

# A MODIFIED QUASI-BOUNDARY VALUE METHOD FOR A CLASS OF ABSTRACT PARABOLIC ILL-POSED PROBLEMS

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We study a final value problem for first-order abstract differential equation with positive self-adjoint unbounded operator coefficient. This problem is ill-posed. Perturbing the final condition, we obtain an approximate nonlocal problem depending on a small parameter. We show that the approximate problems are well posed and that their solutions converge if and only if the original problem has a classical solution. We also obtain estimates of the solutions of the approximate problems and a convergence result of these solutions. Finally, we give explicit convergence rates.

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## 1. Introduction

We consider the following final value problem (FVP)

$$u'(t) + Au(t) = 0, \quad 0 \leq t < T \quad (1.1)$$

$$u(T) = f \quad (1.2)$$

for some prescribed final value  $f$  in a Hilbert space  $H$ ; where  $A$  is a positive self-adjoint operator such that  $0 \in \rho(A)$ . Such problems are not well posed, that is, even if a unique solution exists on  $[0, T]$  it need not depend continuously on the final value  $f$ . We note that this type of problems has been considered by many authors, using different approaches. Such authors as Lavrentiev [8], Lattès and Lions [7], Miller [10], Payne [11], and Showalter [12] have approximated (FVP) by perturbing the operator  $A$ .

In [1, 4, 13] a similar problem is treated in a different way. By perturbing the final value condition, they approximated the problem (1.1), (1.2), with

$$u'(t) + Au(t) = 0, \quad 0 < t < T, \quad (1.3)$$

$$u(T) + \alpha u(0) = f. \quad (1.4)$$

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A similar approach known as the method of auxiliary boundary conditions was given in [6, 9]. Also, we have to mention that the non standard conditions of the form (1.4) for parabolic equations have been considered in some recent papers [2, 3].

In this paper, we perturb the final condition (1.2) to form an approximate nonlocal problem depending on a small parameter, with boundary condition containing a derivative of the same order than the equation, as follows:

$$u'(t) + Au(t) = 0, \quad 0 < t < T, \quad (1.5)$$

$$u(T) - \alpha u'(0) = f. \quad (1.6)$$

Following [4], this method is called quasi-boundary value method, and the related approximate problem is called quasi-boundary value problem (QBVP). We show that the approximate problems are well posed and that their solutions  $u_\alpha$  converge in  $C^1([0, T], H)$  if and only if the original problem has a classical solution. We show that this method gives a better approximation than many other quasi reversibility type methods, for example, [1, 4, 7]. Finally, we obtain several other results, including some explicit convergence rates. The case where the operator  $A$  has discrete spectrum has been treated in [5].

### 2. The approximate problem

*Definition 2.1.* A function  $u : [0, T] \rightarrow H$  is called a classical solution of the (FVP) problem (resp., (QBVP) problem) if  $u \in C^1([0, T], H)$ ,  $u(t) \in D(A)$  for every  $t \in [0, T]$  and satisfies (1.1) and the final condition (1.2) (resp., the boundary condition (1.6)).

Now, let  $\{E_\lambda\}_{\lambda>0}$  be a spectral measure associated to the operator  $A$  in the Hilbert space  $H$ , then for all  $f \in H$ , we can write

$$f = \int_0^\infty dE_\lambda f. \quad (2.1)$$

If the (FVP) problem (resp., (QBVP) problem) admits a solution  $u$  (resp.,  $u_\alpha$ ), then this solution can be represented by

$$u(t) = \int_0^\infty e^{\lambda(T-t)} dE_\lambda f, \quad (2.2)$$

respectively,

$$u_\alpha(t) = \int_0^\infty \frac{e^{-\lambda t}}{\alpha\lambda + e^{-\lambda T}} dE_\lambda f. \quad (2.3)$$

**THEOREM 2.2.** *For all  $f \in H$ , the functions  $u_\alpha$  given by (2.3) are classical solutions to the (QBVP) problem and we have the following estimate*

$$\|u_\alpha(t)\| \leq \frac{T}{\alpha(1 + \ln(T/\alpha))} \|f\|, \quad \forall t \in [0, T], \quad (2.4)$$

where  $\alpha < eT$ .

*Proof.* If we assume that the functions  $u_\alpha$  given in (2.3) are defined for all  $t \in [0, T]$ , then, it is easy to show that  $u_\alpha \in C^1([0, T], H)$  and

$$u'_\alpha(t) = \int_0^\infty \frac{-\lambda e^{-\lambda t}}{\alpha\lambda + e^{-\lambda T}} dE_\lambda f. \quad (2.5)$$

From

$$\|Au_\alpha(t)\|^2 = \int_0^\infty \frac{\lambda^2 e^{-2\lambda t}}{(\alpha\lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2 \leq \frac{1}{\alpha^2} \int_0^\infty d\|E_\lambda f\|^2 = \frac{1}{\alpha^2} \|f\|^2, \quad (2.6)$$

we get  $u_\alpha(t) \in D(A)$  and so  $u_\alpha \in C([0, T], D(A))$ . This shows that the function  $u_\alpha$  is a classical solution to the (QBVP) problem.

Now, using (2.3), we have

$$\|u_\alpha(t)\|^2 \leq \int_0^\infty \frac{1}{(\alpha\lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2, \quad (2.7)$$

if we put

$$h(\lambda) = (\alpha\lambda + e^{-\lambda T})^{-1}, \quad \text{for } \lambda > 0, \quad (2.8)$$

then,

$$\sup_{\lambda > 0} h(\lambda) = h\left(\frac{\ln(T/\alpha)}{T}\right), \quad (2.9)$$

and this yields

$$\|u_\alpha(t)\|^2 \leq \left[ \frac{T}{\alpha(1 + \ln(T/\alpha))} \right]^2 \int_0^\infty d\|E_\lambda f\|^2 = \left[ \frac{T}{\alpha(1 + \ln(T/\alpha))} \right]^2 \|f\|^2. \quad (2.10)$$

This shows that the integral defining  $u_\alpha(t)$  exists for all  $t \in [0, T]$  and we have the desired estimate.  $\square$

*Remark 2.3.* One advantage of this method of regularization is that the order of the error, introduced by small changes in the final value  $f$ , is less than the order given in [4].

Now, we give the following convergence result.

**THEOREM 2.4.** *For every  $f \in H$ ,  $u_\alpha(T)$  converges to  $f$  in  $H$ , as  $\alpha$  tends to zero.*

*Proof.* Let  $\varepsilon > 0$ , choose  $\eta > 0$  for which

$$\int_\eta^\infty d\|E_\lambda f\|^2 < \frac{\varepsilon}{2}. \quad (2.11)$$

From (2.3), we have

$$\|u_\alpha(T) - f\|^2 \leq \alpha^2 \int_0^\eta \frac{\lambda^2}{(\alpha\lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2 + \frac{\varepsilon}{2}, \quad (2.12)$$

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so by choosing  $\alpha$  such that

$$\alpha^2 < \varepsilon \left( 2 \int_0^\eta \lambda^2 e^{2\lambda T} \|E_\lambda f\|^2 \right)^{-1}, \quad (2.13)$$

we obtain the desired result.  $\square$

**THEOREM 2.5.** *For every  $f \in H$ , the (FVP) problem has a classical solution  $u$  given by (2.2), if and only if the sequence  $(u'_\alpha(0))_{\alpha>0}$  converge in  $H$ . Furthermore, we then have that  $u_\alpha(t)$  converges to  $u(t)$  in  $C^1([0, T], H)$  as  $\alpha$  tends to zero.*

*Proof.* If we assume that the (FVP) problem has a classical solution  $u$ , then we have

$$\begin{aligned} \|u'_\alpha(0) - u'(0)\|^2 &= \int_0^\infty \frac{\alpha^2 \lambda^4 e^{2\lambda T}}{(\alpha\lambda + e^{-\lambda T})^2} \|dE_\lambda f\|^2 \\ &\leq \alpha^2 \int_0^\eta \lambda^4 e^{4\lambda T} d\|E_\lambda f\|^2 + \int_\eta^\infty \frac{\alpha^2 \lambda^4 e^{2\lambda T}}{\alpha^2 \lambda^2} d\|E_\lambda f\|^2 \\ &< \alpha^2 \int_0^\eta \lambda^4 e^{4\lambda T} d\|E_\lambda f\|^2 + \frac{\varepsilon}{2}, \end{aligned} \quad (2.14)$$

so by choosing  $\alpha$  such that  $\alpha^2 < \varepsilon (2 \int_0^\eta \lambda^4 e^{4\lambda T} d\|E_\lambda f\|^2)^{-1}$ , we obtain

$$\|u'_\alpha(0) - u'(0)\|^2 < \varepsilon, \quad (2.15)$$

this shows that  $\|u'_\alpha(0) - u'(0)\|$  tends to zero as  $\alpha$  tends to zero. Since

$$\begin{aligned} \|u'_\alpha(t) - u'(t)\|^2 &\leq \int_0^\infty \lambda^2 \left( \frac{1}{\alpha\lambda + e^{-\lambda T}} - e^{\lambda T} \right)^2 d\|E_\lambda f\|^2 \\ &= \|u'_\alpha(0) - u'(0)\|^2, \end{aligned} \quad (2.16)$$

then  $u'_\alpha(t)$  converges to  $u'(t)$  uniformly in  $[0, T]$  as  $\alpha$  tends to zero.

Since

$$\|u_\alpha(0) - u(0)\|^2 \leq \alpha^2 \int_0^\eta \lambda^2 e^{4\lambda T} d\|E_\lambda f\|^2 + \frac{\varepsilon}{2}, \quad (2.17)$$

for  $\eta$  quite large. Then by choosing  $\alpha$  such that  $\alpha^2 < (2 \int_0^\eta \lambda^2 e^{4\lambda T} d\|E_\lambda f\|^2)^{-1}$ , we get

$$\|u_\alpha(0) - u(0)\|^2 < \varepsilon. \quad (2.18)$$

Thus  $u_\alpha(0)$  converges to  $u(0)$ , which in turn gives that  $u_\alpha(t)$  converges to  $u(t)$  uniformly in  $[0, T]$  as  $\alpha$  tends to zero. Combining all these convergence results, we conclude that  $u_\alpha(t)$  converges to  $u(t)$  in  $C^1([0, T], H)$ .

Now, assume that  $(u'_\alpha(0))_{\alpha>0}$  converges in  $H$ . Since  $u_\alpha$  is a classical solution to the (QBVP) problem, then we have

$$\|u'_\alpha(0)\|^2 = \int_0^\infty \frac{\lambda^2}{(\alpha\lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2, \quad (2.19)$$

and it is easy to show that

$$\left\| \lim_{\alpha \downarrow 0} u'_\alpha(0) \right\|^2 = \int_0^\infty \lambda^2 e^{2\lambda T} d\|E_\lambda f\|^2, \quad (2.20)$$

and so the function  $u(t)$  defined by

$$u(t) = \int_0^\infty e^{\lambda(T-t)} dE_\lambda f, \quad (2.21)$$

is a classical solution to the (FVP) problem. This ends the proof of the theorem.  $\square$

**THEOREM 2.6.** *If the function  $u$  given by (2.2) is a classical solution of the (FVP) problem, and  $u_\alpha^\delta$  is a solution of the (QBVP) problem for  $f = f_\delta$ , such that  $\|f - f_\delta\| < \delta$ , then we have*

$$\|u(0) - u_\alpha^\delta(0)\| \leq c \left(1 + \ln \frac{T}{\delta}\right)^{-1}, \quad (2.22)$$

where  $c = T(1 + \|Au(0)\|)$ .

*Proof.* Suppose that the function  $u$  given by (2.2) is a classical solution to the (FVP) problem, and let's denote by  $u_\alpha^\delta$  a solution of the (QBVP) problem for  $f = f_\delta$ , such that

$$\|f - f_\delta\| < \delta. \quad (2.23)$$

Then,  $u_\alpha^\delta(t)$  is given by

$$u_\alpha^\delta(t) = \int_0^\infty \frac{e^{-\lambda t}}{\alpha\lambda + e^{-\lambda T}} dE_\lambda f_\delta, \quad \forall t \in [0, T]. \quad (2.24)$$

From (2.2) and (2.24), we have

$$\|u(0) - u_\alpha^\delta(0)\| \leq \Delta_1 + \Delta_2, \quad (2.25)$$

where  $\Delta_1 = \|u(0) - u_\alpha(0)\|$ , and  $\Delta_2 = \|u_\alpha(0) - u_\alpha^\delta(0)\|$ . Using (2.9), we get

$$\begin{aligned} \Delta_1 &\leq \frac{T}{(1 + \ln(T/\alpha))} \left( \int_0^\infty \lambda^2 e^{2\lambda T} d\|E_\lambda f\|^2 \right)^{1/2}, \\ \Delta_2 &\leq \frac{T}{\alpha(1 + \ln(T/\alpha))} \|f - f_\delta\|, \end{aligned} \quad (2.26)$$

then,

$$\begin{aligned} \Delta_1 &\leq \frac{T\|Au(0)\|}{1 + \ln(T/\alpha)}, \\ \Delta_2 &\leq \frac{T\delta}{\alpha(1 + \ln(T/\alpha))}. \end{aligned} \quad (2.27)$$

From (2.27), we obtain

$$\|u_\alpha(0) - u_\alpha^\delta(0)\|^2 \leq \frac{T\|Au(0)\|}{(1 + \ln(T/\alpha))} + \frac{T\delta}{\alpha(1 + \ln(T/\alpha))}, \quad (2.28)$$

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then, for the choice  $\alpha = \delta$ , we get

$$\|u_\alpha(0) - u_\alpha^\delta(0)\|^2 \leq \frac{T(1 + \|Au(0)\|)}{(1 + \ln(T/\alpha))}. \quad (2.29)$$

□

*Remark 2.7.* From (2.22), for  $T > e^{-1}$  we get

$$\|u(0) - u_\alpha^\delta(0)\| \leq c \left( \ln \frac{1}{\delta} \right)^{-1}, \quad (2.30)$$

*Remark 2.8.* Under the hypothesis of the above theorem, if we denote by  $U_\alpha^\delta$  the solution of the approximate (FVP) problem for  $f = f_\delta$ , using the quasireversibility method [7], we obtain the following estimate

$$\|u(0) - U_\alpha^\delta(0)\| \leq c_1 \left( \ln \frac{1}{\delta} \right)^{-2/3}. \quad (2.31)$$

*Proof.* A proof can be given in a similar way as in [9]. □

**THEOREM 2.9.** *If there exists an  $\varepsilon \in ]0, 2[$  so that*

$$\int_0^\infty \lambda^\varepsilon e^{\varepsilon \lambda T} \|dE_\lambda f\|^2, \quad (2.32)$$

*converges, then  $u_\alpha(T)$  converges to  $f$  with order  $\alpha^\varepsilon \varepsilon^{-2}$  as  $\alpha$  tends to zero.*

*Proof.* Let  $\varepsilon \in ]0, 2[$  such that  $\int_0^\infty \lambda^\varepsilon e^{\varepsilon \lambda T} \|dE_\lambda f\|^2$  converges, and let  $\beta \in ]0, 2[$ . For a fix  $\lambda > 0$ , and if we define a function  $g_\lambda(\alpha) = \alpha^\beta / (\alpha \lambda + e^{-\lambda T})^2$ . Then we can show that

$$g_\lambda(\alpha) \leq g_\lambda(\alpha_0), \quad \forall \alpha > 0, \quad (2.33)$$

where  $\alpha_0 = \beta e^{-\lambda T} / (2 - \beta) \lambda$ . Furthermore, from (2.3), we have

$$\|u_\alpha(T) - f\|^2 = \alpha^{2-\beta} \int_0^\infty \lambda^2 g_\lambda(\alpha) dE_\lambda f. \quad (2.34)$$

Hence from (2.33) and (2.34) we obtain

$$\|u_\alpha(T) - f\|^2 \leq \alpha^{2-\beta} \left( \frac{\beta}{2-\beta} \right)^\beta \int_0^\infty \lambda^{2-\beta} e^{(2-\beta)\lambda T} d\|E_\lambda f\|^2. \quad (2.35)$$

If we choose  $\beta = (2 - \varepsilon)$ , we have

$$\|u_\alpha(T) - f\|^2 \leq \alpha^\varepsilon \varepsilon^{-2} \left( 4 \int_0^\infty \lambda^\varepsilon e^{\varepsilon \lambda T} d\|E_\lambda f\|^2 \right), \quad (2.36)$$

hence

$$\|u_\alpha(T) - f\|^2 \leq c_\varepsilon \alpha^\varepsilon \varepsilon^{-2} \quad (2.37)$$

with  $c_\varepsilon = 4 \int_0^\infty \lambda^\varepsilon e^{\varepsilon \lambda T} d\|E_\lambda f\|^2$ . □

Now, we give the following corollary.

**COROLLARY 2.10.** *If there exists an  $\varepsilon \in ]0, 2[$  so that*

$$\int_0^\infty \lambda^{(\varepsilon+2\gamma)} e^{(\varepsilon+2)\lambda T} d\|E_\lambda f\|^2, \quad (2.38)$$

where  $\gamma = \overline{0, 1}$ , converges, then  $u_\alpha$  converges to  $u$  in  $C^1([0, T], H)$  with order of convergence  $\alpha^\varepsilon \varepsilon^{-2}$ .

*Proof.* If we assume that (2.38) is satisfied, then

$$\int_0^\infty \lambda^2 e^{2\lambda T} d\|E_\lambda f\|^2, \quad (2.39)$$

converges, and so the function  $u(t)$  given by (2.2) is a classical solution of the (FVP) problem. Let  $u_\alpha^{(\gamma)}$ ,  $u^{(\gamma)}$  denote the derivatives of order  $\gamma$  ( $\gamma = \overline{0, 1}$ ) of the functions  $u_\alpha$  and  $u$ , respectively. Using the following inequalities

$$\begin{aligned} \left\| u_\alpha^{(\gamma)}(0) - u^{(\gamma)}(0) \right\|^2 &= \int_0^\infty \frac{\alpha^2 \lambda^{(2+2\gamma)} e^{2\lambda T}}{(\alpha \lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2 \\ &\leq \alpha^{2-\beta} \left( \frac{\beta}{2-\beta} \right)^\beta \int_0^\infty \lambda^{(2+2\gamma-\beta)} e^{(4-\beta)\lambda T} d\|E_\lambda f\|^2, \end{aligned} \quad (2.40)$$

and setting  $\beta = 2 - \varepsilon$ , in (2.40), we obtain

$$\left\| u_\alpha^{(\gamma)}(0) - u^{(\gamma)}(0) \right\|^2 \leq c_{\varepsilon, \gamma} \alpha^\varepsilon \varepsilon^{-2}, \quad (2.41)$$

where  $c_{\varepsilon, \gamma} = 4 \int_0^\infty \lambda^{(\varepsilon+2\gamma)} e^{(\varepsilon+2)\lambda T} d\|E_\lambda f\|^2$ .

And since

$$\left\| u_\alpha^{(\gamma)}(t) - u^{(\gamma)}(t) \right\|^2 \leq \left\| u_\alpha^{(\gamma)}(0) - u^{(\gamma)}(0) \right\|^2, \quad (2.42)$$

then  $u_\alpha^{(\gamma)}(t)$  converges to  $u^{(\gamma)}(t)$  uniformly in  $[0, T]$ , with order of convergence  $\alpha^\varepsilon \varepsilon^{-2}$ , and so  $u_\alpha$  converges to  $u$  in  $C^1([0, T], H)$ , with order  $\alpha^\varepsilon \varepsilon^{-2}$ .  $\square$

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