-\Psi}\delta < 1, \qquad He^{-\Psi}\delta - \alpha e^{-2\Psi}\delta < 1. \tag{3.29}$$

Then, using the second inequality in (3.29), we find

$$e^v = e^{\Psi + He^{-\Psi}\delta - \alpha e^{-2\Psi}\delta} < e^{\Psi}\left[1 + He^{-\Psi}\delta - \alpha e^{-2\Psi}\delta + e(He^{-\Psi}\delta)^2 + e(\alpha e^{-2\Psi}\delta)^2\right]. \tag{3.30}$$

Hence, using the first inequality in (3.29), we get

$$\begin{aligned} f(v) = e^{v+e^v} &< e^{\Psi + He^{-\Psi}\delta - \alpha e^{-2\Psi}\delta + e^{\Psi} + H\delta - \alpha e^{-\Psi}\delta + eH^2 e^{-\Psi}\delta^2 + e\alpha^2 e^{-3\Psi}\delta^2} \\ &< f(\Psi)e^{H\delta + M_6 e^{-\Psi}\delta - \alpha e^{-\Psi}\delta} < f(\Psi)\left[1 + H\delta + M_7 e^{-\Psi}\delta - \alpha e^{-\Psi}\delta + (\alpha e^{-\Psi}\delta)^2\right]. \end{aligned} \tag{3.31}$$

Comparing the last estimate with (3.28) we have

$$\Delta v > f(v) \tag{3.32}$$

provided

$$1 + H\delta - M_4 e^{-\Psi}\delta - \alpha M_5 e^{-2\Psi}\delta > 1 + H\delta + M_7 e^{-\Psi}\delta - \alpha e^{-\Psi}\delta + (\alpha e^{-\Psi}\delta)^2. \tag{3.33}$$

Rearranging, this inequality reads as

$$\alpha[1 - \alpha e^{-\Psi}\delta - M_5 e^{-\Psi}] > M_4 + M_7. \tag{3.34}$$

Of course, (3.34) and (3.29) hold provided α is large and δ is small enough. Using the left-hand side of (3.25), decreasing δ_0 and increasing α if necessary, one proves that $v(x) - u(x) \leq 0$ at all points in Ω with $\delta(x) = \delta_0$. Moreover, using (3.25) again we observe that $v(x) - u(x) \leq 0$ on $\partial\Omega$. Therefore, by (3.32) it follows that $v(x)$ is a subsolution on $\{x \in \Omega : \delta(x) < \delta_0\}$. The theorem is proved. $\qquad \square$

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