

MULTIPLE POSITIVE SOLUTIONS OF SINGULAR p -LAPLACIAN PROBLEMS BY VARIATIONAL METHODS

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We obtain multiple positive solutions of singular p -Laplacian problems using variational methods. The techniques are applicable to other types of singular problems as well.

1. Introduction

We consider the singular quasilinear elliptic boundary value problem

$$\begin{aligned} -\Delta_p u &= a(x)u^{-\gamma} + \lambda f(x, u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded C^2 domain in \mathbb{R}^n , $n \geq 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian, $1 < p < \infty$, $a \geq 0$ is a nontrivial measurable function, $\gamma > 0$ is a constant, $\lambda > 0$ is a parameter, and f is a Carathéodory function on $\Omega \times [0, \infty)$ satisfying

$$\sup_{(x,t) \in \Omega \times [0,T]} |f(x,t)| < \infty \quad \forall T > 0. \tag{1.2}$$

The semilinear case $p = 2$ with $\gamma < 1$ and $f = 0$ has been studied extensively in both bounded and unbounded domains (see [5, 6, 7, 10, 11, 12, 14, 20] and their references). In particular, Lair and Shaker [11] showed the existence of a unique (weak) solution when Ω is bounded and $a \in L^2(\Omega)$. Their result was extended to the sublinear case $f(t) = t^\beta$, $0 < \beta \leq 1$ by Shi and Yao [15] and Wiegner [18]. In the superlinear case $1 < \beta < 2^* - 1$ and for small λ , Coclite and Palmieri [4] obtained a solution when $a = 1$ and Sun et al. [16] obtained two solutions using the Ekeland's variational principle for more general a 's. Zhang [19] extended their multiplicity result to more general superlinear terms $f(t) \geq 0$ using critical point theory on closed convex sets. The ODE case $n = 1$ was studied by Agarwal and O'Regan [1] using fixed point theory and by Agarwal et al. [2] using variational methods. The purpose of the present paper is to treat the general quasilinear case $p \in (1, \infty)$, $\gamma \in (0, \infty)$, and f is allowed to change sign. We use a simple cutoff argument and only the basic critical point theory. Our results seem to be new even for $p = 2$.

First we assume

(H_1) $\exists \varphi \geq 0$ in $C_0^1(\overline{\Omega})$ and $q > n$ such that $a\varphi^{-\gamma} \in L^q(\Omega)$.

This does not require $\gamma < 1$ as usually assumed in the literature. For example, when Ω is the unit ball, $a(x) = (1 - |x|^2)^\sigma$, $\sigma \geq 0$, and $\gamma < \sigma + 1/n$, we can take $\varphi(x) = 1 - |x|^2$ and $q < 1/(\gamma - \sigma)$ (resp., q with no additional restrictions) if $\gamma > \sigma$ (resp., $\gamma \leq \sigma$).

THEOREM 1.1. *If (H_1) and (1.2) hold and $f \geq 0$, then $\exists \lambda_0 > 0$ such that problem (1.1) has a solution $\forall \lambda \in (0, \lambda_0)$.*

COROLLARY 1.2. *Problem (1.1) with $f = 0$ has a solution if (H_1) holds.*

Next we allow f to change sign, but strengthen (H_1) to

(H_2) $a \in L^\infty(\Omega)$ with $a_0 := \inf_\Omega a > 0$ and $\gamma < 1/n$.

This implies that $a\varphi^{-\gamma} \in L^q(\Omega)$ for any φ whose interior normal derivative $\partial\varphi/\partial\nu > 0$ on $\partial\Omega$ and $q < 1/\gamma$.

THEOREM 1.3. *If (H_2) and (1.2) hold, then $\exists \lambda_0 > 0$ such that problem (1.1) has a solution $\forall \lambda \in (0, \lambda_0)$.*

Finally we assume that f is C^1 in t , satisfies

$$|f_t(x, t)| \leq C(t^{r-2} + 1) \quad (1.3)$$

for some $2 \leq r < p^*$, and p -superlinear:

$$0 < \theta F(x, t) \leq t f(x, t), \quad t \text{ large} \quad (1.4)$$

for some $\theta > p$. Here $p^* = np/(n - p)$ (resp., ∞) if $p < n$ (resp., $p \geq n$) is the critical Sobolev exponent and C denotes a generic positive constant.

THEOREM 1.4. *If $p \geq 2$, (H_1) , (1.3), and (1.4) hold, and $f \geq 0$, then $\exists \lambda_0 > 0$ such that problem (1.1) has two solutions $\forall \lambda \in (0, \lambda_0)$.*

THEOREM 1.5. *If $p \geq 2$ and (H_2) , (1.3), and (1.4) hold, then $\exists \lambda_0 > 0$ such that problem (1.1) has two solutions $\forall \lambda \in (0, \lambda_0)$.*

2. Preliminaries on the p -Laplacian

Consider the problem

$$\begin{aligned} -\Delta_p u &= g(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (2.1)$$

PROPOSITION 2.1. *If $g \in L^q(\Omega)$ for some $q > n$, then (2.1) has a unique weak solution $u \in C_0^1(\overline{\Omega})$. If, in addition, $g \geq 0$ is nontrivial, then*

$$u > 0 \quad \text{in } \Omega, \quad \partial u / \partial \nu > 0 \quad \text{on } \partial\Omega. \quad (2.2)$$

Proof. The existence of a unique solution $u \in W_0^{1,p}(\Omega)$ is well-known. The problem

$$\begin{aligned} -\Delta v &= g(x) \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (2.3)$$

has a unique solution $v \in W^{2,q}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$, $\alpha = 1 - n/q$. Then u satisfies

$$\begin{aligned} \operatorname{div}(|\nabla u|^{p-2} \nabla u - G(x)) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (2.4)$$

where $G = \nabla v \in C^\alpha(\overline{\Omega})$, and u is bounded by Guedda and Véron [8] since $q > n/p$ if $p \leq n$, so $u \in C_0^1(\overline{\Omega})$ by Lieberman [13]. The rest now follows from Vázquez [17]. \square

3. Proofs of Theorems 1.1 and 1.3

Proof of Theorem 1.1. Since $a \in L^q(\Omega)$ by (H_1) , the problem

$$\begin{aligned} -\Delta_p v &= a(x) \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (3.1)$$

has a unique positive solution $v \in C_0^1(\overline{\Omega})$ with $\partial v / \partial \nu > 0$ on $\partial\Omega$ by Proposition 2.1. Then $\inf_\Omega (v/\varphi) > 0$ and hence $av^{-\gamma} \in L^q(\Omega)$. Fix $0 < \varepsilon \leq 1$ so small that $\underline{u} := \varepsilon^{1/(p-1)} v \leq 1$. Then

$$-\Delta_p \underline{u} - a(x) \underline{u}^{-\gamma} - \lambda f(x, \underline{u}) \leq -(1 - \varepsilon)a(x) \leq 0, \quad (3.2)$$

so \underline{u} is a subsolution of (1.1).

Since $a\underline{u}^{-\gamma} \in L^q(\Omega)$, the problem

$$\begin{aligned} -\Delta_p u &= a(x) \underline{u}(x)^{-\gamma} + 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (3.3)$$

has a unique solution $\overline{u} \in C_0^1(\overline{\Omega})$ by Proposition 2.1, and $\overline{u} \geq \underline{u}$ since

$$-\Delta_p \overline{u} \geq a(x) \geq \varepsilon a(x) = -\Delta_p \underline{u}. \quad (3.4)$$

Then

$$-\Delta_p \overline{u} - a(x) \overline{u}^{-\gamma} - \lambda f(x, \overline{u}) \geq 1 - \lambda \sup_{x \in \Omega, t \leq \max_\Omega \overline{u}} f(x, t), \quad (3.5)$$

so $\exists \lambda_0 > 0$ such that \overline{u} is a supersolution of (1.1) $\forall \lambda \in (0, \lambda_0)$ by (1.2).

Let

$$g_{\lambda, \bar{u}}(x, t) = \begin{cases} a(x)\bar{u}(x)^{-\gamma} + \lambda f(x, \bar{u}(x)), & t > \bar{u}(x) \\ a(x)t^{-\gamma} + \lambda f(x, t), & \underline{u}(x) \leq t \leq \bar{u}(x) \\ a(x)\underline{u}(x)^{-\gamma} + \lambda f(x, \underline{u}(x)), & t < \underline{u}(x), \end{cases} \quad (3.6)$$

$$G_{\lambda, \bar{u}}(x, t) = \int_0^t g_{\lambda, \bar{u}}(x, s) ds,$$

$$\Phi_{\lambda, \bar{u}}(u) = \int_{\Omega} |\nabla u|^p - p G_{\lambda, \bar{u}}(x, u), \quad u \in W_0^{1,p}(\Omega).$$

Since

$$0 \leq g_{\lambda, \bar{u}}(x, t) \leq a(x)\underline{u}(x)^{-\gamma} + \lambda \sup_{x \in \Omega, t \leq \max_{\Omega} \bar{u}} f(x, t), \quad \forall (x, t) \in \Omega \times \mathbb{R}, \quad (3.7)$$

and $a\underline{u}^{-\gamma} \in L^q(\Omega)$, $\Phi_{\lambda, \bar{u}}$ is bounded from below and has a global minimizer u_0 , which then is a solution of (1.1) in the order interval $[\underline{u}, \bar{u}]$. \square

Proof of Theorem 1.3. The problem

$$\begin{aligned} -\Delta_p v &= a_0 & \text{in } \Omega, \\ v &= 0 & \text{on } \partial\Omega \end{aligned} \quad (3.8)$$

has a unique positive solution $v \in C_0^1(\bar{\Omega})$ with $\partial v / \partial \nu > 0$ on $\partial\Omega$. Fix $0 < \varepsilon < 1$ so small that $\underline{u} := \varepsilon^{1/(p-1)} v \leq 1$. Then

$$-\Delta_p \underline{u} - a(x)\underline{u}^{-\gamma} - \lambda f(x, \underline{u}) \leq -(1 - \varepsilon)a_0 + \lambda \sup_{x \in \Omega, t \leq \max_{\Omega} \underline{u}} |f(x, t)|, \quad (3.9)$$

so $\exists \lambda_0 > 0$ such that \underline{u} is a subsolution of (1.1) $\forall \lambda \in (0, \lambda_0)$. The rest of the proof now proceeds as above. \square

4. Proofs of Theorems 1.4 and 1.5

Proof of Theorem 1.4. Define a Carathéodory function on $\Omega \times \mathbb{R}$ by

$$g_{\lambda}(x, t) = \begin{cases} a(x)t^{-\gamma} + \lambda f(x, t), & t \geq \underline{u}(x) \\ a(x)\underline{u}(x)^{-\gamma} + \lambda f(x, \underline{u}(x)), & t < \underline{u}(x) \end{cases} \quad (4.1)$$

and consider the problem

$$\begin{aligned} -\Delta_p u &= g_{\lambda}(x, u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (4.2)$$

Every solution of (4.2) is $\geq \underline{u}$ and hence also a solution of (1.1). By (1.3),

$$0 \leq g_{\lambda}(x, t) \leq a(x)\underline{u}(x)^{-\gamma} + \lambda C \left((t^+)^{r-1} + 1 \right), \quad \forall (x, t) \in \Omega \times \mathbb{R} \quad (4.3)$$

so solutions of (4.2) are the critical points of the C^1 functional

$$\Phi_\lambda(u) = \int_\Omega |\nabla u|^p - pG_\lambda(x, u), \quad u \in W_0^{1,p}(\Omega), \quad (4.4)$$

where $G_\lambda(x, t) = \int_0^t g_\lambda(x, s) ds$.

Since u_0 solves

$$\begin{aligned} -\Delta_p u &= g_{\lambda, \bar{u}}(x, u_0(x)) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (4.5)$$

and $g_{\lambda, \bar{u}}(\cdot, u_0(\cdot)) \in L^q(\Omega)$ by (3.7), $u_0 \in C_0^1(\bar{\Omega})$ by Proposition 2.1. Note that, with a possibly smaller λ_0 , $2\bar{u}$ is also a supersolution of (1.1) $\forall \lambda \in (0, \lambda_0)$. We assume that u_0 is the global minimizer of the corresponding functional $\Phi_{\lambda, 2\bar{u}}$ also, for otherwise we are done. Since

$$u_0 \leq \bar{u} < 2\bar{u} \quad \text{in } \Omega, \quad \partial u_0 / \partial \nu \leq \partial \bar{u} / \partial \nu < \partial(2\bar{u}) / \partial \nu \quad \text{on } \partial\Omega, \quad (4.6)$$

$\Phi_{\lambda, 2\bar{u}} = \Phi_\lambda$ in a $C_0^1(\bar{\Omega})$ -neighborhood of u_0 , so u_0 is a local minimizer of $\Phi_\lambda|_{C_0^1(\bar{\Omega})}$, and hence also of Φ_λ by Brezis and Nirenberg [3] for $p = 2$ and by Guo and Zhang [9] for $p > 2$. The mountain pass lemma now gives a second critical point as (1.4) implies that Φ_λ satisfies the (PS) condition and $\Phi_\lambda(tu) \rightarrow -\infty$ as $t \rightarrow \infty$. \square

Proof of Theorem 1.5 is similar and therefore omitted.

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