

ON A PERIODIC BOUNDARY VALUE PROBLEM FOR SECOND-ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Unimprovable efficient sufficient conditions are established for the unique solvability of the periodic problem $u''(t) = \ell(u)(t) + q(t)$ for $0 \leq t \leq \omega$, $u^{(i)}(0) = u^{(i)}(\omega)$ ($i = 0, 1$), where $\omega > 0$, $\ell : C([0, \omega]) \rightarrow L([0, \omega])$ is a linear bounded operator, and $q \in L([0, \omega])$.

1. Introduction

Consider the equation

$$u''(t) = \ell(u)(t) + q(t) \quad \text{for } 0 \leq t \leq \omega \quad (1.1)$$

with the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(\omega) \quad (i = 0, 1), \quad (1.2)$$

where $\omega > 0$, $\ell : C([0, \omega]) \rightarrow L([0, \omega])$ is a linear bounded operator and $q \in L([0, \omega])$.

By a solution of the problem (1.1), (1.2) we understand a function $u \in \tilde{C}'([0, \omega])$, which satisfies (1.1) almost everywhere on $[0, \omega]$ and satisfies the conditions (1.2).

The periodic boundary value problem for functional differential equations has been studied by many authors (see, for instance, [1, 2, 3, 4, 5, 6, 8, 9] and the references therein). Results obtained in this paper on the one hand generalise the well-known results of Lasota and Opial (see [7, Theorem 6, page 88]) for linear ordinary differential equations, and on the other hand describe some properties which belong only to functional differential equations. In the paper [8], it was proved that the problem (1.1), (1.2) has a unique solution if the inequality

$$\int_0^\omega |\ell(1)(s)| ds \leq \frac{d}{\omega} \quad (1.3)$$

with $d = 16$ is fulfilled. Moreover, there was also shown that the condition (1.3) is non-improvable. This paper attempts to find a specific subset of the set of linear monotone operators, in which the condition (1.3) guarantees the unique solvability of the problem

(1.1), (1.2) even for $d \geq 16$ (see Corollary 2.3). It turned out that if A satisfies some conditions dependent only on the constants d and ω , then $K_{[0,\omega]}(A)$ (see Definition 1.2) is such a subset of the set of linear monotone operators.

The following notation is used throughout.

N is the set of all natural numbers.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$C([a,b])$ is the Banach space of continuous functions $u : [a,b] \rightarrow R$ with the norm $\|u\|_C = \max\{|u(t)| : a \leq t \leq b\}$.

$\tilde{C}'([a,b])$ is the set of functions $u : [a,b] \rightarrow R$ which are absolutely continuous together with their first derivatives.

$L([a,b])$ is the Banach space of Lebesgue integrable functions $p : [a,b] \rightarrow R$ with the norm $\|p\|_L = \int_a^b |p(s)| ds$.

If $x \in R$, then $[x]_+ = (|x| + x)/2$, $[x]_- = (|x| - x)/2$.

Definition 1.1. We will say that an operator $\ell : C([a,b]) \rightarrow L([a,b])$ is *nonnegative* (*nonpositive*), if for any nonnegative $x \in C([a,b])$ the inequality

$$\ell(x)(t) \geq 0 \quad (\ell(x)(t) \leq 0) \quad \text{for } a \leq t \leq b \quad (1.4)$$

is satisfied.

We will say that an operator ℓ is *monotone* if it is nonnegative or nonpositive.

Definition 1.2. Let $A \subset [a,b]$ be a nonempty set. We will say that a linear operator $\ell : C([a,b]) \rightarrow L([a,b])$ belongs to the set $K_{[a,b]}(A)$ if for any $x \in C([a,b])$, satisfying

$$x(t) = 0 \quad \text{for } t \in A, \quad (1.5)$$

the equality

$$\ell(x)(t) = 0 \quad \text{for } a \leq t \leq b \quad (1.6)$$

holds.

We will say that $K_{[a,b]}(A)$ is the set of operators *concentrated* on the set $A \subset [a,b]$.

2. Main results

Define, for any nonempty set $A \subseteq R$, the continuous (see Lemma 3.1) functions:

$$\rho_A(t) = \inf \{|t - s| : s \in A\}, \quad \sigma_A(t) = \rho_A(t) + \rho_A\left(t + \frac{\omega}{2}\right) \quad \text{for } t \in R. \quad (2.1)$$

THEOREM 2.1. Let $A \subset [0,\omega]$, $A \neq \emptyset$ and a linear monotone operator $\ell \in K_{[0,\omega]}(A)$ be such that the conditions

$$\int_0^\omega \ell(1)(s) ds \neq 0, \quad (2.2)$$

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right) \int_0^\omega |\ell(1)(s)| ds \leq \frac{16}{\omega} \quad (2.3)$$

are satisfied, where

$$\delta = \min \left\{ \sigma_A(t) : 0 \leq t \leq \frac{\omega}{2} \right\}. \quad (2.4)$$

Then the problem (1.1), (1.2) has a unique solution.

Example 2.2. The example below shows that condition (2.3) in Theorem 2.1 is optimal and it cannot be replaced by the condition

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right) \int_0^\omega |\ell(1)(s)| ds \leq \frac{16}{\omega} + \varepsilon, \quad (2.3_\varepsilon)$$

no matter how small $\varepsilon \in]0, 1]$ would be. Let $\omega = 1$, $\varepsilon_0 \in]0, 1/16[$, $\delta_1 \in]0, 1/4 - 2\varepsilon_0[$ and μ_i, ν_i ($i = 1, 2$) be the numbers given by the equalities

$$\mu_i = \frac{1 - 2\delta_1}{4} + (-1)^i \varepsilon_0, \quad \nu_i = \frac{3 + 2\delta_1}{4} + (-1)^i \varepsilon_0 \quad (i = 1, 2). \quad (2.5)$$

Let, moreover, the functions $x \in \tilde{C}'([\mu_1, \mu_2])$, $y \in \tilde{C}'([\nu_1, \nu_2])$ be such that

$$x(\mu_1) = x(\mu_2) = 1, \quad x'(\mu_1) = \frac{1}{\mu_1}, \quad x'(\mu_2) = -\frac{1}{\mu_1 + \delta_1}, \quad (2.5_1)$$

$$x''(t) \leq 0 \quad \text{for } \mu_1 \leq t \leq \mu_2,$$

$$y(\nu_1) = y(\nu_2) = -1, \quad y'(\nu_1) = -\frac{1}{\mu_1 + \delta_1}, \quad y'(\nu_2) = \frac{1}{\mu_1}, \quad (2.5_2)$$

$$y''(t) \geq 0 \quad \text{for } \nu_1 \leq t \leq \nu_2.$$

Define a function

$$u_0(t) = \begin{cases} \frac{t}{\mu_1} & \text{for } 0 \leq t \leq \mu_1 \\ x(t) & \text{for } \mu_1 < t < \mu_2 \\ \frac{1 - 2t}{\nu_1 - \mu_2} & \text{for } \mu_2 \leq t \leq \nu_1 \\ y(t) & \text{for } \nu_1 < t < \nu_2 \\ \frac{t - 1}{\mu_1} & \text{for } \nu_2 \leq t \leq 1. \end{cases} \quad (2.6)$$

Obviously, $u_0 \in \tilde{C}'([0, \omega])$. Now let $A = \{\mu_1, \nu_2\}$, the function $\tau : [0, \omega] \rightarrow A$ and the operator $\ell : C([0, \omega]) \rightarrow L([0, \omega])$ be given by the equalities:

$$\tau(t) = \begin{cases} \mu_1 & \text{if } u_0''(t) \geq 0 \\ \nu_2 & \text{if } u_0''(t) < 0, \end{cases} \quad \ell(z)(t) = |u_0''(t)|z(\tau(t)). \quad (2.7)$$

It is clear from the definition of the functions τ and σ_A that the nonnegative operator ℓ is concentrated on the set A and the condition (2.4) is satisfied with $\delta = \delta_1 + 2\varepsilon_0$. In view

of (2.5₁), (2.5₂), and (2.7) we obtain

$$\int_0^\omega \ell(1)(s)ds = \int_{\nu_1}^{\nu_2} y''(s)ds - \int_{\mu_1}^{\mu_2} x''(s)ds = 2 \frac{2\mu_1 + \delta_1}{\mu_1(\mu_1 + \delta_1)} = 16 \frac{1 - 4\varepsilon_0}{(1 - 4\varepsilon_0)^2 - 4\delta_1^2}. \quad (2.8)$$

When ε is small enough, the last equality it implies the existence of ε_0 such that

$$0 < \int_0^\omega \ell(1)(s)ds = \frac{16 + \varepsilon}{1 - 4\delta_1^2}. \quad (2.9)$$

Thus, because $\delta_1 < \delta$, all the assumptions of Theorem 2.1 are satisfied except (2.3), and instead of (2.3) the condition (2.3_ε) is fulfilled with $\omega = 1$. On the other hand, from the definition of the function u_0 and from (2.7), it follows that $\ell(u_0)(t) = |u_0''(t)|u_0(\tau(t)) = |u_0''(t)|\operatorname{sign} u_0''(t)$, that is, u_0 is a nontrivial solution of the homogeneous problem $u''(t) = \ell(u)(t)$, $u^{(i)}(0) = u^{(i)}(1)$ ($i = 1, 2$) which contradicts the conclusion of Theorem 2.1.

COROLLARY 2.3. *Let the set $A \subset [0, \omega]$, number $d \geq 16$, and a linear monotone operator $\ell \in K_{[0, \omega]}(A)$ be such that the conditions (2.2)*

$$\int_0^\omega |\ell(1)(s)|ds \leq \frac{d}{\omega}, \quad (2.10)$$

are satisfied and

$$\sigma_A(t) \geq \frac{\omega}{2} \sqrt{1 - \frac{16}{d}} \quad \text{for } 0 \leq t \leq \frac{\omega}{2}. \quad (2.11)$$

Then the problem (1.1), (1.2) has a unique solution.

COROLLARY 2.4. *Let $\alpha \in [0, \omega]$, $\beta \in [\alpha, \omega]$, and a linear monotone operator $\ell \in K_{[0, \omega]}(A)$ be such that the conditions (2.2) and (2.3) are satisfied, where*

$$A = [\alpha, \beta], \quad \delta = \left[\frac{\omega}{2} - (\beta - \alpha) \right]_+ \quad (2.11_1)$$

or

$$A = [0, \alpha] \cup [\beta, \omega], \quad \delta = \left[\frac{\omega}{2} - (\beta - \alpha) \right]_- \quad (2.11_2)$$

Then the problem (1.1), (1.2) has a unique solution.

Consider the equation with deviating arguments

$$u''(t) = p(t)u(\tau(t)) + q(t) \quad \text{for } 0 \leq t \leq \omega, \quad (2.12)$$

where $p \in L([0, \omega])$ and $\tau : [0, \omega] \rightarrow [0, \omega]$ is a measurable function.

COROLLARY 2.5. *Let there exist $\sigma \in \{-1, 1\}$ such that*

$$\sigma p(t) \geq 0 \quad \text{for } 0 \leq t \leq \omega, \quad (2.13)$$

$$\int_0^\omega p(s)ds \neq 0. \quad (2.14)$$

Moreover, let $\delta \in [0, \omega/2]$ and the function p be such that

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right) \int_0^\omega |p(s)| ds \leq \frac{16}{\omega}, \quad (2.15)$$

and let at least one of the following items be fulfilled:

(a) the set $A \subset [0, \omega]$ is such that the condition (2.4) holds and

$$p(t) = 0 \quad \text{if } \tau(t) \notin A \quad (2.16)$$

on $[0, \omega]$;

(b) the constants $\alpha \in [0, \omega]$, $\beta \in [\alpha, \omega]$ are such that

$$\tau(t) \in [\alpha, \beta] \quad \text{for } 0 \leq t \leq \omega, \quad (2.17)$$

$$\delta = \left[\frac{\omega}{2} - (\beta - \alpha) \right]_+. \quad (2.18)$$

Then the problem (2.12), (1.2) has a unique solution.

Now consider the ordinary differential equation

$$u''(t) = p(t)u(t) + q(t) \quad \text{for } 0 \leq t \leq \omega, \quad (2.19)$$

where $p, q \in L([0, \omega])$.

COROLLARY 2.6. *Let*

$$p(t) \leq 0 \quad \text{for } 0 \leq t \leq \omega. \quad (2.20)$$

Moreover, let $\delta \in [0, \omega/2]$ and the function p be such that the conditions (2.14), (2.15) hold, and let at least one of the following items be fulfilled:

(a) the set $A \subset [0, \omega]$ is such that $\text{mes } A \neq 0$, the condition (2.4) holds and

$$p(t) = 0 \quad \text{for } t \notin A; \quad (2.21)$$

(b) the constants $\alpha \in [0, \omega]$, $\beta \in [\alpha, \omega]$ are such that

$$p(t) = 0 \quad \text{for } t \in [0, \alpha \cup \beta, \omega], \quad (2.22)$$

and $\delta \in [0, \omega/2]$ satisfies (2.18). Then the problem (2.19), (1.2) has a unique solution.

Remark 2.7. As for the case where $p(t) \geq 0$ for $0 \leq t \leq \omega$, the necessary and sufficient condition for the unique solvability of (2.19), (1.2) is $p(t) \neq 0$ (see [2, Proposition 1.1, page 72]).

3. Auxiliary propositions

LEMMA 3.1. *The function $\rho_A : R \rightarrow R$ defined by the equalities (2.1), is continuous and*

$$\rho_{\bar{A}}(t) = \rho_A(t) \quad \text{for } t \in R, \quad (3.1)$$

where \bar{A} is the closure of the set A .

Proof. Since $A \subseteq \bar{A}$, it is clear that

$$\rho_{\bar{A}}(t) \leq \rho_A(t) \quad \text{for } t \in R. \quad (3.2)$$

Let $t_0 \in R$ be an arbitrary point, $s_0 \in \bar{A}$, and the sequence $s_n \in A$ ($n \in N$) be such that $\lim_{n \rightarrow \infty} s_n = s_0$. Then $\rho_A(t_0) \leq \lim_{n \rightarrow \infty} |t_0 - s_n| = |t_0 - s_0|$, that is,

$$\rho_{\bar{A}}(t) \geq \rho_A(t) \quad \text{for } t \in R. \quad (3.3)$$

From the last relation and (3.2) we get the equality (3.1).

For arbitrary $s \in A$, $t_1, t_2 \in R$, we have

$$\rho_A(t_i) \leq |t_i - s| \leq |t_2 - t_1| + |t_{3-i} - s| \quad (i = 1, 2). \quad (3.4)$$

Consequently $\rho_A(t_i) - |t_2 - t_1| \leq \rho_A(t_{3-i})$ ($i = 1, 2$). Thus the function ρ_A is continuous. \square

LEMMA 3.2. *Let $A \subseteq [0, \omega]$ be a nonempty set, $A_1 = \{t + \omega : t \in A\}$, $B = A \cup A_1$, and*

$$\min \left\{ \sigma_A(t) : 0 \leq t \leq \frac{\omega}{2} \right\} = \delta. \quad (3.5)$$

Then

$$\min \left\{ \sigma_B(t) : 0 \leq t \leq \frac{3\omega}{2} \right\} = \delta. \quad (3.6)$$

Proof. Let $\alpha = \inf A$, $\beta = \sup A$, and let $t_0 \in [0, 3\omega/2]$ be such that

$$\sigma_B(t_0) = \min \left\{ \sigma_B(t) : 0 \leq t \leq \frac{3\omega}{2} \right\}. \quad (3.7)$$

Assume that $t_1 \in [0, 3\omega/2]$ is such that $t_1 \notin \bar{B}$, $t_1 + \omega/2 \notin \bar{B}$. Then

$$\varepsilon = \min \{ \rho_B(t_1), \rho_B(t_1 + \omega/2) \} > 0, \quad (3.8)$$

and either

$$\sigma_B(t_1 - \varepsilon) \leq \sigma_B(t_1) \quad \text{and} \quad \rho_B(t_1 - \varepsilon) = 0 \quad \text{or} \quad \rho_B\left(t_1 + \frac{\omega}{2} - \varepsilon\right) = 0 \quad (3.9)$$

or

$$\sigma_B(t_1 + \varepsilon) \leq \sigma_B(t_1) \quad \text{and} \quad \rho_B(t_1 + \varepsilon) = 0 \quad \text{or} \quad \rho_B\left(t_1 + \frac{\omega}{2} + \varepsilon\right) = 0. \quad (3.10)$$

In view of this fact, without loss of generality we can assume that

$$t_0 \in \bar{B} \quad \text{or} \quad t_0 + \frac{\omega}{2} \in \bar{B}. \quad (3.11)$$

From (3.5) and the condition $A \subseteq [0, \omega]$, we have

$$\min \left\{ \sigma_A(t) : 0 \leq t \leq \frac{3\omega}{2} \right\} = \delta. \quad (3.12)$$

First suppose that $0 \leq t_0 \leq \beta - \omega/2$. From this inequality by the inclusion $\beta \in \bar{A}$, we get

$$\inf \left\{ \left| t_0 + \frac{\omega i}{2} - s \right| : s \in B \right\} = \inf \left\{ \left| t_0 + \frac{\omega i}{2} - s \right| : s \in A \right\} \quad (3.12_i)$$

for $i = 0, 1$. Then $\sigma_B(t_0) = \sigma_A(t_0)$ and in view of (3.12)

$$\sigma_B(t_0) \geq \delta. \quad (3.13)$$

Let now

$$\beta - \frac{\omega}{2} < t_0 \leq \beta. \quad (3.14)$$

Obviously, either

$$\left(t_0 + \frac{\omega}{2} \right) - \beta \leq \alpha + \omega - \left(t_0 + \frac{\omega}{2} \right), \quad (3.14_1)$$

or

$$\left(t_0 + \frac{\omega}{2} \right) - \beta > \alpha + \omega - \left(t_0 + \frac{\omega}{2} \right). \quad (3.14_2)$$

If (3.14₁) is satisfied, then, in view of (3.14) and $\beta \in \bar{A}$, the equalities (3.12_i) ($i = 0, 1$) hold. Therefore $\sigma_B(t_0) = \sigma_A(t_0)$ and, in view of (3.12), the inequality (3.13) is fulfilled. Let now (3.14₂) be satisfied. If $\alpha + \omega > t_0 + \omega/2$, then, in view of (3.14), we have $t_0 + \omega/2 \notin \bar{B}$. Consequently, from (3.12) and (3.14₂) by virtue of (3.11) and the inclusions $\alpha, \beta \in \bar{A}$, we get

$$\sigma_B(t_0) = \rho_B \left(t_0 + \frac{\omega}{2} \right) = \alpha + \frac{\omega}{2} - t_0 \geq \rho_A \left(\alpha + \frac{\omega}{2} \right) \geq \delta. \quad (3.15)$$

If $\alpha + \omega \leq t_0 + \omega/2$, then $t_0 + \omega/2 \in \bar{A}_1$ and

$$\inf \left\{ \left| t_0 + \frac{\omega}{2} - s \right| : s \in B \right\} = \inf \left\{ \left| t_0 - \frac{\omega}{2} - s \right| : s \in A \right\}, \quad (3.16)$$

that is, $\rho_B(t_0 + \omega/2) = \rho_A(t_0 - \omega/2)$ and in view of (3.12), (3.14) we get

$$\sigma_B(t_0) = \rho_A(t_0) + \rho_A \left(t_0 - \frac{\omega}{2} \right) = \sigma_A \left(t_0 - \frac{\omega}{2} \right) \geq \delta. \quad (3.17)$$

Consequently the inequality (3.13) is fulfilled as well.

Further, let $\beta \leq t_0 \leq t_0 + \omega/2 \leq \alpha + \omega$. Then $t_0 - \alpha \leq \alpha + \omega - t_0$, and also $t_0 - \beta \leq \alpha + \omega - t_0$. On account of (3.12) and $\beta \in \bar{A}$ we have

$$\sigma_B(t_0) = \alpha + \frac{\omega}{2} - \beta \geq \rho_A\left(\alpha + \frac{\omega}{2}\right) = \sigma_A(\alpha) \geq \delta. \quad (3.18)$$

Thus the inequality (3.13) is fulfilled.

Let now

$$\beta \leq t_0 \leq \alpha + \omega \leq t_0 + \frac{\omega}{2}. \quad (3.19)$$

From (3.19) it follows that

$$\begin{aligned} \inf \left\{ \left| t_0 + \frac{\omega}{2} - s \right| : s \in B \right\} &= \inf \left\{ \left| t_0 + \frac{\omega}{2} - s \right| : s \in A_1 \right\} \\ &= \inf \left\{ \left| t_0 - \frac{\omega}{2} - s \right| : s \in A \right\} \geq \inf \left\{ \left| t_0 - \frac{\omega}{2} - s \right| : s \in B \right\}, \end{aligned} \quad (3.20)$$

and therefore,

$$\sigma_B(t_0) \geq \rho_B\left(t_0 - \frac{\omega}{2}\right) + \rho_B(t_0) = \sigma_B\left(t_0 - \frac{\omega}{2}\right). \quad (3.21)$$

The inequalities (3.19) imply $t_0 - \omega/2 \leq \alpha + \omega$ and, according to the case considered above, we have $\sigma_B(t_0 - \omega/2) \geq \delta$. Consequently, (3.21) results in (3.13).

Finally, if $\alpha + \omega \leq t_0$, the validity of (3.13) can be proved analogously to the previous cases. Then we have

$$\sigma_B(t) \geq \delta \quad \text{for } 0 \leq t \leq \frac{3\omega}{2}. \quad (3.22)$$

On the other hand, since $A \subset B$, it is clear that

$$\sigma_B(t) \leq \sigma_A(t) \quad \text{for } 0 \leq t \leq \frac{3\omega}{2}. \quad (3.23)$$

The last two relations and (3.5) yields the equality (3.6). \square

LEMMA 3.3. *Let $\sigma \in \{-1, 1\}$, $D \subset [a, b]$, $D \neq \emptyset$, $\ell_1 \in K_{[a,b]}(D)$, and let $\sigma\ell_1$ be nonnegative. Then, for an arbitrary $v \in C([a, b])$,*

$$\begin{aligned} \min \{v(s) : s \in \bar{D}\} |\ell_1(1)(t)| \\ \leq \sigma\ell_1(v)(t) \leq \max \{v(s) : s \in \bar{D}\} |\ell_1(1)(t)| \quad \text{for } a \leq t \leq b. \end{aligned} \quad (3.24)$$

Proof. Let $\alpha = \inf D, \beta = \sup D$,

$$v_0(t) = \begin{cases} v(\alpha) & \text{for } t \in [a, \alpha[\\ v(t) & \text{for } t \in \bar{D} \\ \frac{v(\mu(t)) - v(\nu(t))}{\mu(t) - \nu(t)} (t - \nu(t)) + v(\nu(t)) & \text{for } t \in [\alpha, \beta] \setminus \bar{D} \\ v(\beta) & \text{for } t \in]\beta, b], \end{cases} \quad (3.25)$$

where

$$\mu(t) = \min\{s \in \bar{D} : t \leq s\}, \quad \nu(t) = \max\{s \in \bar{D} : t \geq s\} \quad \text{for } \alpha \leq t \leq \beta. \quad (3.26)$$

It is clear that $v_0 \in C([a, b])$ and

$$\begin{aligned} \min\{v(s) : s \in \bar{D}\} \leq v_0(t) \leq \max\{v(s) : s \in \bar{D}\} & \quad \text{for } a \leq t \leq b, \\ v_0(t) = v(t) & \quad \text{for } t \in D. \end{aligned} \quad (3.27)$$

Since $\ell_1 \in K_{[a, b]}(D)$ and the operator $\sigma\ell_1$ is nonnegative, it follows from (3.27) that (3.24) is true. \square

LEMMA 3.4. *Let $a \in [0, \omega]$, $D \subset [a, a + \omega]$, $c \in [a, a + \omega]$, and $\delta \in [0, \omega/2]$ be such that*

$$\sigma_D(t) \geq \delta \quad \text{for } a \leq t \leq a + \frac{\omega}{2}, \quad (3.28)$$

$$A_c = \bar{D} \cap [a, c] \neq \emptyset, \quad B_c = \bar{D} \cap [c, a + \omega] \neq \emptyset. \quad (3.29)$$

Then the estimate

$$\left(\frac{(c - t_1)(t_1 - a)(a + \omega - t_2)(t_2 - c)}{(c - a)(a + \omega - c)} \right)^{1/2} \leq \frac{\omega^2 - 4\delta^2}{8\omega} \quad (3.30)$$

for all $t_1 \in A_c$, $t_2 \in B_c$ is satisfied.

Proof. Put $b = a + \omega$ and

$$\sigma_1 = \rho_D\left(\frac{a + c}{2}\right), \quad \sigma_2 = \rho_D\left(\frac{c + b}{2}\right). \quad (3.31)$$

Then, from the condition (3.28) it is clear

$$\sigma_1 + \sigma_2 \geq \delta. \quad (3.32)$$

Obviously, either

$$\max(\sigma_1, \sigma_2) \geq \delta \quad (3.32_1)$$

or

$$\max(\sigma_1, \sigma_2) < \delta. \quad (3.32_2)$$

First note that from (3.29) and (3.31) the equalities

$$\begin{aligned}\max \{(c-t_1)(t_1-a) : t_1 \in A_c\} &= (c-t'_1)(t'_1-a), \\ \max \{(b-t_2)(t_2-c) : t_2 \in B_c\} &= (b-t'_2)(t'_2-c),\end{aligned}\tag{3.33}$$

follow, where $t'_1 = (a+c)/2 - \sigma_1$, $t'_2 = (c+b)/2 - \sigma_2$. Hence, on account of well-known inequality

$$d_1 d_2 \leq \frac{(d_1 + d_2)^2}{4},\tag{3.34}$$

we have

$$\begin{aligned}\left(\frac{(c-t_1)(t_1-a)(b-t_2)(t_2-c)}{(c-a)(b-c)} \right)^{1/2} \\ \leq \left(\frac{c-a}{4} - \frac{\sigma_1^2}{c-a} \right)^{1/2} \left(\frac{b-c}{4} - \frac{\sigma_2^2}{b-c} \right)^{1/2} \leq \frac{1}{2} \left(\frac{\omega}{4} - \frac{\sigma_1^2}{c-a} - \frac{\sigma_2^2}{b-c} \right)\end{aligned}\tag{3.35}$$

for all $t_1 \in A_c$, $t_2 \in B_c$. In the case, where inequality (3.32₁) is fulfilled, we have

$$\frac{\omega}{4} - \frac{\sigma_1^2}{c-a} - \frac{\sigma_2^2}{b-c} \leq \frac{\omega}{4} - \frac{(\max(\sigma_1, \sigma_2))^2}{\omega} \leq \frac{\omega^2 - 4\delta^2}{4\omega}.\tag{3.36}$$

This, together with (3.35), yields the estimate (3.30). Suppose now that the condition (3.32₂) is fulfilled. Then in view of Lemma 3.1, we can choose $\alpha, \beta \in \bar{D}$ such that

$$\rho_D\left(\frac{a+c}{2}\right) = \left| \frac{a+c}{2} - \alpha \right|, \quad \rho_D\left(\frac{c+b}{2}\right) = \left| \frac{c+b}{2} - \beta \right|,\tag{3.37}$$

which together with (3.31) yields

$$\frac{\omega}{4} - \frac{\sigma_1^2}{c-a} - \frac{\sigma_2^2}{b-c} = \omega - (\beta - \alpha) - \eta(c),\tag{3.38}$$

where $\eta(t) = (\alpha - a)^2/(t - a) + (b - \beta)^2/(b - t)$. It is not difficult to verify that the function η achieves its minimum at the point $t_0 = ((\alpha - a)b + (b - \beta)a)/(\omega - (\beta - \alpha))$. Thus,

$$\omega - (\beta - \alpha) - \eta(c) \leq (\omega - (\beta - \alpha)) \frac{\beta - \alpha}{\omega}.\tag{3.39}$$

Put

$$\sigma = \min(\sigma_1, \sigma_2).\tag{3.40}$$

Then it follows from (3.37) that either

$$\alpha \leq \frac{a+c}{2} - \sigma \quad (3.40_1)$$

or

$$\alpha \geq \frac{a+c}{2} + \sigma, \quad (3.40_2)$$

and either

$$\beta \geq \frac{c+b}{2} + \sigma \quad (3.40_3)$$

or

$$\beta \leq \frac{c+b}{2} - \sigma. \quad (3.40_4)$$

Consider now the case where α satisfies the inequality (3.40₁) and assume that β satisfies the inequality (3.40₄). Then from (3.37), (3.40₁), and (3.40₄) we get

$$\rho_D\left(\frac{a+c}{2} - \sigma\right) = \rho_D\left(\frac{a+c}{2}\right) - \sigma, \quad \rho_D\left(\frac{c+b}{2} - \sigma\right) = \rho_D\left(\frac{c+b}{2}\right) - \sigma. \quad (3.41)$$

These equalities in view of (3.31) and (3.40) yield

$$\sigma_D\left(\frac{a+c}{2} - \sigma\right) = (\sigma_1 - \sigma) + (\sigma_2 - \sigma) = \max(\sigma_1, \sigma_2) - \sigma, \quad (3.42)$$

but in view of (3.32₂) this contradicts the condition (3.28). Consequently, β satisfies the inequality (3.40₃). Then from (3.31), (3.37), by (3.40₁) and (3.40₃), we get $\sigma_1 = (a+c)/2 - \alpha$, $\sigma_2 = \beta - (c+b)/2$, that is,

$$\beta - \alpha = \sigma_1 + \sigma_2 + \frac{\omega}{2}. \quad (3.42_1)$$

Now suppose that (3.40₂) holds. It can be proved in a similar manner as above that, in this case, the inequality (3.40₄) is satisfied. Therefore, from (3.31), (3.37), (3.40₂), and (3.40₄) we obtain

$$\beta - \alpha = \frac{\omega}{2} - (\sigma_1 + \sigma_2). \quad (3.42_2)$$

Then, on account of (3.32), in both (3.42₁) and (3.42₂) cases we have

$$(\omega - (\beta - \alpha)) \frac{\beta - \alpha}{\omega} = \frac{\omega^2 - 4(\sigma_1 + \sigma_2)^2}{4\omega} \leq \frac{\omega^2 - 4\delta^2}{4\omega}. \quad (3.43)$$

Consequently from (3.35), (3.38), (3.39), and (3.43) we obtain the estimate (3.30), also in case where the inequality (3.32₂) holds. \square

4. Proof of the main results

Proof of Theorem 2.1. Consider the homogeneous problem

$$v''(t) = \ell(v)(t) \quad \text{for } 0 \leq t \leq \omega, \quad (4.1)$$

$$v^{(i)}(0) = v^{(i)}(\omega) \quad (i = 0, 1). \quad (4.2)$$

It is known from the general theory of boundary value problems for functional differential equations that if ℓ is a monotone operator, then problem (1.1), (1.2) has the Fredholm property (see [3, Theorem 1.1, page 345]). Thus, the problem (1.1), (1.2) is uniquely solvable iff the homogeneous problem (4.1), (4.2) has only the trivial solution.

Assume that, on the contrary, the problem (4.1), (4.2) has a nontrivial solution v . If $v \equiv \text{const}$, then, in view of (4.1) we obtain a contradiction with the condition (2.2). Consequently, $v \not\equiv \text{const}$. Then, in view of the conditions (4.2), there exist subsets I_1 and I_2 from $[0, \omega]$ which have positive measure and

$$v''(t) > 0 \quad \text{for } t \in I_1, \quad v''(t) < 0 \quad \text{for } t \in I_2. \quad (4.3)$$

Assume that v is either nonnegative or nonpositive on the entire set A . Without loss of generality we can suppose $v(t) \geq 0$ for $t \in A$. Then, from Lemma 3.3 with $a = 0$, $b = \omega$, $D = A$, and $\ell_1 \equiv \ell$ we obtain

$$\sigma\ell(v)(t) \geq 0 \quad \text{for } 0 \leq t \leq \omega. \quad (4.4)$$

In view of (4.1), the inequality (4.4) contradicts one of the inequalities in (4.3). Therefore, the function v changes its sign on the set A , that is, there exist $t'_1, t_1 \in \bar{A}$ such that

$$v(t'_1) = \min \{v(t) : t \in \bar{A}\}, \quad v(t_1) = \max \{v(t) : t \in \bar{A}\}, \quad (4.5)$$

and $v(t'_1) < 0$, $v(t_1) > 0$. Without loss of generality we can assume that $t'_1 < t_1$. Then, in view of the last inequalities, there exists $a \in]t'_1, t_1[$ such that $v(a) = 0$.

Let us set $C_\omega([a, a + \omega]) = \{x \in C([a, a + \omega]) : x(a) = x(a + \omega)\}$, and let the continuous operators $\gamma : L([0, \omega]) \rightarrow L([a, a + \omega])$, $\ell_1 : C_\omega([a, a + \omega]) \rightarrow L([a, a + \omega])$ and the function $v_0 \in C([a, a + \omega])$ be given by the equalities

$$\gamma(x)(t) = \begin{cases} x(t) & \text{for } a \leq t \leq \omega \\ x(t - \omega) & \text{for } \omega < t \leq a + \omega, \end{cases} \quad (4.6)$$

$$v_0(t) = \gamma(v(t)), \quad \ell_1(x)(t) = \gamma(\ell(\gamma^{-1}(x)))(t) \quad \text{for } a \leq t \leq a + \omega.$$

Let, moreover, $t_2 = t'_1 + \omega$ and $D = A \cup \{t + \omega : t \in A\} \cap [a, a + \omega]$. Then (4.1), (4.2) with regard for (4.6) and the definitions of a , t'_1 , t_1 , imply that $v_0 \in \tilde{C}'([a, a + \omega])$, $t_1, t_2 \in D$,

$$v_0''(t) = \ell_1(v_0)(t) \quad \text{for } a \leq t \leq a + \omega, \quad (4.7)$$

$$v_0(a) = 0, \quad v_0(a + \omega) = 0, \quad (4.8)$$

$$v_0(t_1) = \max \{v_0(t) : t \in \bar{D}\}, \quad v_0(t_2) = \min \{v_0(t) : t \in \bar{D}\}, \quad (4.9)$$

$$v_0(t_1) > 0, \quad v_0(t_2) < 0, \quad (4.10)$$

and there exists $c \in]t_1, t_2[$ such that

$$\nu_0(c) = 0. \quad (4.11)$$

It is not difficult to verify that the condition $\ell \in K_{[0,\omega]}(A)$ implies

$$\ell_1 \in K_{[a,a+\omega]}(D). \quad (4.12)$$

Since $D \subset A \cup \{t + \omega : t \in A\}$, it follows from condition (2.4) and Lemma 3.2 that

$$\sigma_D(t) \geq \delta \quad \text{for } a \leq t \leq a + \frac{\omega}{2}. \quad (4.13)$$

Thus, from the general theory of ordinary differential equations (see [6, Theorem 1.1, page 2348]), in view of (4.7), (4.8), (4.9), and (4.11), we obtain the representations

$$\nu_0(t_1) = - \int_a^c |G_1(t_1, s)| \ell_1(\nu_0)(s) ds, \quad (4.13_1)$$

$$|\nu_0(t_2)| = \int_c^{a+\omega} |G_2(t_2, s)| \ell_1(\nu_0)(s) ds, \quad (4.13_2)$$

where $G_1(G_2)$ is Green's function of the problem

$$\begin{aligned} z''(t) &= 0 \quad \text{for } a \leq t \leq c \ (c \leq t \leq a + \omega), \\ z(a) &= 0, \quad z(c) = 0 \quad (z(c) = 0, z(a + \omega) = 0). \end{aligned} \quad (4.14)$$

If ℓ is a nonnegative operator, then from (4.6) it is clear that ℓ_1 is also nonnegative. Then, from (4.13₁) and (4.13₂), by Lemma 3.3 and relations (4.9), (4.10), and (4.12), we get the strict estimates

$$\begin{aligned} 0 &< \frac{\nu_0(t_1)}{|\nu_0(t_2)|} < \frac{(t_1 - a)(c - t_1)}{c - a} \int_a^c \ell_1(1)(s) ds, \\ 0 &< \frac{|\nu_0(t_2)|}{\nu_0(t_1)} < \frac{(t_2 - c)(a + \omega - t_2)}{a + \omega - c} \int_c^{a+\omega} \ell_1(1)(s) ds, \end{aligned} \quad (4.15)$$

respectively. By multiplying these estimates and applying the numerical inequality (3.34), we obtain

$$1 < \frac{1}{2} \left(\frac{(t_1 - a)(c - t_1)(t_2 - c)(a + \omega - t_2)}{(c - a)(a + \omega - c)} \right)^{1/2} \int_a^{a+\omega} |\ell_1(1)(s)| ds. \quad (4.16)$$

Reasoning analogously, we can show that the estimate (4.16) is valid also in case where the operator ℓ is nonpositive.

From the definitions of t_1, t_2, c , and (4.13), it follows that all the conditions of Lemma 3.4 are satisfied. In view of the estimate (3.30) and the definition of the operator ℓ_1 , the inequality (4.16) contradicts the condition (2.3). \square

Proof of Corollary 2.3. Let $\delta = \omega/2(1 - 16/d)^{1/2}$. Then, on account of (2.10) and (2.11), we obtain that the conditions (2.3) and (2.4) of Theorem 2.1 are fulfilled. Consequently, all the assumptions of Theorem 2.1 are satisfied. \square

Proof of Corollary 2.4. It is not difficult to verify that if $A = [\alpha, \beta]$ ($A = [0, \alpha] \cup [\beta, \omega]$), then

$$\sigma_A(t) \geq \left[\frac{\omega}{2} - \beta + \alpha \right]_+ \quad \left(\sigma_A(t) \geq \left[\frac{\omega}{2} - \beta + \alpha \right]_- \right) \quad \text{for } 0 \leq t \leq \frac{\omega}{2}. \quad (4.17)$$

Consequently, in view of the condition (2.11₁), (2.11₂), all the assumptions of Theorem 2.1 are satisfied. \square

Proof of Corollary 2.5. Let $\ell(u)(t) \equiv p(t)u(\tau(t))$. On account of (2.13), (2.14), and (2.15) we see that the operator ℓ is monotone and the conditions (2.2) and (2.3) are satisfied.

(a) It is not difficult to verify that from the condition (2.16) it follows that $\ell \in K_{[0, \omega]}(A)$. Consequently, all the assumptions of Theorem 2.1 are satisfied.

(b) Let $A = [\alpha, \beta]$. Then in view of the condition (2.17) the inclusion $\ell \in K_{[0, \omega]}(A)$ is satisfied. The inequality (4.17) obtained in the proof of Corollary 2.4, by virtue of (2.18), implies the inequality (2.4). Consequently, all the assumptions of Theorem 2.1 are satisfied. \square

Proof of Corollary 2.6. The validity of this assertion follows immediately from Corollary 2.5(a). \square

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