

DELAY DYNAMIC EQUATIONS WITH STABILITY

DOUGLAS R. ANDERSON, ROBERT J. KRUEGER, AND ALLAN C. PETERSON

Received 13 August 2005; Accepted 23 October 2005

We first give conditions which guarantee that every solution of a first order linear delay dynamic equation for isolated time scales vanishes at infinity. Several interesting examples are given. In the last half of the paper, we give conditions under which the trivial solution of a nonlinear delay dynamic equation is asymptotically stable, for arbitrary time scales.

Copyright © 2006 Douglas R. Anderson et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Preliminaries

The unification and extension of continuous calculus, discrete calculus, q -calculus, and indeed arbitrary real-number calculus to time-scale calculus, where a time scale is simply any nonempty closed set of real numbers, were first accomplished by Hilger in [4]. Since then, time-scale calculus has made steady inroads in explaining the interconnections that exist among the various calculi, and in extending our understanding to a new, more general and overarching theory. The purpose of this work is to illustrate this new understanding by extending some continuous and discrete delay equations to certain time scales. Examples will include specific cases in differential equations, difference equations, q -difference equations, and harmonic-number equations. The definitions that follow here will serve as a short primer on the time-scale calculus; they can be found in [1, 2] and the references therein.

Definition 1.1. Define the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ (resp., $\rho(t)$ at t for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\}), \quad \forall t \in \mathbb{T}. \quad (1.1)$$

Also define $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T} < \infty$, and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, if $\inf \mathbb{T} > -\infty$. Define the graininess function $\mu : \mathbb{T} \rightarrow \mathbb{R}$ by $\mu(t) = \sigma(t) - t$.

2 Delay dynamic equations with stability

Throughout this work the assumption is made that \mathbb{T} is unbounded above and has the topology that it inherits from the standard topology on the real numbers \mathbb{R} . Also assume throughout that $a < b$ are points in \mathbb{T} and define the time scale interval $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. Other time scale intervals are defined similarly. The jump operators σ and ρ allow the classification of points in a time scale in the following way: if $\sigma(t) > t$ then call the point t right-scattered; while if $\rho(t) < t$ then we say t is left-scattered. If $\sigma(t) = t$ then call the point t right-dense; while if $t > \inf \mathbb{T}$ and $\rho(t) = t$ then we say t is left-dense. We next define the so-called delta derivative. The novice could skip this definition and look at the results stated in Theorem 1.4. In particular in part (2) of Theorem 1.4 we see what the delta derivative is at right-scattered points and in part (3) of Theorem 1.4 we see that at right-dense points the derivative is similar to the definition given in calculus.

Definition 1.2. Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \rightarrow \mathbb{R}$. Define $y^{\Delta}(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood U of t such that, for all $s \in U$,

$$| [y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s] | \leq \epsilon |\sigma(t) - s|. \quad (1.2)$$

Call $y^{\Delta}(t)$ the (delta) derivative of $y(t)$ at t .

Definition 1.3. If $F^{\Delta}(t) = f(t)$ then define the (Cauchy) delta integral by

$$\int_a^t f(s) \Delta s = F(t) - F(a). \quad (1.3)$$

The following theorem is due to Hilger [4].

THEOREM 1.4. *Assume that $f : \mathbb{T} \rightarrow \mathbb{R}$ and let $t \in \mathbb{T}$.*

- (1) *If f is differentiable at t , then f is continuous at t .*
- (2) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}. \quad (1.4)$$

- (3) *If f is differentiable and t is right-dense, then*

$$f^{\Delta}(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}. \quad (1.5)$$

- (4) *If f is differentiable at t , then $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$.*

Next we define the important concept of right-dense continuity. An important fact concerning right-dense continuity is that every right-dense continuous function has a delta antiderivative [1, Theorem 1.74]. This implies that the delta definite integral of any right-dense continuous function exists.

Definition 1.5. We say that $f : \mathbb{T} \rightarrow \mathbb{R}$ is right-dense continuous (and write $f \in C_{rd}(\mathbb{T}; \mathbb{R})$) provided f is continuous at every right-dense point $t \in \mathbb{T}$, and $\lim_{s \rightarrow t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

We say p is regressive provided $1 + \mu(t)p(t) \neq 0$, $\forall t \in \mathbb{T}$. Let

$$\mathcal{R} := \{p \in C_{\text{rd}}(\mathbb{T}; \mathbb{R}) : 1 + \mu(t)p(t) \neq 0, t \in \mathbb{T}\}. \quad (1.6)$$

Also, $p \in \mathcal{R}^+$ if and only if $p \in \mathcal{R}$ and $1 + \mu(t)p(t) > 0$, $\forall t \in \mathbb{T}$. Then if $p \in \mathcal{R}$, $t_0 \in \mathbb{T}$, one can define the generalized exponential function $e_p(t, t_0)$ to be the unique solution of the initial value problem

$$x^\Delta = p(t)x, \quad x(t_0) = 1. \quad (1.7)$$

We will use many of the properties of this generalized exponential function $e_p(t, t_0)$ listed in Theorem 1.6.

THEOREM 1.6 ([1, Theorem 2.36]). *If $p, q \in \mathcal{R}$ and $s, t \in \mathbb{T}$, then*

- (1) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
- (2) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- (3) $1/e_p(t, s) = e_{\ominus p}(t, s)$, where $\ominus p := -p/(1 + \mu p)$;
- (4) $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$;
- (5) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
- (6) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$, where $p \oplus q := p + q + \mu pq$;
- (7) $e_p(t, s)/e_q(t, s) = e_{p \ominus q}(t, s)$.

2. Introduction to a delay dynamic equation

Since we are interested in the asymptotic properties of solutions we assume as mentioned earlier that our time scale \mathbb{T} is unbounded above. Consider the delay dynamic equation

$$x^\Delta(t) = -a(t)x(\delta(t))\delta^\Delta(t), \quad t \in [t_0, \infty)_\mathbb{T}, \quad (2.1)$$

where the delay function $\delta : [t_0, \infty)_\mathbb{T} \rightarrow [\delta(t_0), \infty)_\mathbb{T}$ is strictly increasing and delta differentiable with $\delta(t) < t$ for $t \in [t_0, \infty)_\mathbb{T}$ and $\lim_{t \rightarrow \infty} \delta(t) = \infty$. For example, if $\mathbb{T} = [-m, \infty)$, and $\delta(t) := t - m$, $t \in [0, \infty)$, where $m > 0$, then (2.1) becomes the well-studied delay differential equation

$$x'(t) = -a(t)x(t - m). \quad (2.2)$$

If $\mathbb{T} = \{-m, -m + 1, \dots, 0, 1, 2, \dots\}$, and $\delta(t) := t - m$, $t \in \mathbb{N}_0$, where m is a positive integer, then (2.1) becomes

$$\Delta x(t) = -a(t)x(t - m), \quad (2.3)$$

where Δ is the forward difference operator defined by $\Delta x(t) = x(t + 1) - x(t)$. If $\mathbb{T} = q^{\mathbb{N}_0} \cup \{q^{-1}, q^{-2}, \dots, q^{-m}\}$ where $q^{\mathbb{N}_0} := \{1, q, q^2, \dots\}$, $q > 1$, and $\delta(t) := (1/q^m)t$, $t \in q^{\mathbb{N}_0}$, where $m \in \mathbb{N}$, then (2.1) becomes the delay quantum equation

$$D_q x(t) = -\frac{1}{q^m}a(t)x\left(\frac{1}{q^m}t\right), \quad (2.4)$$

4 Delay dynamic equations with stability

where

$$D_q x(t) := \frac{x(qt) - x(t)}{(q-1)t} \quad (2.5)$$

is the so-called quantum derivative studied in Kac and Cheung [5]. More examples will be given later. We will use the following three lemmas to prove Theorem 3.1.

LEMMA 2.1 (chain rule). *Assume \mathbb{T} is an isolated time scale, and $g(\sigma(t)) = \sigma(g(t))$ for $t \in \mathbb{T}$. If $g : \mathbb{T} \rightarrow \mathbb{T}$ and $h : \mathbb{T} \rightarrow \mathbb{R}$, then*

$$\left(\int_{t_0}^{g(t)} h(s) \Delta s \right)^\Delta = h(g(t)) g^\Delta(t). \quad (2.6)$$

Proof. Since t is right-scattered,

$$\begin{aligned} \left(\int_{t_0}^{g(t)} h(s) \Delta s \right)^\Delta &= \frac{1}{\mu(t)} \left(\int_{t_0}^{g(\sigma(t))} h(s) \Delta s - \int_{t_0}^{g(t)} h(s) \Delta s \right) \\ &= \frac{1}{\mu(t)} \int_{g(t)}^{g(\sigma(t))} h(s) \Delta s \\ &= \frac{1}{\mu(t)} \int_{g(t)}^{\sigma(g(t))} h(s) \Delta s \\ &= \frac{1}{\mu(t)} h(g(t)) (\sigma(g(t)) - g(t)) \\ &= h(g(t)) \frac{g(\sigma(t)) - g(t)}{\mu(t)} \\ &= h(g(t)) g^\Delta(t). \end{aligned} \quad \square \quad (2.7)$$

LEMMA 2.2. *Assume \mathbb{T} is an isolated time scale and the delay δ satisfies $\delta \circ \sigma = \sigma \circ \delta$, or $\mathbb{T} = \mathbb{R}$. Then the delay equation (2.1) is equivalent to the delay equation*

$$x^\Delta(t) = -a(\delta^{-1}(t))x(t) + \left(\int_{\delta(t)}^t a(\delta^{-1}(s))x(s) \Delta s \right)^\Delta. \quad (2.8)$$

Proof. Assume x is a solution of (2.8). Then using the chain rule (Lemma 2.1) for isolated time scales or the regular chain rule for $\mathbb{T} = \mathbb{R}$,

$$\begin{aligned} x^\Delta(t) &= -a(\delta^{-1}(t))x(t) + \left(\int_{\delta(t)}^t a(\delta^{-1}(s))x(s) \Delta s \right)^\Delta \\ &= -a(\delta^{-1}(t))x(t) + a(\delta^{-1}(t))x(t) - a(t)x(\delta(t))\delta^\Delta(t) \\ &= -a(t)x(\delta(t))\delta^\Delta(t). \end{aligned} \quad (2.9)$$

Hence x is a solution of (2.1). Reversing the above steps, we obtain the desired result. \square

LEMMA 2.3. If x is a solution of (2.1) with initial function ψ , then

$$\begin{aligned} x(t) &= e_{-a(\delta^{-1})}(t, t_0)\psi(t_0) + \int_{\delta(t)}^t a(\delta^{-1}(s))x(s)\Delta s \\ &\quad - e_{-a(\delta^{-1})}(t, t_0) \int_{\delta(t_0)}^{t_0} a(\delta^{-1}(s))\psi(s)\Delta s \\ &\quad - \int_{t_0}^t \frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} e_{-a(\delta^{-1})}(t, \tau) \left(\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s \right) \Delta \tau. \end{aligned} \quad (2.10)$$

Proof. We use the variation of constants formula [1, page 77] for (2.8), to obtain

$$x(t) = e_{-a(\delta^{-1})}(t, t_0)x(t_0) + \int_{t_0}^t e_{-a(\delta^{-1})}(t, \sigma(\tau)) \left(\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s \right)^{\Delta \tau} \Delta \tau. \quad (2.11)$$

Using integration by parts [1, page 28],

$$\begin{aligned} x(t) &= e_{-a(\delta^{-1})}(t, t_0)x(t_0) + e_{-a(\delta^{-1})}(t, \tau) \int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s \Big|_{t_0}^t \\ &\quad - \int_{t_0}^t e_{-a(\delta^{-1})}^{\Delta \tau}(t, \tau) \left(\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s \right) \Delta \tau. \end{aligned} \quad (2.12)$$

It follows from Theorem 1.6 that

$$\begin{aligned} x(t) &= e_{-a(\delta^{-1})}(t, t_0)x(t_0) + \int_{\delta(t)}^t a(\delta^{-1}(s))x(s)\Delta s \\ &\quad - e_{-a(\delta^{-1})}(t, t_0) \int_{\delta(t_0)}^{t_0} a(\delta^{-1}(s))x(s)\Delta s \\ &\quad - \int_{t_0}^t e_{\Theta(-a(\delta^{-1}))}^{\Delta \tau}(\tau, t) \left(\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s \right) \Delta \tau \\ &= e_{-a(\delta^{-1})}(t, t_0)x(t_0) + \int_{\delta(t)}^t a(\delta^{-1}(s))x(s)\Delta s \\ &\quad - e_{-a(\delta^{-1})}(t, t_0) \int_{\delta(t_0)}^{t_0} a(\delta^{-1}(s))x(s)\Delta s \\ &\quad - \int_{t_0}^t \Theta(-a(\delta^{-1}))(\tau) e_{\Theta(-a(\delta^{-1}))}(\tau, t) \left(\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s \right) \Delta \tau. \end{aligned} \quad (2.13)$$

Finally, using Theorem 1.6 once again and $x(t) = \psi(t)$ for $t \in [\delta(t_0), t_0]$,

$$\begin{aligned} x(t) &= e_{-a(\delta^{-1})}(t, t_0)\psi(t_0) + \int_{\delta(t)}^t a(\delta^{-1}(s))x(s)\Delta s \\ &\quad - e_{-a(\delta^{-1})}(t, t_0) \int_{\delta(t_0)}^{t_0} a(\delta^{-1}(s))\psi(s)\Delta s \\ &\quad - \int_{t_0}^t \frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} e_{-a(\delta^{-1})}(t, \tau) \left(\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s \right) \Delta \tau. \end{aligned} \quad (2.14)$$

□

6 Delay dynamic equations with stability

3. Asymptotic properties of the delay equation

The results in this section generalize some of the results by Raffoul in [9]. Let $\psi : [\delta(t_0), t_0]_{\mathbb{T}} \rightarrow \mathbb{R}$ be rd-continuous and let $x(t) := x(t, t_0, \psi)$ be the solution of (2.1) on $[t_0, \infty)_{\mathbb{T}}$ with $x(t) = \psi(t)$ on $[\delta(t_0), t_0]_{\mathbb{T}}$. Let $\|\phi\| = \sup |\phi(t)|$ for $t \in [\delta(t_0), \infty)_{\mathbb{T}}$, and define the Banach space $B = \{\phi \in C([\delta(t_0), \infty)_{\mathbb{T}} : \phi(t) \rightarrow 0 \text{ as } t \rightarrow \infty\}$, with

$$S := \left\{ \phi \in B : \phi(t) = \psi(t) \quad \forall t \in [\delta(t_0), t_0]_{\mathbb{T}} \right\}. \quad (3.1)$$

In the following we assume

$$e_{-a(\delta^{-1})}(t, t_0) \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (3.2)$$

and take $D : [t_0, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ to be the function

$$\begin{aligned} D(t) := & \int_{t_0}^t \left(\left| \frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} \right| |e_{-a(\delta^{-1})}(t, \tau)| \int_{\delta(\tau)}^{\tau} |a(\delta^{-1}(s))| \Delta s \right) \Delta \tau \\ & + \int_{\delta(t)}^t |a(\delta^{-1}(s))| \Delta s. \end{aligned} \quad (3.3)$$

To enable the use of the contraction mapping theorem, we in fact assume there exists $\alpha \in (0, 1)$ such that

$$D(t) \leq \alpha, \quad t \in [t_0, \infty)_{\mathbb{T}}. \quad (3.4)$$

THEOREM 3.1. *Assume $\mathbb{T} = \mathbb{R}$ or \mathbb{T} is an isolated time scale. If (3.2) and (3.4) hold and $\delta \circ \sigma = \sigma \circ \delta$, then every solution of (2.1) goes to zero at infinity.*

Proof. Assume \mathbb{T} is an isolated time scale. Fix $\psi : [\delta(t_0), t_0] \rightarrow \mathbb{R}$ and define $P : S \rightarrow B$ by $(P\phi)(t) := \psi(t)$ for $t \leq t_0$ and for $t \geq t_0$,

$$\begin{aligned} (P\phi)(t) = & \psi(t_0) e_{-a(\delta^{-1})}(t, t_0) + \int_{\delta(t_0)}^t a(\delta^{-1}(s)) \phi(s) \Delta s \\ & - e_{-a(\delta^{-1})}(t, t_0) \int_{\delta(t_0)}^{t_0} a(\delta^{-1}(s)) \psi(s) \Delta s \\ & - \int_{t_0}^t \left(\frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} e_{-a(\delta^{-1})}(t, \tau) \int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s)) \phi(s) \Delta s \right) \Delta \tau. \end{aligned} \quad (3.5)$$

Then by Lemma 2.3, it suffices to show that P has a fixed point. We will use the contraction mapping theorem to show P has a fixed point. To show that $(P\phi)(t) \rightarrow 0$ as $t \rightarrow \infty$, note that the first and third terms on the right-hand side of $(P\phi)(t)$ go to zero by (3.2). From (3.3) and (3.4) and the fact that $\phi(t) \rightarrow 0$ as $t \rightarrow \infty$, we have that

$$|\phi(t)| \int_{\delta(t)}^t |a(\delta^{-1}(s))| \Delta s \leq |\phi(t)| \alpha \rightarrow 0, \quad t \rightarrow \infty. \quad (3.6)$$

Let $\epsilon > 0$ be given and choose $t^* \in \mathbb{T}$ so that

$$\alpha \|\phi\| |e_{-a(\delta^{-1})}(t, T)| < \frac{\epsilon}{2}, \quad \forall t > t^*, \quad (3.7)$$

for some large $t^* > T$. For the same T it is possible to make

$$\alpha \|\phi\|_{[\delta(T), \infty)_{\mathbb{T}}} < \frac{\epsilon}{2}, \quad (3.8)$$

where $\|\phi\|_{[\delta(T), \infty)_{\mathbb{T}}} = \sup\{|\phi(t)|, t \in [\delta(T), \infty)_{\mathbb{T}}\}$. By (2.10) and (3.2), for $t \geq T$,

$$\begin{aligned} & \int_{t_0}^t \left(\left| \frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} \right| |e_{-a(\delta^{-1})}(t, \tau)| \int_{\delta(\tau)}^{\tau} |a(\delta^{-1}(s))\phi(s)| \Delta s \right) \Delta \tau \\ &= \left(\int_{t_0}^T + \int_T^t \right) \left(\left| \frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} \right| |e_{-a(\delta^{-1})}(t, \tau)| \right. \\ & \quad \left. \times \int_{\delta(\tau)}^{\tau} |a(\delta^{-1}(s))\phi(s)| \Delta s \right) \Delta \tau \\ &= \int_{t_0}^T \left(\left| \frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} \right| |e_{-a(\delta^{-1})}(t, T)e_{-a(\delta^{-1})}(T, \tau)| \right. \\ & \quad \left. \times \int_{\delta(\tau)}^{\tau} |a(\delta^{-1}(s))\phi(s)| \Delta s \right) \Delta \tau \\ & \quad + \int_T^t \left(\left| \frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} \right| |e_{-a(\delta^{-1})}(t, \tau)| \int_{\delta(\tau)}^{\tau} |a(\delta^{-1}(s))\phi(s)| \Delta s \right) \Delta \tau \\ &\leq |e_{-a(\delta^{-1})}(t, T)| \|\phi\| \int_{t_0}^T \left(\left| \frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} \right| |e_{-a(\delta^{-1})}(T, \tau)| \right. \\ & \quad \left. \times \int_{\delta(\tau)}^{\tau} |a(\delta^{-1}(s))| \Delta s \right) \Delta \tau \\ & \quad + \|\phi\|_{[\delta(T), \infty)_{\mathbb{T}}} \int_T^t \left(\left| \frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} \right| |e_{-a(\delta^{-1})}(t, \tau)| \int_{\delta(\tau)}^{\tau} |a(\delta^{-1}(s))| \Delta s \right) \Delta \tau \\ &\leq \alpha |e_{-a(\delta^{-1})}(t, T)| \|\phi\| + \alpha \|\phi\|_{[\delta(T), \infty)_{\mathbb{T}}} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \quad (3.9)$$

8 Delay dynamic equations with stability

Hence $(P\phi)(t) \rightarrow 0$ as $t \rightarrow \infty$ and therefore, P maps S into S . It remains to show that P is a contraction under the sup norm. Let $x, y \in S$. Then

$$\begin{aligned}
& |(Px)(t) - (Py)(t)| \\
& \leq \int_{t_0}^t \left(\left| \frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} \right| |e_{-a(\delta^{-1})}(t, \tau)| \int_{\delta(\tau)}^{\tau} |a(\delta^{-1}(s))| |x(s) - y(s)| \Delta s \right) \Delta \tau \\
& \quad + \int_{\delta(t)}^t |a(\delta^{-1}(s))| |(x(s) - y(s))| \Delta s \\
& \leq \|x - y\| \left[\int_{\delta(t)}^t |a(\delta^{-1}(s))| \Delta s \right. \\
& \quad \left. + \int_{t_0}^t \left(\left| \frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} \right| |e_{-a(\delta^{-1})}(t, \tau)| \int_{\delta(\tau)}^{\tau} |a(\delta^{-1}(s))| \Delta s \right) \Delta \tau \right] \\
& \leq \alpha \|x - y\|. \tag{3.10}
\end{aligned}$$

Therefore, by the contraction mapping principle [6, page 300], P has a unique fixed point in S . This completes the proof in the isolated time scale case. See Raffoul [9] for the proof of the $\mathbb{T} = \mathbb{Z}$ case and a reference for a proof of the continuous case. \square

Example 3.2. For any real number $q > 1$ and positive integer m , define

$$\mathbb{T} = \{q^{-m}, q^{-m+1}, \dots, q^{-1}, 1, q, q^2, \dots\}. \tag{3.11}$$

We show if $0 < c < q^m/2m(q - 1)$, then for any initial function $\psi(t)$, $t \in [q^{-m}, 1]_{\mathbb{T}}$, the solution of the delay initial value problem

$$D_q x(t) = -\frac{1}{q^m} \frac{c}{t} x\left(\frac{1}{q^m} t\right), \quad t \in [1, \infty]_{\mathbb{T}}, \tag{3.12}$$

$$x(t) = \psi(t), \quad t \in [q^{-m}, 1]_{\mathbb{T}} \tag{3.13}$$

goes to zero as $t \rightarrow \infty$.

To obtain (3.12) from (2.1), take $a(t) = c/t$ and $\delta(t) = q^{-m}t$ which implies $a(\delta^{-1}(t)) = c/q^m t$ and $\delta^{\Delta}(t) = q^{-m}$. To use Theorem 3.1, we verify that conditions (3.2) and (3.4) hold. Note that

$$e_{-a(\delta^{-1})}(t, 1) = \prod_{s \in [1, t]_{\mathbb{T}}} [1 - s(q - 1)a(q^m s)] = (1 - q^{-m}c(q - 1))^n \tag{3.14}$$

for $t = q^n$. If $c \in (0, q^m/2m(q - 1))$, then $c \in (0, 2q^m/(q - 1))$ so that $1 - q^{-m}c(q - 1) \in (-1, 1)$ and

$$\lim_{t \rightarrow \infty} e_{-a(\delta^{-1})}(t, 1) = \lim_{n \rightarrow \infty} (1 - q^{-m}c(q - 1))^n = 0. \tag{3.15}$$

Thus, (3.2) is satisfied. Now consider $D(t)$ as defined in (3.3). We seek $\alpha \in (0, 1)$ such that $D(t) \leq \alpha$, $\forall t \in [1, \infty)_{\mathbb{T}}$. Here we have $t_0 = 1$, $\mu(t) = (q-1)t$, and

$$e_{-a(\delta^{-1})}(t, \tau) = (1 - q^{-m}c(q-1))^{n-k} \quad (3.16)$$

for $t = q^n$, $\tau = q^k$ with $k < n$. For the second integral in $D(t)$, note that

$$\int_u^{qu} f(\zeta) \Delta \zeta = (qu - u) f(u), \quad (3.17)$$

whence

$$\begin{aligned} \int_{\delta(t)}^t a(\delta^{-1}(s)) \Delta s &= \left(\int_{q^{-m}t}^{q^{-m+1}t} + \int_{q^{-m+1}t}^{q^{-m+2}t} + \cdots + \int_{q^{-1}t}^t \right) \left(\frac{c}{q^m s} \right) \Delta s \\ &= \left(\frac{mc}{q^m(q^{-m}t)} q^{-m}t \right) (q-1) \\ &= \frac{mc}{q^m} (q-1), \end{aligned} \quad (3.18)$$

which is independent of t . It follows that

$$\begin{aligned} D(t) &= \frac{mc}{q^m} (q-1) + \frac{mc}{q^m} (q-1) \int_1^t \left(\frac{c}{q^m \tau} \cdot \frac{1}{1 - (q-1)\tau c/q^m \tau} e_{-a(\delta^{-1})}(t, \tau) \right) \Delta \tau \\ &= \frac{mc}{q^m} (q-1) + \frac{mc}{q^m} (q-1) \sum_{k=0}^{n-1} \frac{c}{q^m q^k - q^k (q-1)c} (1 - q^{-m}c(q-1))^{n-k} (q-1) q^k \\ &= \frac{mc}{q^m} (q-1) + \frac{mc}{q^m} \sum_{k=0}^{n-1} \frac{c(q-1)^2}{q^m - (q-1)c} (1 - q^{-m}c(q-1))^{n-k} \\ &= \frac{mc}{q^m} (q-1) + \frac{mc^2 q^{-m} (q-1)^2}{q^m - c(q-1)} \frac{1 - q^{-m}c(q-1)}{-q^{-m}(q-1)c} \left((1 - q^{-m}c(q-1))^n - 1 \right) \\ &= \frac{mc(q-1)}{q^m} + \frac{mcq^{-m}(q-1)}{1 - cq^{-m}(q-1)} (1 - q^{-m}c(q-1)) \left(1 - (1 - q^{-m}c(q-1))^n \right). \end{aligned} \quad (3.19)$$

Consequently,

$$D(t) = \frac{mc(q-1)}{q^m} \left[2 - (1 - q^{-m}c(q-1))^n \right] < \frac{2mc(q-1)}{q^m}, \quad \forall t = q^n \in [1, \infty)_{\mathbb{T}}. \quad (3.20)$$

Since $0 < c < q^m/2m(q-1)$, by taking $\alpha := 2mc(q-1)/q^m$ condition (3.4) is satisfied by

$$D(t) < \alpha < 1, \quad \forall t \in [1, \infty)_{\mathbb{T}}. \quad (3.21)$$

Thus (3.2) and (3.4) are met, so that by Theorem 3.1, the solution of the IVP (3.12), (3.13) goes to zero as $t \rightarrow \infty$.

10 Delay dynamic equations with stability

Example 3.3. Consider the time scale of harmonic numbers

$$\mathbb{T} = \{H_{-m}, H_{-m+1}, \dots, H_0, H_1, \dots\} \quad (3.22)$$

for some $m \in \mathbb{N}$, where $H_0 := 0$, $H_n := \sum_{j=1}^n (1/j)$ and $H_{-n} := -H_n$ for $n \in \mathbb{N}$. We will show that if

$$0 < c < \frac{H_m}{2m}, \quad (3.23)$$

then for any initial function $\psi(t)$, $t \in [H_{-m}, 0]_{\mathbb{T}}$, the solution of the delay initial value problem

$$\Delta_n x(H_n) = -\frac{(n-m+1)c}{H_m} x(H_{n-m}) \Delta_n H_{n-m}, \quad n \in \mathbb{N}_0, \quad (3.24)$$

$$x(H_n) = \psi(H_n), \quad n = 0, -1, \dots, -m, \quad (3.25)$$

goes to zero as $t \rightarrow \infty$.

To get (3.24) from (2.1), take

$$a(t) = a(H_n) = \frac{(n-m+1)c}{H_m}, \quad \delta(t) = \delta(H_n) = H_{n-m}. \quad (3.26)$$

It follows that

$$e_{-a(\delta^{-1})}(H_n, 0) = \left(1 - \frac{c}{H_m}\right)^n \quad \text{for } n \in \mathbb{N}_0. \quad (3.27)$$

If we restrict $c \in (0, 2H_m)$,

$$\lim_{t \rightarrow \infty} e_{-a(\delta^{-1})}(t, 0) = \lim_{n \rightarrow \infty} \left(1 - \frac{c}{H_m}\right)^n = 0, \quad (3.28)$$

satisfying (3.2). Simplifying (3.3),

$$\begin{aligned} D(t) &= \int_0^{H_n} \left(\frac{(\tau+1)c}{H_m} \frac{1}{1 - (\tau+1)c/(\tau+1)H_m} \left(1 - \frac{c}{H_m}\right)^{n-\tau} \int_{H_{\tau-m}}^{H_{\tau}} \frac{(s+1)c}{H_m} \Delta s \right) \Delta \tau \\ &+ \int_{H_{n-m}}^{H_n} \frac{(s+1)c}{H_m} \Delta s \\ &= \frac{cm}{H_m} + \frac{c^2 m}{H_m} \cdot \frac{1}{H_m - c} \sum_{\tau=0}^{n-1} \left(1 - \frac{c}{H_m}\right)^{n-\tau} \\ &= \frac{cm}{H_m} + \frac{c^2 m}{H_m(H_m - c)} \left(\frac{H_m - c}{H_m} \right) \left(\frac{-H_m}{c} \right) \left[\left(1 - \frac{c}{H_m}\right)^n - 1 \right] \\ &= \frac{cm}{H_m} \left[2 - \left(1 - \frac{c}{H_m}\right)^n \right] < \frac{2cm}{H_m} \end{aligned} \quad (3.29)$$

for all $t = H_n \in [0, \infty)_{\mathbb{T}}$. By choosing $c \in (0, H_m/2m)$, $D(t) < \alpha := 2cm/H_m < 1$, $\forall t = H_n \in [0, \infty)_{\mathbb{T}}$, satisfying (3.4). Thus by Theorem 3.1, for any given initial function ψ , the solution of the IVP (2.1), (3.25) goes to zero as $t = H_n$ goes to infinity.

Example 3.4. Let $\mathbb{T} = \{-mh, \dots, -h, 0, h, 2h, \dots\}$ where $m \in \mathbb{N}$, then (2.1) becomes

$$\Delta_h x(t) = -cx(t - hm), \quad t \in h\mathbb{N}, \quad (3.30)$$

where $\Delta_h x(t) := (x(t + h) - x(t))/h$. Our results give that if

$$0 < c < \frac{1}{2mh}, \quad (3.31)$$

then all solutions go to zero as $t \rightarrow \infty$.

In this case, $a(t) = c$, $\delta(t) = t - mh$, and $\delta^{-1}(t) = t + mh$. It can be shown that

$$e_{-a(\delta^{-1})}(t, 0) = (1 - ch)^{t/h}. \quad (3.32)$$

Note for $0 < ch < 2$, condition (3.2) is satisfied because

$$e_{-a(\delta^{-1})}(t, 0) \rightarrow 0 \quad (3.33)$$

as $t \rightarrow \infty$. It also can be shown that

$$D(t) = mhc + mhc(1 - |1 - ch|^{t/h}) \leq 2mhc \quad (3.34)$$

if $0 < ch < 1$. Hence (3.4) holds if $0 < c < 1/2mh$. Therefore, we use Theorem 3.1 to conclude that all solutions of (3.30) go to zero as $t \rightarrow \infty$. It can be shown that if $c \leq 0$, then there is $\lambda \geq 1$ such that $x(t) = \lambda^{ht}$ is a solution of (3.30). However, $x(t)$ does not approach zero as $t \rightarrow \infty$. Hence our lower estimate for c is sharp. If $m = 1$, then it can be shown that all solutions of (3.30) go to zero if $0 < c < 1/h$. If $m = 2$, our example shows that if $0 < c < 1/4h$ all solutions go to zero as $t \rightarrow \infty$. It can be shown that if $c = 1/2h$, then there is a solution that does not go to zero as $t \rightarrow \infty$. Next we give an elementary example where \mathbb{T} is the real interval $[-m, \infty)$, $m > 0$. Delay differential equations have been studied extensively; for example, see [8].

Example 3.5. Let $\mathbb{T} = [-m, \infty)$. For the delay differential equation

$$x'(t) = -cx(t - m), \quad t \in [0, \infty), \quad (3.35)$$

if $0 < c < 1/2m$, then all solutions $x(t)$ approach zero as $t \rightarrow \infty$.

Note that (2.1) reduces to (3.35) if $a(t) = c$, $\delta(t) = t - m$, and $\mathbb{T} = [-m, \infty)$. It can be shown that

$$e_{-a(\delta^{-1})}(t, \tau) = e^{-c(t-\tau)}, \quad D(t) = cm(2 - e^{-ct}). \quad (3.36)$$

Note that $c > 0$ implies $e_{-a(\delta^{-1})}(t, 0) \rightarrow 0$ as $t \rightarrow \infty$ and $D(t) \leq 2cm < 1$ if $c < 1/2m$. Our result follows from Theorem 3.1. Also note that if $c = \pi/2m$, then $x(t) = \sin((\pi/2m)t)$ is a solution that does not approach zero as $t \rightarrow \infty$. It is well known that if $0 < c < \pi/2m$, then all solutions of (3.35) approach zero as $t \rightarrow \infty$.

4. Asymptotic stability of a nonlinear delay dynamic equation

In this section we consider, on arbitrary time scales, the nonlinear delay dynamic equation

$$x^\Delta(t) = - \int_{\delta(t)}^t \left(\sum_{i=1}^n f_i(\tau, x(\tau)) \right) \Delta \tau, \quad t \in [t_0, \infty)_\mathbb{T}, \quad (4.1)$$

where $f_i(t, x)$ for each fixed $t \in \mathbb{T}$ is continuous with respect to x . In addition, we always suppose

$$(H1) \quad x f_i(t, x) \geq 0 \text{ and } \sum_{i=1}^n f_i(t, x) = 0 \Leftrightarrow x = 0, \quad t \in [t_0, \infty)_\mathbb{T},$$

$$(H2) \quad \delta : \mathbb{T} \rightarrow \mathbb{T} \text{ is continuous and nondecreasing, with } \delta(t) \leq t \text{ and } \lim_{t \rightarrow \infty} \delta(t) = \infty.$$

The initial condition associated with (4.1) takes the form

$$x(t) = \psi(t), \quad t \in [\delta(t_0), t_0], \quad \psi \text{ is rd-continuous on } [\delta(t_0), t_0]. \quad (4.2)$$

Equation (4.1) is studied extensively in [7] in the case when $\mathbb{T} = \mathbb{R}$; indeed many of our techniques in this section are motivated by those in [7]. See also a related discussion in [3].

THEOREM 4.1. *Assume there exists $M > 0$ such that for $|c| \leq M$,*

- (i) $\int_{\delta(t)}^{\sigma(t)} \sum_{i=1}^n (1/c) f_i(\tau, c) (\tau - \delta(\tau)) \Delta \tau \leq \xi < 1, c \neq 0$, for $t \in [t_1, \infty)_\mathbb{T}$;
- (ii) $\int_{t_0}^\infty \sum_{i=1}^n |f_i(\tau, c)| (\tau - \delta(\tau)) \Delta \tau = \infty$;
- (iii) $|\sum_{i=1}^n f_i(\tau, c)| \leq \sum_{i=1}^n a_i(\tau) |c|$ where the a_i are rd-continuous and nonnegative for $\tau \geq t_0$;
- (iv) $|\sum_{i=1}^n f_i(\tau, x)| \leq \sum_{i=1}^n f_i(\tau, y)$ for $|x| \leq y \leq M$, $\tau \in [\delta(t_0), \infty)_\mathbb{T}$.

Then for any $0 < \epsilon \leq M$, there is an $\eta(\epsilon) > 0$ such that for any rd-continuous initial function ψ with $\|\psi\|_{[\delta(t_0), t_0]_\mathbb{T}} < \eta(\epsilon)$, the solution x of (4.1), (4.2) satisfies

$$|x(t)| < \epsilon \quad \forall t \in [\delta(t_0), \infty)_\mathbb{T}, \quad \lim_{t \rightarrow \infty} x(t) = 0. \quad (4.3)$$

In other words, the trivial solution of (4.1) is asymptotically stable.

Proof. Let $\epsilon \in (0, M]$ and $t_1 \in [t_0, \infty)_\mathbb{T}$ be given. Also let $p(t) := \sum_{i=1}^n a_i(t)(t - \delta(t))$; by (H2) and (iii), $p \in \mathcal{R}^+$. Define $\eta(\epsilon) := \epsilon/e_p(t_1, t_0)$ and take an initial function ψ with $\|\psi\|_{[\delta(t_0), t_0]_\mathbb{T}} < \eta(\epsilon)$. Integrating (4.1) from t_0 to $t \in [t_0, t_1]_\mathbb{T}$, we get

$$x(t) = x(t_0) - \int_{t_0}^t \left(\int_{\delta(\tau)}^{\sigma(s)} \sum_{i=1}^n f_i(\tau, x(\tau)) \Delta \tau \right) \Delta s. \quad (4.4)$$

Taking absolute values,

$$|x(t)| \leq |x(t_0)| + \int_{t_0}^t \left(\left| \int_{\delta(\tau)}^{\sigma(s)} \sum_{i=1}^n f_i(\tau, x(\tau)) \Delta \tau \right| \Delta s \right) \Delta s. \quad (4.5)$$

By property (iii),

$$|x(t)| \leq |x(t_0)| + \int_{t_0}^t \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^n a_i(\tau) |x(s)| \Delta s \right) \Delta \tau. \quad (4.6)$$

Let $\tilde{x}(t) := \max\{|x(s)| : s \in [\delta(t_0), t]\}$. Then rewriting (4.6), we have

$$\tilde{x}(t) \leq \tilde{x}(t_0) + \int_{t_0}^t \tilde{x}(\tau) \sum_{i=1}^n a_i(\tau) (\tau - \delta(\tau)) \Delta \tau. \quad (4.7)$$

By Gronwall's inequality [1, page 257],

$$\tilde{x}(t) \leq \tilde{x}(t_0) e_p(t, t_0), \quad \forall t \in [t_0, \infty)_{\mathbb{T}}. \quad (4.8)$$

Therefore by the definition of $\eta(\epsilon)$ and the choice of ψ , it follows that $\tilde{x}(t_0) < \eta(\epsilon)$ and $|x(t)| \leq \tilde{x}(t) < \epsilon$ for $t \in [t_0, t_1]_{\mathbb{T}}$. Suppose there exists $t^* \geq t_1$ such that $x(t^*) = \epsilon$ and $x^{\Delta}(t^*) \geq 0$. Note the case when $x(t^*) = -\epsilon$ and $x^{\Delta}(t^*) \leq 0$ is similar. By (4.1), there exists $\bar{t} \in [\delta(t^*), t^*]_{\mathbb{T}}$ such that $x(\bar{t}) \leq 0$. Integrating (4.1) from \bar{t} to t^* , we obtain

$$\epsilon - x(\bar{t}) = - \int_{\bar{t}}^{t^*} \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^n f_i(\tau, x(s)) \Delta s \right) \Delta \tau. \quad (4.9)$$

Since $x(\bar{t}) \leq 0$,

$$\epsilon \leq \int_{\delta(t^*)}^{t^*} \left(\int_{\delta(\tau)}^{\tau} \left| \sum_{i=1}^n f_i(\tau, x(s)) \right| \Delta s \right) \Delta \tau. \quad (4.10)$$

By (H1) and (iv),

$$\epsilon \leq \int_{\delta(t^*)}^{t^*} \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^n |f_i(\tau, \epsilon)| \Delta s \right) \Delta \tau, \quad (4.11)$$

so that

$$1 \leq \frac{1}{\epsilon} \int_{\delta(t^*)}^{t^*} (\tau - \delta(\tau)) \sum_{i=1}^n f_i(\tau, \epsilon) \Delta \tau, \quad (4.12)$$

which is a contradiction of (i). Therefore no such t^* exists. Now suppose there exists $t^* \in [t_1, \infty)_{\mathbb{T}}$ such that t^* is right scattered, $|x(t)| \leq \epsilon$ for all $t \in [\delta(t_0), t^*]_{\mathbb{T}}$, and $x(t^*) \in (-\epsilon, \epsilon)$ but $|x(\sigma(t^*))| \geq \epsilon$. Without loss of generality, assume

$$x(\sigma(t^*)) \geq \epsilon. \quad (4.13)$$

By (4.1),

$$\int_{\delta(t^*)}^{t^*} \left(\sum_{i=1}^n f_i(t^*, x(s)) \right) \Delta s < 0. \quad (4.14)$$

14 Delay dynamic equations with stability

Therefore by (H1) and (4.1), there exists $\bar{t} \in [\delta(t^*), t^*]_{\mathbb{T}}$ such that $x(\bar{t}) \leq 0$. Integrate (4.1) from \bar{t} to $\sigma(t^*)$ and use (4.13) to see that

$$\epsilon \leq x(\sigma(t^*)) \leq x(\sigma(t^*)) - x(\bar{t}) = - \int_{\bar{t}}^{\sigma(t^*)} \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^n f_i(\tau, x(s)) \Delta s \right) \Delta \tau. \quad (4.15)$$

Thus, comparing the extremities,

$$\epsilon \leq \int_{\delta(t^*)}^{\sigma(t^*)} \left(\int_{\delta(\tau)}^{\tau} \left| \sum_{i=1}^n f_i(\tau, x(s)) \right| \Delta s \right) \Delta \tau \leq \int_{\delta(t^*)}^{\sigma(t^*)} \sum_{i=1}^n f_i(\tau, \epsilon) (\tau - \delta(\tau)) \Delta \tau; \quad (4.16)$$

division by ϵ yields

$$1 \leq \frac{1}{\epsilon} \int_{\delta(t^*)}^{\sigma(t^*)} \sum_{i=1}^n f_i(\tau, \epsilon) (\tau - \delta(\tau)) \Delta \tau, \quad (4.17)$$

which also contradicts (i) and provides the desired result. Now we show the limit of $x(t)$ goes to zero as $t \rightarrow \infty$.

Case 1. Let x be a nonoscillatory function. Assume $\xi < 1$ but there exists a solution x of (4.1) such that

$$\lim_{t \rightarrow \infty} x(t) \neq 0. \quad (4.18)$$

Since x is nonoscillatory, there exists $T_1 > t_0$ such that $x(t)x(T_1) > 0$ for all $t \in [T_1, \infty]_{\mathbb{T}}$. Without loss of generality, assume $x(t) > 0$ for all $t \in [T_1, \infty]_{\mathbb{T}}$. From (4.1) and (H1), $x^{\Delta}(t) < 0$. Hence there exists a constant $x^* > 0$ such that

$$\lim_{t \rightarrow \infty} x(t) = x^* \quad (4.19)$$

and there exists $T^* \in \mathbb{T}$ such that for all $t \in [T_1, \infty]_{\mathbb{T}}$, $x(t) \in [x^*, 3x^*/2]$. Integrate (4.1) from $\delta^{-1}(T^*)$ to t to obtain

$$x(\delta^{-1}(T^*)) - x(t) = \int_{\delta^{-1}(T^*)}^t \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^n f_i(\tau, x(s)) \Delta s \right) \Delta \tau. \quad (4.20)$$

For $t \geq \delta^{-1}(T^*)$,

$$x(\delta^{-1}(T^*)) - x(t) \leq \frac{x^*}{2}, \quad (4.21)$$

but

$$\begin{aligned} \int_{\delta^{-1}(T^*)}^t \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^n f_i(\tau, x(s)) \Delta s \right) \Delta \tau &\geq \int_{\delta^{-1}(T^*)}^t \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^n f_i(\tau, x^*) \Delta s \right) \Delta \tau \\ &= \int_{\delta^{-1}(T^*)}^t \left(\sum_{i=1}^n f_i(\tau, x^*) (\tau - \delta(\tau)) \right) \Delta \tau, \end{aligned} \quad (4.22)$$

which goes to ∞ as $t \rightarrow \infty$ by (H1) and properties (ii) and (iv). This contradicts (4.20) and (4.21), so when x is nonoscillatory, the limit of $x(t)$ goes to zero as $t \rightarrow \infty$.

Case 2. Now assume x is oscillatory. Pick $\epsilon \in (0, M]$ and $T_2 > t_0$ such that

$$\int_{\delta(t)}^t \left(\sum_{i=1}^n \frac{1}{c} f_i(\tau, c)(\tau - \delta(\tau)) \right) \Delta \tau < \xi \quad (4.23)$$

for $0 < |c| \leq \epsilon$ and $\forall t \in [T_2, \infty)_{\mathbb{T}}$. Assume

$$\bar{x} := \limsup_{t \rightarrow \infty} |x(t)| \neq 0. \quad (4.24)$$

Without loss of generality, we will assume

$$\limsup_{t \rightarrow \infty} x(t) = \bar{x} \neq 0. \quad (4.25)$$

Since the solution is oscillatory, there exists a sequence $\{t_j\}_{j=1}^{\infty}$ in \mathbb{T} such that $\lim_{j \rightarrow \infty} t_j = \infty$, $|x(\sigma(t_j))| \rightarrow \bar{x} \leq \epsilon$ as $j \rightarrow \infty$, and $x^{\Delta}(t_j) \geq 0$. Let $\beta := (\bar{x}/2)(1 - \xi)$ and $T_3 > T_2$ such that $|x(t)| < \beta + \bar{x}$ for $t \in [T_3, \infty)_{\mathbb{T}}$. Further assume the t_j were chosen such that $\delta(t_j) > T_3$ and $x(\sigma(t_j)) > \bar{x} - \beta$. From (4.1) and the choice of t_j , we see that

$$0 \leq x^{\Delta}(t_j) = - \int_{\delta(t_j)}^{t_j} \left(\sum_{i=1}^n f_i(t_j, x(s)) \right) \Delta s. \quad (4.26)$$

By (H1), there exists $t^* \in [\delta(t_j), t_j]_{\mathbb{T}}$ such that $x(t^*) \leq 0$. Then

$$\begin{aligned} \bar{x} - \beta &< x(\sigma(t_j)) \leq x(\sigma(t_j)) - x(t^*) = - \int_{t^*}^{\sigma(t_j)} \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^n f_i(\tau, x(s)) \Delta s \right) \Delta \tau \\ \bar{x} - \beta &< \int_{t^*}^{\sigma(t_j)} \left(\int_{\delta(\tau)}^{\tau} \left| \sum_{i=1}^n f_i(\tau, x(s)) \right| \Delta s \right) \Delta \tau \\ &\leq \int_{t^*}^{\sigma(t_j)} \sum_{i=1}^n f_i(\tau, \beta + \bar{x})(\tau - \delta(\tau)) \Delta \tau \\ &\leq \int_{\delta(t_j)}^{\sigma(t_j)} \sum_{i=1}^n f_i(\tau, \beta + \bar{x})(\tau - \delta(\tau)) \Delta \tau \\ &< (\beta + \bar{x})\xi \end{aligned} \quad (4.27)$$

by (i). But then $\beta > ((1 - \xi)/(1 + \xi))\bar{x}$, a contradiction of our selection of β . Therefore $\bar{x} = 0$, in other words, the solution goes to zero as $t \rightarrow \infty$. \square

Example 4.2. Let $\mathbb{T} = \mathbb{Z}$, the set of integers, the delay function $\delta(t) := t - m$ for some positive integer m , $n = 1$ and $f_1(t, x) := kx$ for some constant k . Then the four conditions of Theorem 4.1 are met if $0 < k \leq \xi/m(m+1)$ for any $\xi \in (0, 1)$, whereby the trivial solution of (4.1) is asymptotically stable.

16 Delay dynamic equations with stability

For $\mathbb{T} = \mathbb{Z}$ and f_1 as above, (4.1) becomes the (delay) difference equation

$$x(t+1) = x(t) - \sum_{s=t-m}^{t-1} kx(s), \quad t \in \mathbb{Z}. \quad (4.28)$$

Fix $\xi \in (0, 1)$, $m \in \mathbb{N}$, and $0 < k \leq \xi/m(m+1)$. To check (i), note that

$$\sum_{\tau=t-m}^t \frac{1}{c} kc(\tau - (\tau - m)) = km(m+1) \leq \xi < 1, \quad t \in \mathbb{Z}. \quad (4.29)$$

In (ii), $\sum_{\tau=t_0}^{\infty} k|c|m = \infty$. For (iii), $|f_1(\tau, c)| = k|c|$, so we take $a_1(\tau) \equiv k > 0$, which is rd-continuous. Finally, (iv) is met as $|f_1(\tau, x)| = k|x| \leq ky = f_1(\tau, y)$ for $|x| \leq y \leq M$ with $\tau \in [t_0, \infty)_{\mathbb{T}}$. Therefore the trivial solution of (4.1) is asymptotically stable by Theorem 4.1.

LEMMA 4.3. *Assume x is a global solution for (4.1), (4.2). Then either x is bounded or x is oscillatory.*

Proof. If x is nonoscillatory, then there exists $T > 0$ such that for $t > T$, x does not change sign. Without loss of generality, we suppose $x(t) > 0$ for $t > T$. By (H1), $\sum_{i=1}^n f_i(t, x(t)) > 0$ for $t > T$, which together with (4.1) yields that $x^{\Delta}(t) < 0$ for large t . Therefore, x is strictly decreasing on each interval, so by continuity, x is bounded. \square

THEOREM 4.4. *Assume there exists $M > 0$ and $t_1 \geq t_0$ such that for $|c| \geq M$,*

- (i) $\int_{\delta(t)}^{\sigma(t)} \sum_{i=1}^n (1/c) f_i(\tau, c) (\tau - \delta(\tau)) \Delta \tau \leq \xi < 1$, for $t \in [t_1, \infty)_{\mathbb{T}}$;
- (ii) $|\sum_{i=1}^n f_i(\tau, c)| \leq \sum_{i=1}^n a_i(\tau) |c|$ where the a_i are rd-continuous and nonnegative for $\tau \geq t_0$;
- (iii) $|\sum_{i=1}^n f_i(t, u)| \leq \sum_{i=1}^n f_i(t, y)$ for $|u| \leq y$, $t \in [\delta(t_0), \infty)_{\mathbb{T}}$.

Then every solution x of (4.1), (4.2) for bounded initial function ψ is bounded and satisfies

$$\limsup_{t \rightarrow \infty} |x(t)| \leq M. \quad (4.30)$$

Proof. Appealing to assumption (ii), $x^{\Delta}(t)$ exists and is finite for each $t \in [t_0, \infty)_{\mathbb{T}}$, so that solutions of (4.1), (4.2) are global. Suppose x is an unbounded solution of (4.1), (4.2) with bounded initial function ψ . By Lemma 4.3, x is also oscillatory. As in Case 2 of the proof of Theorem 4.1, without loss of generality there exists a sequence $\{t_j\}_{j=1}^{\infty}$ in \mathbb{T} such that $\lim_{j \rightarrow \infty} t_j = \infty$, $M \leq x(\sigma(t_j))$ with $|x(s)| \leq x(\sigma(t_j))$ for all $s \in [\delta^2(t_j), t_j]$, and $x(\sigma(t_j)) \rightarrow \infty$ as $j \rightarrow \infty$. Moreover there exist corresponding $t_j^* \in [\delta(t_j), t_j]_{\mathbb{T}}$ satisfying $x(t_j^*) \leq 0$. Then

$$x(\sigma(t_j)) \leq x(\sigma(t_j)) - x(t_j^*) = - \int_{t_j^*}^{\sigma(t_j)} \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^n f_i(s, x(s)) \Delta s \right) \Delta \tau, \quad (4.31)$$

so that by (iii) we have

$$\begin{aligned} x(\sigma(t_j)) &\leq \int_{t_j^*}^{\sigma(t_j)} \left(\int_{\delta(\tau)}^{\tau} \left| \sum_{i=1}^n f_i(\tau, x(s)) \right| \Delta s \right) \Delta \tau \\ &\leq \int_{\delta(t_j)}^{\sigma(t_j)} \int_{\delta(\tau)}^{\tau} \sum_{i=1}^n f_i(\tau, x(\sigma(t_j))) \Delta s \Delta \tau. \end{aligned} \quad (4.32)$$

But then

$$\xi < 1 \leq \int_{\delta(t_j)}^{\sigma(t_j)} \frac{\tau - \delta(\tau)}{x(\sigma(t_j))} \sum_{i=1}^n f_i(\tau, x(\sigma(t_j))) \Delta \tau, \quad (4.33)$$

a contradiction of (i). Therefore x must be bounded. To prove the last assertion of the theorem, suppose

$$\limsup_{t \rightarrow \infty} |x(t)| = \bar{M} > M. \quad (4.34)$$

As in the proof of Theorem 4.1 there are two cases to consider, nonoscillatory and oscillatory. Assuming the former leads to a contradiction; in the latter case, there exists a sequence $\{\bar{t}_j\}_{j=1}^{\infty}$ as before, which would likewise lead to a contradiction, whereby the conclusion of the theorem holds. \square

Finally we consider the general nonlinear delay dynamic equation

$$x^{\Delta}(t) = - \int_{\delta(t)}^t \left(\sum_{i=1}^n f_i(t, x(s)) \right) \Delta_s g(t, s), \quad t \in [t_0, \infty)_{\mathbb{T}}, \quad (4.35)$$

where for simplicity we define

$$\int_a^b f(t, s) \Delta_s g(t, s) := \int_a^b f(t, s) g^{\Delta_s}(t, s) \Delta s \quad (4.36)$$

and assume for each fixed $t \in \mathbb{T}$ that $f(t, s) g^{\Delta_s}(t, s)$ is rd-continuous. Then it is straightforward to generalize Theorem 4.1 to get the following result.

THEOREM 4.5. *Assume there exists $M > 0$ such that for $|c| \leq M$,*

- (i) $\int_{\delta(t)}^{\sigma(t)} \sum_{i=1}^n (1/c) f_i(\tau, c) (g(\tau, \tau) - g(\tau, \delta(\tau))) \Delta \tau \leq \xi < 1$, $c \neq 0$, for $t \in [t_1, \infty)_{\mathbb{T}}$;
- (ii) $\int_{t_0}^{\infty} \sum_{i=1}^n |f_i(\tau, c)| (g(\tau, \tau) - g(\tau, \delta(\tau))) \Delta \tau = \infty$;
- (iii) $|\sum_{i=1}^n f_i(\tau, c)| \leq \sum_{i=1}^n a_i(\tau) |c|$ where the a_i are rd-continuous and nonnegative for $\tau \geq t_0$;
- (iv) $|\sum_{i=1}^n f_i(\tau, x)| \leq \sum_{i=1}^n f_i(\tau, y)$ for $|x| \leq y \leq M$, $\tau \in [\delta(t_0), \infty)_{\mathbb{T}}$;
- (v) $g(t, t) > g(t, \delta(t))$, $t \in [t_0, \infty)_{\mathbb{T}}$.

Then the trivial solution of (4.35) is asymptotically stable.

Example 4.6. Equation (2.1) is a special case of (4.35) if \mathbb{T} is an isolated time scale.

18 Delay dynamic equations with stability

If \mathbb{T} is an isolated time scale, then the delay dynamic equation (4.35) with $f_1(t, x) = a(t)\delta^\Delta(t)x$ and $f_i(t, x) = 0$, $2 \leq i \leq n$ becomes

$$x^\Delta(t) = - \int_{\tau(t)}^t a(s)\delta^\Delta(s)x(s)\Delta_s g(t, s), \quad (4.37)$$

where $\tau(t) \leq \delta(t) < t$ are delay functions and g is (the Heaviside function) defined by

$$g(t, s) = \begin{cases} 0, & \tau(t) \leq s \leq \delta(t), \\ 1, & s > \delta(t). \end{cases} \quad (4.38)$$

Then (the Dirac delta function)

$$g^{\Delta_s}(t, s) = \begin{cases} \frac{1}{\mu(s)}, & s = \delta(t), \\ 0, & \text{otherwise.} \end{cases} \quad (4.39)$$

Using the above we obtain

$$\begin{aligned} x^\Delta(t) &= - \int_{\tau(t)}^t a(s)\delta^\Delta(s)x(s)\Delta_s g(t, s) \\ &= -a(t)\delta^\Delta(t) \int_{\tau(t)}^t x(s)g^{\Delta_s}(t, s)\Delta s \\ &= -a(t)\delta^\Delta(t) \int_{\delta(t)}^{\sigma(\delta(t))} x(s)g^{\Delta_s}(t, s)\Delta s \\ &= -a(t)\delta^\Delta(t)x(\delta(t)) \frac{1}{\mu(\delta(t))}\mu(\delta(t)) \\ &= -a(t)\delta^\Delta(t)x(\delta(t)). \end{aligned} \quad (4.40)$$

Acknowledgment

This research was supported by NSF Grant 0354281.

References

- [1] M. Bohner and A. C. Peterson, *Dynamic Equations on Time Scales. An Introduction with Applications*, Birkhäuser Boston, Massachusetts, 2001.
- [2] M. Bohner and A. C. Peterson (eds.), *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, Massachusetts, 2003.
- [3] J. R. Haddock and Y. Kuang, *Asymptotic theory for a class of nonautonomous delay differential equations*, Journal of Mathematical Analysis and Applications **168** (1992), no. 1, 147–162.
- [4] S. Hilger, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, Results in Mathematics **18** (1990), no. 1-2, 18–56.
- [5] V. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer, New York, 2002.
- [6] W. G. Kelley and A. C. Peterson, *The Theory of Differential Equations, Classical and Qualitative*, Pearson Prentice Hall, New Jersey, 2004.

- [7] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Mathematics in Science and Engineering, vol. 191, Academic Press, Massachusetts, 1993.
- [8] G. S. Ladde, V. Lakshmikantham, and B. G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 110, Marcel Dekker, New York, 1987.
- [9] Y. Raffoul, *Stability and Periodicity in Completely Delayed Equations*, to appear in *Journal of Mathematical Analysis and Applications*.

Douglas R. Anderson: Department of Mathematics, Concordia College, Moorhead, MN 56562, USA
E-mail address: andersod@cord.edu

Robert J. Krueger: Department of Mathematics, Concordia University, St. Paul, MN 55104, USA
E-mail address: rkrueger@csp.edu

Allan C. Peterson: Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588, USA
E-mail address: apeterso@math.unl.edu

Special Issue on Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at <http://www.hindawi.com/journals/mpe/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil ; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru