

BOUNDEDNESS IN FUNCTIONAL DYNAMIC EQUATIONS ON TIME SCALES

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Using nonnegative definite Lyapunov functionals, we prove general theorems for the boundedness of all solutions of a functional dynamic equation on time scales. We apply our obtained results to linear and nonlinear Volterra integro-dynamic equations on time scales by displaying suitable Lyapunov functionals.

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1. Introduction

In this paper, we consider the boundedness of solutions of equations of the form

$$x^\Delta(t) = G(t, x(s); 0 \leq s \leq t) := G(t, x(\cdot)) \quad (1.1)$$

on a time scale \mathbb{T} (a nonempty closed subset of real numbers), where $x \in \mathbb{R}^n$ and $G : [0, \infty) \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a given nonlinear continuous function in t and x . For a vector $x \in \mathbb{R}^n$, we take $\|x\|$ to be the Euclidean norm of x . We refer the reader to [8] for the continuous case, that is, $\mathbb{T} = \mathbb{R}$.

In [6], the boundedness of solutions of

$$x^\Delta(t) = G(t, x(t)), \quad x(t_0) = x_0, \quad t_0 \geq 0, \quad x_0 \in \mathbb{R} \quad (1.2)$$

is considered by using a type I Lyapunov function. Then, in [5], the authors considered nonnegative definite Lyapunov functions and obtained sufficient conditions for the exponential stability of the zero solution. However, the results in either [5] or [6] do not apply to the equations similar to

$$x^\Delta = a(t)x + \int_0^t B(t, s)f(x(s))\Delta s, \quad (1.3)$$

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which is the Volterra integro-dynamic equation. In particular, we are interested in applying our results to (1.3) with $f(x) = x^n$, where n is positive and rational. The authors are confident that there is nothing in the literature that deals with the qualitative analysis of Volterra integro-dynamic equations on time scales. Thus, this paper is going to play a major role in any future research that is related to Volterra integro-dynamic equations.

Let $\phi : [0, t_0] \rightarrow \mathbb{R}^n$ be continuous, we define $|\phi| = \sup\{\|\phi(t)\| : 0 \leq t \leq t_0\}$.

We say that solutions of (1.1) are *bounded* if any solution $x(t, t_0, \phi)$ of (1.1) satisfies

$$\|x(t, t_0, \phi)\| \leq C(|\phi|, t_0), \quad \forall t \geq t_0, \quad (1.4)$$

where C is a constant and depends on t_0 . Moreover, solutions of (1.1) are *uniformly bounded* if C is independent of t_0 . Throughout this paper, we assume $0 \in \mathbb{T}$ and $[0, \infty) = \{t \in \mathbb{T} : 0 \leq t < \infty\}$.

Next, we generalize a “type I Lyapunov function” which is defined by Peterson and Tisdell [6] to Lyapunov functionals. We say $V : [0, \infty) \times \mathbb{R}^n \rightarrow [0, \infty)$ is a *type I Lyapunov functional* on $[0, \infty) \times \mathbb{R}^n$ when

$$V(t, x) = \sum_{i=1}^n (V_i(x_i) + U_i(t)), \quad (1.5)$$

where each $V_i : \mathbb{R} \rightarrow \mathbb{R}$ and $U_i : [0, \infty) \rightarrow \mathbb{R}$ are continuously differentiable. Next, we extend the definition of the derivative of a type I Lyapunov function to type I Lyapunov functionals. If V is a type I Lyapunov functional and x is a solution of (1.1), then (2.11) gives

$$\begin{aligned} [V(t, x)]^\Delta &= \sum_{i=1}^n (V_i(x_i(t)) + U_i(t))^\Delta \\ &= \int_0^1 \nabla V[x(t) + h\mu(t)G(t, x(\cdot))] \cdot G(t, x(\cdot)) dh + \sum_{i=1}^n U_i^\Delta(t), \end{aligned} \quad (1.6)$$

where $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ is the gradient operator. This motivates us to define $\dot{V} : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\dot{V}(t, x) = [V(t, x)]^\Delta. \quad (1.7)$$

Continuing in the spirit of [6], we have

$$\dot{V}(t, x) = \begin{cases} \sum_{i=1}^n \frac{V_i(x_i + \mu(t)G_i(t, x(\cdot))) - V_i(x_i)}{\mu(t)} + \sum_{i=1}^n U_i^\Delta(t), & \text{when } \mu(t) \neq 0, \\ \nabla V(x) \cdot G(t, x(\cdot)) + \sum_{i=1}^n U_i^\Delta(t), & \text{when } \mu(t) = 0. \end{cases} \quad (1.8)$$

We also use a continuous strictly increasing function $W_i : [0, \infty) \rightarrow [0, \infty)$ with $W_i(0) = 0$, $W_i(s) > 0$, if $s > 0$ for each $i \in \mathbb{Z}^+$.

We make use of the above expression in our examples.

Example 1.1. Assume $\phi(t, s)$ is right-dense continuous (rd-continuous) and let

$$V(t, x) = x^2 + \int_0^t \phi(t, s) W(|x(s)|) \Delta s. \quad (1.9)$$

If x is a solution of (1.1), then we have by using (2.10) and Theorem 2.2 that

$$\begin{aligned} \dot{V}(t, x) &= 2x \cdot G(t, x(\cdot)) + \mu(t) G^2(t, x(\cdot)) \\ &\quad + \int_0^t \phi^\Delta(t, s) W(|x(s)|) \Delta s + \phi(\sigma(t), t) W(|x(t)|), \end{aligned} \quad (1.10)$$

where $\phi^\Delta(t, s)$ denotes the derivative of ϕ with respect to the first variable.

We say that a type I Lyapunov functional $V : [0, \infty) \times \mathbb{R}^n \mapsto [0, \infty)$ is *negative definite* if $V(t, x) > 0$ for $x \neq 0$, $x \in \mathbb{R}^n$, $V(t, x) = 0$ for $x = 0$ and along the solutions of (1.1), we have $\dot{V}(t, x) \leq 0$. If the condition $\dot{V}(t, x) \leq 0$ does not hold for all $(t, x) \in \mathbb{T} \times \mathbb{R}^n$, then the Lyapunov functional is said to be *nonnegative definite*.

In the case of differential equations or difference equations, it is known that if one can display a negative definite Lyapunov function, or functionals, for (1.1), then boundedness of all solutions follows. In [8], the second author displayed nonnegative Lyapunov functionals and proved boundedness of all solutions of (1.1), in the case $\mathbb{T} = \mathbb{R}$.

2. Calculus on time scales

In this section, we introduce a calculus on time scales including preliminary results. An introduction with applications and advances in dynamic equations are given in [2, 3]. Our aim is not only to unify some results when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ but also to extend them for other time scales such as $h\mathbb{Z}$, where $h > 0$, $q^{\mathbb{N}_0}$, where $q > 1$ and so on. We define the *forward jump operator* σ on \mathbb{T} by

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\} \in \mathbb{T} \quad (2.1)$$

for all $t \in \mathbb{T}$. In this definition, we put $\inf(\emptyset) = \sup \mathbb{T}$. The *backward jump operator* ρ on \mathbb{T} is defined by

$$\rho(t) := \sup\{s < t : s \in \mathbb{T}\} \in \mathbb{T} \quad (2.2)$$

for all $t \in \mathbb{T}$. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$, we say t is *left-scattered*. If $\sigma(t) = t$, we say t is *right-dense*, while if $\rho(t) = t$, we say t is *left-dense*. The *graininess function* $\mu : \mathbb{T} \mapsto [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t. \quad (2.3)$$

\mathbb{T} has left-scattered maximum point m , then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Assume $x : \mathbb{T} \mapsto \mathbb{R}^n$. Then we define $x^\Delta(t)$ to be the vector (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$| [x_i(\sigma(t)) - x_i(s)] - x_i^\Delta(t) [\sigma(t) - s] | \leq \epsilon |\sigma(t) - s| \quad (2.4)$$

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for all $s \in U$ and for each $i = 1, 2, \dots, n$. We call $x^\Delta(t)$ the *delta derivative* of $x(t)$ at t , and it turns out that $x^\Delta(t) = x'(t)$ if $\mathbb{T} = \mathbb{R}$ and $x^\Delta(t) = x(t+1) - x(t)$ if $\mathbb{T} = \mathbb{Z}$. If $G^\Delta(t) = g(t)$, then the Cauchy integral is defined by

$$\int_a^t g(s) \Delta s = G(t) - G(a). \quad (2.5)$$

It can be shown that if $f : \mathbb{T} \rightarrow \mathbb{R}^n$ is continuous at $t \in \mathbb{T}$ and t is right-scattered, then

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}, \quad (2.6)$$

while if t is right-dense, then

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}, \quad (2.7)$$

if the limit exists. If $f, g : \mathbb{T} \rightarrow \mathbb{R}^n$ are differentiable at $t \in \mathbb{T}$, then the product and quotient rules are as follows:

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t), \quad (2.8)$$

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)} \quad \text{if } g(t)g^\sigma(t) \neq 0. \quad (2.9)$$

If f is differentiable at t , then

$$f^\sigma(t) = f(t) + \mu(t)f^\Delta(t), \quad \text{where } f^\sigma = f \circ \sigma. \quad (2.10)$$

We say $f : \mathbb{T} \rightarrow \mathbb{R}$ is *rd-continuous* provided f is continuous at each right-dense point $t \in \mathbb{T}$ and whenever $t \in \mathbb{T}$ is left-dense, $\lim_{s \rightarrow t^-} f(s)$ exists as a finite number. We say that $p : \mathbb{T} \rightarrow \mathbb{R}$ is *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. We define the set \mathcal{R} of all regressive and rd-continuous functions. We define the set \mathcal{R}^+ of all positively regressive elements of \mathcal{R} by $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

The following chain rule is due to Poetzsche and the proof can be found in [2, Theorem 1.90].

THEOREM 2.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t)) dh \right\} g^\Delta(t) \quad (2.11)$$

holds.

We use the following result [2, Theorem 1.117] to calculate the derivative of the Lyapunov function in further sections.

THEOREM 2.2. *Let $t_0 \in \mathbb{T}^\kappa$ and assume $k : \mathbb{T} \times \mathbb{T}^\kappa \rightarrow \mathbb{R}$ is continuous at (t, t) , where $t \in \mathbb{T}^\kappa$ with $t > t_0$. Also assume that $k(t, \cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose for each $\epsilon > 0$,*

there exists a neighborhood of t , independent U of $\tau \in [t_0, \sigma(t)]$, such that

$$|k(\sigma(t), \tau) - k(s, \tau) - k^\Delta(t, \tau)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s| \quad \forall s \in U, \quad (2.12)$$

where k^Δ denotes the derivative of k with respect to the first variable. Then

$$\begin{aligned} g(t) &:= \int_{t_0}^t k(t, \tau) \Delta \tau \quad \text{implies} \quad g^\Delta(t) = \int_{t_0}^t k^\Delta(t, \tau) \Delta \tau + k(\sigma(t), t); \\ h(t) &:= \int_t^b k(t, \tau) \Delta \tau \quad \text{implies} \quad h^\Delta(t) = \int_t^b k^\Delta(t, \tau) \Delta \tau - k(\sigma(t), t). \end{aligned} \quad (2.13)$$

We apply the following Cauchy-Schwarz inequality in [2, Theorem 6.15] to prove Theorem 4.1.

THEOREM 2.3. Let $a, b \in \mathbb{T}$. For rd-continuous $f, g : [a, b] \mapsto \mathbb{R}$,

$$\int_a^b |f(t)g(t)| \Delta t \leq \sqrt{\left\{ \int_a^b |f(t)|^2 \Delta t \right\} \left\{ \int_a^b |g(t)|^2 \Delta t \right\}}. \quad (2.14)$$

If $p : \mathbb{T} \mapsto \mathbb{R}$ is rd-continuous and regressive, then the exponential function $e_p(t, t_0)$ is for each fixed $t_0 \in \mathbb{T}$ the unique solution of the initial value problem

$$x^\Delta = p(t)x, \quad x(t_0) = 1 \quad (2.15)$$

on \mathbb{T} . Under the addition on \mathcal{R} defined by

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t), \quad t \in \mathbb{T}, \quad (2.16)$$

is an Abelian group (see [2]), where the additive inverse of p , denoted by $\ominus p$, is defined by

$$(\ominus p)(t) = \frac{-p(t)}{1 + \mu(t)p(t)}, \quad t \in \mathbb{T}. \quad (2.17)$$

We use the following properties of the exponential function $e_p(t, s)$ which are proved in Bohner and Peterson [2].

THEOREM 2.4. If $p, q \in \mathcal{R}$, then for $t, s, r, t_0 \in \mathbb{T}$,

- (i) $e_p(t, t) \equiv 1$ and $e_0(t, s) \equiv 1$;
- (ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- (iii) $1/e_p(t, s) = e_{\ominus p}(t, s) = e_p(s, t)$;
- (iv) $e_p(t, s)/e_q(t, s) = e_{p \ominus q}(t, s)$;
- (v) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$.

Moreover, the following can be found in [1].

THEOREM 2.5. Let $t_0 \in \mathbb{T}$.

- (i) If $p \in \mathcal{R}^+$, then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.
- (ii) If $p \geq 0$, then $e_p(t, t_0) \geq 1$ for all $t \geq t_0$. Therefore, $e_{\ominus p}(t, t_0) \leq 1$ for all $t \geq t_0$.

3. Boundedness of solutions

In this section, we use a nonnegative definite type I Lyapunov functional and establish sufficient conditions to obtain boundedness of solutions of (1.1).

THEOREM 3.1. *Let $D \subset \mathbb{R}^n$. Suppose that there exists a type I Lyapunov functional $V : [0, \infty) \times D \mapsto [0, \infty)$ such that for all $(t, x) \in [0, \infty) \times D$,*

$$\lambda_1 W_1(|x|) \leq V(t, x) \leq \lambda_2 W_2(|x|) + \lambda_2 \int_0^t \phi_1(t, s) W_3(|x(s)|) \Delta s, \quad (3.1)$$

$$\dot{V}(t, x) \leq \frac{-\lambda_3 W_4(|x|) - \lambda_3 \int_0^t \phi_2(t, s) W_5(|x(s)|) \Delta s + L}{1 + \mu(t)(\lambda_3/\lambda_2)}, \quad (3.2)$$

where $\lambda_1, \lambda_2, \lambda_3$, and L are positive constants and $\phi_i(t, s) \geq 0$ is rd-continuous function for $0 \leq s \leq t < \infty, i = 1, 2$ such that

$$W_2(|x|) - W_4(|x|) + \int_0^t (\phi_1(t, s) W_3(|x(s)|) - \phi_2(t, s) W_5(|x(s)|)) \Delta s \leq \gamma, \quad (3.3)$$

where $\gamma \geq 0$. If $\int_0^t \phi_1(t, s) \Delta s \leq B$ for some $B \geq 0$, then all solutions of (1.1) staying in D are uniformly bounded.

Proof. Let x be a solution of (1.1) with $x(t) = \phi(t)$ for $0 \leq t \leq t_0$. Set $M = \lambda_3/\lambda_2$. By (2.8) and (2.10) and inequalities (3.1), (3.2), and (3.3) we obtain

$$\begin{aligned} [V(t, x(t)) e_M(t, t_0)]^\Delta &= \dot{V}(t, x(t)) e_M^\sigma(t, t_0) + M V(t, x(t)) e_M(t, t_0) \\ &= [\dot{V}(t, x(t)) (1 + \mu(t)M) + M V(t, x(t))] e_M(t, t_0) \\ &\leq \left[-\lambda_3 W_4(|x|) - \lambda_3 \int_0^t \phi_2(t, s) W_5(|x(s)|) \Delta s + L \right] e_M(t, t_0) \\ &\quad + \left[\lambda_3 W_2(|x|) + \lambda_3 \int_0^t \phi_1(t, s) W_3(|x(s)|) \Delta s \right] e_M(t, t_0) \\ &\leq [\lambda_3 \gamma + L] e_M(t, t_0) =: K e_M(t, t_0), \end{aligned} \quad (3.4)$$

where we used Theorem 2.5(i). Integrating both sides from t_0 to t , we have

$$\begin{aligned} V(t, x(t)) e_M(t, t_0) &\leq V(t_0, \phi) + \frac{K}{M} \int_{t_0}^t e_M^\Delta(\tau, t_0) \Delta \tau \\ &= V(t_0, \phi) + \frac{K}{M} (e_M(t, t_0) - 1) \leq V(t_0, \phi) + \frac{K}{M} e_M(t, t_0). \end{aligned} \quad (3.5)$$

It follows from Theorem 2.4(iii) that for all $t \geq t_0$,

$$V(t, x(t)) \leq V(t_0, \phi) e_{\ominus M}(t, t_0) + \frac{K}{M}. \quad (3.6)$$

From inequality (3.1), we have

$$\begin{aligned} W_1(|x|) &\leq \frac{1}{\lambda_1} \left(V(t_0, \phi) e_{\ominus M}(t, t_0) + \frac{K}{M} \right) \\ &\leq \frac{1}{\lambda_1} \left[\lambda_2 W_2(|\phi|) + \lambda_2 W_3(|\phi|) \int_0^{t_0} \phi_1(t_0, s) \Delta s + \frac{K}{M} \right], \end{aligned} \quad (3.7)$$

where we used the fact Theorem 2.5(ii). Therefore, we obtain

$$|x| \leq W_1^{-1} \left\{ \frac{1}{\lambda_1} \left[\lambda_2 W_2(|\phi|) + \lambda_2 W_3(|\phi|) \int_0^{t_0} \phi_1(t_0, s) \Delta s + \frac{K}{M} \right] \right\} \quad (3.8)$$

for all $t \geq t_0$. This concludes the proof. \square

In the next theorem, we give sufficient conditions to show that solutions of (1.1) are bounded.

THEOREM 3.2. *Let $D \subset \mathbb{R}^n$. Suppose that there exists a type I Lyapunov functional $V : [0, \infty) \times D \mapsto [0, \infty)$ such that for all $(t, x) \in [0, \infty) \times D$,*

$$\begin{aligned} \lambda_1(t) W_1(|x|) &\leq V(t, x) \leq \lambda_2(t) W_2(|x|) + \lambda_2(t) \int_0^t \phi_1(t, s) W_3(|x(s)|) \Delta s, \\ \dot{V}(t, x) &\leq \frac{-\lambda_3(t) W_4(|x|) - \lambda_3(t) \int_0^t \phi_2(t, s) W_5(|x(s)|) \Delta s + L}{1 + \mu(t)(\lambda_3(t)/\lambda_2(t))}, \end{aligned} \quad (3.9)$$

where $\lambda_1, \lambda_2, \lambda_3$ are positive continuous functions, L is a positive constant, λ_1 is nondecreasing, and $\phi_i(t, s) \geq 0$ is rd-continuous for $0 \leq s \leq t < \infty$, $i = 1, 2$, such that

$$W_2(|x|) - W_4(|x|) + \int_0^t (\phi_1(t, s) W_3(|x|) - \phi_2(t, s) W_5(|x(s)|)) \Delta s \leq \gamma, \quad (3.10)$$

where $\gamma \geq 0$. If $\int_0^t \phi_1(t, s) \Delta s \leq B$ and $\lambda_3(t) \leq N$ for $t \in [0, \infty)$ and some positive constants B and N , then all solutions of (1.1) staying in D are bounded.

Proof. Let $M := \inf_{t \geq 0} (\lambda_3(t)/\lambda_2(t)) > 0$ and let x be any solution of (1.1) with $x(t_0) = \phi(t_0)$. Then we obtain

$$\begin{aligned} [V(t, x(t)) e_M(t, t_0)]^\Delta &= \dot{V}(t, x(t)) e_M^\sigma(t, t_0) + M V(t, x(t)) e_M(t, t_0) \\ &= [\dot{V}(t, x(t)) (1 + \mu(t)M) + M V(t, x(t))] e_M(t, t_0) \\ &\leq \left[-\lambda_3(t) W_4(|x|) - \lambda_3(t) \int_0^t \phi_2(t, s) W_5(|x(s)|) \Delta s + L \right] e_M(t, t_0) \\ &\quad + \left[M \lambda_2(t) W_2(|x|) + M \lambda_2(t) \int_0^t \phi_1(t, s) W_3(|x(s)|) \Delta s \right] e_M(t, t_0) \\ &\leq [\lambda_3(t) \gamma + L] e_M(t, t_0) \leq (N \gamma + L) e_M(t, t_0) =: K e_M(t, t_0), \end{aligned} \quad (3.11)$$

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because of $M \leq \lambda_3(t)/\lambda_2(t)$, $\lambda_3(t) \leq N$, for $t \in [0, \infty)$ and Theorem 2.5(i). Integrating both sides from t_0 to t , we obtain

$$V(t, x(t))e_M(t, t_0) \leq V(t_0, \phi) + \frac{K}{M}e_M(t, t_0). \quad (3.12)$$

This implies from Theorem 2.4(iii) that for all $t \geq t_0$,

$$V(t, x(t)) \leq V(t_0, \phi)e_{\ominus M}(t, t_0) + \frac{K}{M}. \quad (3.13)$$

From inequality (3.1), we have

$$W_1(|x|) \leq \frac{1}{\lambda_1(t_0)} \left(\lambda_2(t_0) W_2(|\phi|) + \lambda_2(t_0) W_3(|\phi|) \int_0^{t_0} \phi_1(t_0, s) \Delta s + \frac{K}{M} \right) \quad (3.14)$$

for all $t \geq t_0$, where we used the fact Theorem 2.5(ii) and λ_1 is nondecreasing. \square

The following theorem is the special case of [8, Theorem 2.6].

THEOREM 3.3. *Suppose there exists a continuously differentiable type I Lyapunov functional $V : [0, \infty) \times \mathbb{R}^n \mapsto [0, \infty)$ that satisfies*

$$\lambda_1 \|x\|^p \leq V(t, x), \quad V(t, x) \neq 0 \quad \text{if } x \neq 0, \quad (3.15)$$

$$[V(t, x)]^\Delta \leq -\lambda_2(t) V(t, x) V^\sigma(t, x) \quad (3.16)$$

for some positive constants λ_1 and p are positive constants, and λ_2 is a positive continuous function such that

$$c_1 = \inf_{0 \leq t_0 \leq t} \lambda_2(t). \quad (3.17)$$

Then all solutions of (1.1) satisfy

$$\|x\| \leq \frac{1}{\lambda_1^{1/p}} \left[\frac{1}{1/V(t_0, \phi) + c_1(t - t_0)} \right]^{1/p}. \quad (3.18)$$

Proof. For any $t_0 \geq 0$, let x be the solution of (1.1) with $x(t_0) = \phi(t_0)$. By inequalities (3.16) and (3.17), we have

$$[V(t, x)]^\Delta \leq -c_1 V(t, x) V^\sigma(t, x). \quad (3.19)$$

Let $u(t) = V(t, x(t))$ so that we have

$$\frac{u^\Delta(t)}{u(t)u^\sigma(t)} \leq -c_1. \quad (3.20)$$

Since $(1/u(t))^\Delta = -u^\Delta/u(t)u(\sigma(t))$, we obtain

$$\left(\frac{1}{u(t)}\right)^\Delta \geq c_1. \quad (3.21)$$

Integrating the above inequality from t_0 to t , we have

$$u(t) \leq \frac{1}{1/u(t_0) + c_1(t - t_0)} \quad (3.22)$$

or

$$V(t, x(t)) \leq \frac{1}{1/V(t_0, \phi) + c_1(t - t_0)}. \quad (3.23)$$

Using (3.15), we obtain

$$\|x\| \leq \frac{1}{\lambda_1^{1/p}} \left[\frac{1}{1/V(t_0, \phi) + c_1(t - t_0)} \right]^{1/p}. \quad (3.24)$$

□

The next theorem is an extension of [7, Theorem 2.6].

THEOREM 3.4. *Assume $D \subset \mathbb{R}^n$ and there exists a type I Lyapunov functional $V : [0, \infty) \times D \rightarrow [0, \infty)$ such that for all $(t, x) \in [0, \infty) \times D$,*

$$\lambda_1 \|x\|^p \leq V(t, x), \quad (3.25)$$

$$\dot{V}(t, x) \leq \frac{-\lambda_2 V(x) + L}{1 + \varepsilon \mu(t)}, \quad (3.26)$$

where $\lambda_1, \lambda_2, p > 0$, $L \geq 0$ are constants and $0 < \varepsilon < \lambda_2$. Then all solutions of (1.1) staying in D are bounded.

Proof. For any $t_0 \geq 0$, let x be the solution of (1.1) with $x(t_0) = \phi$. Since $\varepsilon \in \mathcal{R}^+$, $e_\varepsilon(t, 0)$ is well defined and positive. By (3.26), we obtain

$$\begin{aligned} [V(t, x(t))e_\varepsilon(t, 0)]^\Delta &= \dot{V}(t, x(t))e_\varepsilon^\sigma(t, 0) + \varepsilon V(t, x(t))e_\varepsilon(t, 0), \\ &\leq (-\lambda_2 V(t, x(t)) + L)e_\varepsilon(t, 0) + \varepsilon V(t, x(t))e_\varepsilon(t, 0), \\ &= e_\varepsilon(t, 0)[\varepsilon V(t, x(t)) - \lambda_2 V(t, x(t)) + L] \leq L e_\varepsilon(t, 0). \end{aligned} \quad (3.27)$$

Integrating both sides from t_0 to t , we obtain

$$V(t, x(t))e_\varepsilon(t, 0) \leq V(t_0, \phi) + \frac{L}{\varepsilon} e_\varepsilon(t, 0). \quad (3.28)$$

Dividing both sides of the above inequality by $e_\varepsilon(t, 0)$ and then using (3.25) and Theorem 2.5, we obtain

$$\|x\| \leq \left\{ \frac{1}{\lambda_1} \right\}^{1/p} \left[V(t_0, \phi) + \frac{L}{\varepsilon} \right]^{1/p} \quad \text{for all } t \geq t_0. \quad (3.29)$$

This completes the proof. \square

Remark 3.5. In Theorem 3.4, if $V(t_0, \phi)$ is uniformly bounded, then one concludes that all solutions of (1.1) that stay in D are uniformly bounded.

4. Applications to Volterra integro-dynamic equations

In this section, we apply our theorems from the previous section and obtain sufficient conditions that insure the boundedness and uniform boundedness of solutions of Volterra integro-dynamic equations. We begin with the following theorem.

THEOREM 4.1. *Suppose $B(t, s)$ is rd-continuous and consider the scalar nonlinear Volterra integro-dynamic equation*

$$x^\Delta = a(t)x(t) + \int_0^t B(t, s)x^{2/3}(s)\Delta s, \quad t \geq 0, \quad x(t) = \phi(t) \text{ for } 0 \leq t \leq t_0, \quad (4.1)$$

where ϕ is a given bounded continuous initial function on $[0, \infty)$, and a is a continuous function on $[0, \infty)$. Suppose there are positive constants ν, β_1, β_2 , with $\nu \in (0, 1)$, and $\lambda_3 = \min\{\beta_1, \beta_2\}$ such that

$$\begin{aligned} & \left[2a(t) + \mu(t)a^2(t) + \mu(t)|a(t)| \int_0^t |B(t, s)|\Delta s + \int_0^t |B(t, s)|\Delta s \right. \\ & \left. + \nu \int_{\sigma(t)}^\infty |B(u, t)|\Delta u \right] (1 + \mu(t)\lambda_3) \leq -\beta_1, \end{aligned} \quad (4.2)$$

$$\left\{ \frac{2}{3} \left[1 + \mu(t)|a(t)| + \mu(t) \int_0^t |B(t, s)|\Delta s \right] - \nu \right\} (1 + \mu(t)\lambda_3) \leq -\beta_2, \quad (4.3)$$

$$\begin{aligned} & \int_0^t \int_t^\infty |B(u, s)|\Delta u \Delta s < \infty, \quad \int_0^t |B(t, s)|\Delta s < \infty, \\ & |B(t, s)| \geq \nu \int_t^\infty |B(u, s)|\Delta u, \end{aligned} \quad (4.4)$$

then all solutions of (4.1) are uniformly bounded.

Proof. Let

$$V(t, x) = x^2(t) + \nu \int_0^t \int_t^\infty |B(u, s)|\Delta u x^2(s)\Delta s. \quad (4.5)$$

Using Theorem 2.2, we have along the solutions of (4.1) that

$$\begin{aligned}
\dot{V}(t, x) &= 2x(t) \left(a(t)x(t) + \int_0^t B(t, s)x^{2/3}(s)\Delta s \right) \\
&\quad + \mu(t) \left(a(t)x(t) + \int_0^t B(t, s)x^{2/3}(s)\Delta s \right)^2 \\
&\quad - \gamma \int_0^t |B(t, s)| x^2(s) \Delta s + \gamma \int_{\sigma(t)}^\infty |B(u, t)| x^2(t) \Delta u \\
&\leq 2a(t)x^2(t) + 2 \int_0^t |B(t, s)| |x(t)| x^{2/3}(s) \Delta s \\
&\quad + \mu(t)a^2(t)x^2(t) + 2\mu(t) |a(t)| \int_0^t |B(t, s)| |x(t)| x^{2/3}(s) \Delta s \\
&\quad + \mu(t) \left(\int_0^t B(t, s)x^{2/3}(s)\Delta s \right)^2 \\
&\quad + \gamma \int_{\sigma(t)}^\infty |B(u, t)| x^2(t) \Delta u - \gamma \int_0^t |B(t, s)| x^2(s) \Delta s.
\end{aligned} \tag{4.6}$$

Using the fact that $ab \leq a^2/2 + b^2/2$ for any real numbers a and b , we have

$$2 \int_0^t |B(t, s)| |x(t)| x^{2/3}(s) \Delta s \leq \int_0^t |B(t, s)| (x^2(t) + x^{4/3}(s)) \Delta s. \tag{4.7}$$

Also, using Theorem 2.3, one obtains

$$\begin{aligned}
\left(\int_0^t |B(t, s)| x^{2/3}(s) \Delta s \right)^2 &= \left(\int_0^t |B(t, s)|^{1/2} |B(t, s)|^{1/2} x^{2/3}(s) \Delta s \right)^2 \\
&\leq \int_0^t |B(t, s)| \Delta s \int_0^t |B(t, s)| x^{4/3}(s) \Delta s.
\end{aligned} \tag{4.8}$$

A substitution of the above two inequalities into (4.6) yields

$$\begin{aligned}
\dot{V}(t, x) &\leq \left[2a(t) + \mu(t)a^2(t) + \mu(t) |a(t)| \int_0^t |B(t, s)| \Delta s \right. \\
&\quad \left. + \int_0^t |B(t, s)| \Delta s + \gamma \int_{\sigma(t)}^\infty |B(u, t)| \Delta u \right] x^2(t) \\
&\quad + \left[1 + \mu(t) |a(t)| + \mu(t) \int_0^t |B(t, s)| \Delta s \right] \int_0^t |B(t, s)| x^{4/3}(s) \Delta s \\
&\quad - \gamma \int_0^t |B(t, s)| x^2(s) \Delta s.
\end{aligned} \tag{4.9}$$

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To further simplify (4.9), we make use of Young's inequality, which says that for any two nonnegative real numbers w and z , we have

$$wz \leq \frac{w^e}{e} + \frac{z^f}{f}, \quad \text{with } \frac{1}{e} + \frac{1}{f} = 1. \quad (4.10)$$

Thus, for $e = 3/2$ and $f = 3$, we get

$$\begin{aligned} \int_0^t |B(t,s)| x^{4/3}(s) \Delta s &= \int_0^t |B(t,s)|^{1/3} |B(t,s)|^{2/3} x^{4/3}(s) \Delta s \\ &\leq \int_0^t \left(\frac{|B(t,s)|}{3} + \frac{2}{3} |B(t,s)| x^2(s) \right) \Delta s. \end{aligned} \quad (4.11)$$

By substituting the above inequality into (4.9), we arrive at

$$\begin{aligned} \dot{V}(t,x) &\leq \left[2a(t) + \mu(t)a^2(t) + \mu(t)|a(t)| \int_0^t |B(t,s)| \Delta s \right. \\ &\quad \left. + \int_0^t |B(t,s)| \Delta s + \nu \int_{\sigma(t)}^\infty |B(u,t)| \Delta u \right] x^2(t) \\ &\quad + \left[-\nu + \frac{2}{3} \left(1 + \mu(t)|a(t)| + \mu(t) \int_0^t |B(t,s)| \Delta s \right) \right] \int_0^t |B(t,s)| x^2(s) \Delta s \\ &\quad + \frac{1}{3} \left(1 + \mu(t)|a(t)| + \mu(t) \int_0^t |B(t,s)| \Delta s \right) \int_0^t |B(t,s)| \Delta s. \end{aligned} \quad (4.12)$$

Multiplying and dividing the above inequality by $1 + \mu(t)\lambda_3$, and then applying conditions (4.2) and (4.3), $\dot{V}(t,x)$ reduces to

$$\dot{V}(t,x) \leq \frac{-\beta_1 x^2(t) - \beta_2 \int_0^t |B(t,s)| x^2(s) \Delta s + L}{1 + \mu(t)\lambda_3}, \quad (4.13)$$

where $L = 1/3(1 + \mu(t)|a(t)| + \mu(t) \int_0^t |B(t,s)| \Delta s) \int_0^t |B(t,s)| \Delta s (1 + \mu(t)\lambda_3)$. By taking $W_1 = W_2 = W_4 = x^2(t)$, $W_3 = W_5 = x^2(s)$, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \min\{\beta_1, \beta_2\}$, $\phi_1(t,s) = \nu \int_t^\infty |B(u,s)| \Delta u$, and $\phi_2(t,s) = |B(t,s)|$, we see that conditions (3.1) and (3.2) of Theorem 3.1 are satisfied. Next we make sure that condition (3.3) holds. Use (4.4) to obtain

$$\begin{aligned} W_2(|x|) - W_4(|x|) &+ \int_0^t (\phi_1(t,s) W_3(|x(s)|) - \phi_2(t,s) W_5(|x(s)|)) \Delta s \\ &= x^2(t) - x^2(t) + \int_0^t \left(\nu \int_t^\infty |B(u,s)| \Delta u - |B(t,s)| \right) x^2(s) \Delta s \leq 0. \end{aligned} \quad (4.14)$$

Thus condition (3.3) is satisfied with $\gamma = 0$. An application of Theorem 3.1 yields the results. \square

Remark 4.2. In the case $\mathbb{T} = \mathbb{R}$, the second author in [8] took $\nu = 1$ in the displayed Lyapunov functional. On the other hand, in our theorem, we had to incorporate such ν

in the Lyapunov functional, otherwise, condition (4.5) may only hold if $B(t, s) = 0$ for all $t \in \mathbb{T}$ with $0 \leq s \leq t < \infty$ for a particular time scale. For example, if we take $\mathbb{T} = \mathbb{Z}$, then condition (4.5) reduces to $|B(t, s)| \geq \nu \sum_{u=t}^{\infty} |B(u, s)|$, which can only hold if $B(t, s) = 0$ for $\nu = 1$.

Remark 4.3. If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ for all t and hence Theorem 4.1 reduces to [8, Example 2.3].

Remark 4.4. We assert that Theorem 4.1 can be easily generalized to handle scalar nonlinear Volterra integro-dynamic equations of the form

$$x^\Delta = a(t)x(t) + \int_0^t B(t, s)f(s, x(s))\Delta s, \quad (4.15)$$

where $|f(t, x(t))| \leq x^{2/3}(t) + M$ for some positive constant M .

For the next theorem, we consider the scalar Volterra integro-dynamic equation

$$x^\Delta(t) = a(t)x(t) + \int_0^t B(t, s)f(s, x(s))\Delta s + g(t, x(t)), \quad (4.16)$$

where $t \geq 0$, $x(t) = \phi(t)$ for $0 \leq t \leq t_0$, ϕ is a given bounded continuous initial function, $a(t)$ is continuous for $t \geq 0$, and $B(t, s)$ is right-dense continuous for $0 \leq s \leq t < \infty$. We assume $f(t, x)$ and $g(t, x)$ are continuous in x and t and satisfy

$$|g(t, x)| \leq \gamma_1(t) + \gamma_2(t)|x(t)|, \quad |f(t, x)| \leq \gamma(t)|x(t)|, \quad (4.17)$$

where γ and γ_2 are positive and bounded, and γ_1 is nonnegative and bounded.

For the next theorem, we need the identity

$$|x(t)|^\Delta = \frac{x(t) + x^\sigma(t)}{|x(t)| + |x^\sigma(t)|} x^\Delta(t). \quad (4.18)$$

Its proof can be found in [4].

THEOREM 4.5. *Suppose there exist constants $k > 1$ and ε, α with $0 < \varepsilon < \alpha$ such that*

$$\left[a(t) + \gamma_2(t) + k \int_{\sigma(t)}^{\infty} |B(u, t)| \Delta u \gamma(t) \right] (1 + \varepsilon \mu(t)) \leq -\alpha < 0, \quad (4.19)$$

where $k = 1 + \zeta$ for some $\zeta > 0$. Suppose

$$(1 + \mu(t)\varepsilon) |B(t, s)| \geq \lambda \int_t^{\infty} |B(u, s)| \Delta u, \quad (4.20)$$

where $\lambda \geq k\alpha/\zeta$, $0 \leq s < t \leq u < \infty$,

$$\int_0^{t_0} \int_{t_0}^{\infty} |B(u, s)| \Delta u \gamma(s) \Delta s \leq \rho < \infty \quad \forall t_0 \geq 0, \quad (4.21)$$

and for some positive constant L ,

$$\gamma_1(t)(1 + \varepsilon\mu(t)) \leq L. \quad (4.22)$$

Then all solutions of (4.16) are uniformly bounded.

Proof. Define

$$V(t, x(\cdot)) = |x(t)| + k \int_0^t \int_t^\infty |B(u, s)| \Delta u |f(s, x(s))| \Delta s. \quad (4.23)$$

Along the solutions of (4.16), we have

$$\begin{aligned} \dot{V}(t, x) &= \frac{x(t) + x^\sigma(t)}{|x(t)| + |x^\sigma(t)|} x^\Delta(t) + k \int_{\sigma(t)}^\infty |B(u, t)| \Delta u |f(t, x(t))| \\ &\quad - k \int_0^t |B(t, s)| |f(s, x(s))| \Delta s \leq a(t) |x(t)| + \int_0^t |B(t, s)| |f(s, x(s))| \Delta s \\ &\quad + |g(t, x(t))| + k \int_{\sigma(t)}^\infty |B(u, t)| \Delta u |f(t, x(t))| - k \int_0^t |B(t, s)| |f(s, x(s))| \Delta s \\ &\leq \left[a(t) + \gamma_2(t) + k \int_{\sigma(t)}^\infty |B(u, t)| \Delta u \gamma(t) \right] |x(t)| \\ &\quad + (1 - k) \int_0^t |B(t, s)| |f(s, x(s))| \Delta s + \gamma_1(t) \\ &= \left[a(t) + \gamma_2(t) + k \int_{\sigma(t)}^\infty |B(u, t)| \Delta u \gamma(t) \right] |x(t)| \frac{1 + \mu(t)\varepsilon}{1 + \mu(t)\varepsilon} \\ &\quad - \zeta(1 + \mu(t)\varepsilon) \int_0^t |B(t, s)| |f(s, x(s))| \Delta s \frac{1}{1 + \mu(t)\varepsilon} + (1 + \mu(t)\varepsilon) \gamma_1(t) \frac{1}{1 + \mu(t)\varepsilon} \\ &\leq -\alpha |x(t)| \frac{1}{1 + \mu(t)\varepsilon} - \zeta \lambda \int_0^t \int_t^\infty |B(u, s)| \Delta u |f(s, x(s))| \Delta s \frac{1}{1 + \mu(t)\varepsilon} + \frac{L}{1 + \mu(t)\varepsilon} \\ &= -\alpha \left[|x(t)| + k \int_0^t \int_t^\infty |B(u, s)| \Delta u |f(s, x(s))| \Delta s \right] \frac{1}{1 + \mu(t)\varepsilon} + \frac{L}{1 + \mu(t)\varepsilon} \\ &= \frac{-\alpha V(t, x) + L}{1 + \mu(t)\varepsilon}. \end{aligned} \quad (4.24)$$

The results follow from Theorem 3.4 and Remark 3.5. \square

In the next theorem, we establish sufficient conditions that guarantee the boundedness of all solutions of the vector Volterra integro-dynamic equation

$$x^\Delta = Ax(t) + \int_0^t C(t, s)x(s)\Delta s + g(t), \quad (4.25)$$

where $t \geq 0$, $x(t) = \phi(t)$ for $0 \leq t \leq t_0$, ϕ is a given bounded continuous initial $k \times 1$ vector function. Also, A and $C(t, s)$ are $k \times k$ matrix with $C(t, s)$ being continuous on $\mathbb{T} \times \mathbb{T}$, g, x are $k \times 1$ vector functions that are continuous for $t \in \mathbb{T}$. If D is a matrix, then $|D|$ means the sum of the absolute values of the elements.

THEOREM 4.6. *Suppose $C^T(t, s) = C(t, s)$. Let I be the $k \times k$ identity matrix. Assume there exist positive constants $L, \nu, \xi, \beta_1, \beta_2, \lambda_3$, and $k \times k$ positive definite constant symmetric matrix B such that*

$$[A^T B + BA + \mu(t)A^T B A] \leq -\xi I, \quad (4.26)$$

$$\begin{aligned} & \left[-\xi + |A^T B g| + |B g| + \int_0^t |B| |C(t, s)| \Delta s + \mu(t) \int_0^t |A^T B| |C(t, s)| \Delta s \right. \\ & \left. + \nu \int_{\sigma(t)}^\infty |C(u, t)| \Delta u \right] (1 + \mu(t)\lambda_3) \leq -\beta_1, \end{aligned} \quad (4.27)$$

$$\left[|B| - \nu + \mu(t) \left((g^T B)^2 + 1 + |A^T B| + \int_0^t |C(t, s)| \Delta s \right) \right] (1 + \mu(t)\lambda_3) \leq -\beta_2, \quad (4.28)$$

$$(\mu(t) |g^T g| + |B g|) (1 + \mu(t)\lambda_3) + \mu(t) |A^T B g| = L, \quad (4.29)$$

$$|C(t, s)| \geq \nu \int_{\sigma(t)}^\infty |C(u, s)| \Delta u, \quad (4.30)$$

$$\int_0^t \int_t^\infty |C(u, s)| \Delta u \Delta s < \infty, \quad \int_0^t |C(t, s)| \Delta s < \infty. \quad (4.31)$$

Then there exists an $r_1 \in (0, 1]$ such that

$$r_1 x^T x \leq x^T B x \leq x^T x. \quad (4.32)$$

Proof. Let the matrix B be defined by (4.26) and define

$$V(t, x) = x^T B x + \nu \int_0^t \int_t^\infty |C(u, s)| \Delta u x^2(s) \Delta s. \quad (4.33)$$

Here $x^T x = x^2 = (x_1^2 + x_2^2 + \cdots + x_k^2)$. Using the product rule given in (2.8), we have along the solutions of (4.25) that

$$\begin{aligned} \dot{V}(t, x) &= (x^\Delta)^T B x + (x^\sigma)^T B x^\Delta - \nu \int_0^t |C(t, s)| x^2(s) \Delta s + \nu \int_{\sigma(t)}^\infty |C(u, t)| \Delta u x^2 \\ &= (x^\Delta)^T B x + (x + \mu(t)x^\Delta)^T B x^\Delta - \nu \int_0^t |C(t, s)| x^2(s) \Delta s + \nu \int_{\sigma(t)}^\infty |C(u, t)| \Delta u x^2 \\ &= (x^\Delta)^T B x + x^T B x^\Delta + \mu(t) (x^\Delta)^T B x^\Delta - \nu \int_0^t |C(t, s)| x^2(s) \Delta s + \nu \int_{\sigma(t)}^\infty |C(u, t)| \Delta u x^2. \end{aligned} \quad (4.34)$$

Substituting the right-hand side of (4.25) for x^Δ into (4.34) and making use of (4.26), we obtain

$$\begin{aligned}\dot{V}(t, x) &= \left[Ax + \int_0^t C(t, s)x(s)\Delta s + g \right]^T Bx + x^T B \left[Ax + \int_0^t C(t, s)x(s)\Delta s + g \right] \\ &\quad + \mu(t) \left[Ax + \int_0^t C(t, s)x(s)\Delta s + g \right]^T B \left[Ax + \int_0^t C(t, s)x(s)\Delta s + g \right] \\ &\quad - \nu \int_0^t |C(t, s)| x^2(s) \Delta s + \nu \int_{\sigma(t)}^\infty |C(u, t)| \Delta u x^2.\end{aligned}\quad (4.35)$$

By noting that the right side of (4.35) is scalar and by recalling that B is a symmetric matrix, expression (4.35) simplifies to

$$\begin{aligned}\dot{V}(t, x) &= x^T (A^T B + BA + \mu(t)A^T BA)x + 2x^T Bg + 2 \int_0^t x^T BC(t, s)x(s)\Delta s \\ &\quad + \mu(t) \left[2x^T A^T Bg + 2g^T B \int_0^t C(t, s)x(s)\Delta s + 2x^T A^T B \int_0^t C(t, s)x(s)\Delta s \right. \\ &\quad \left. + \int_0^t x^T(s)C(t, s)\Delta s B \int_0^t C(t, s)x(s)\Delta s + g^T Bg \right] \\ &\quad - \nu \int_0^t |C(t, s)| x^2(s) \Delta s + \nu \int_{\sigma(t)}^\infty |C(u, t)| \Delta u x^2 \\ &\leq -\xi x^2 + 2|x^T||Bg| + 2 \int_0^t |x^T||B||C(t, s)||x(s)|\Delta s \\ &\quad + \mu(t) \left[\int_0^t |C(t, s)| 2|g^T B||x(s)|\Delta s + 2 \int_0^t |x^T||A^T B||C(t, s)||x(s)|\Delta s \right. \\ &\quad \left. + \int_0^t x^T(s)C(t, s)B\Delta s \int_0^t C(t, s)x(s)\Delta s + |g^T B| + 2|x^T||A^T Bg| \right] \\ &\quad - \nu \int_0^t |C(t, s)| x^2(s) \Delta s + \nu \int_{\sigma(t)}^\infty |C(u, t)| \Delta u x^2.\end{aligned}\quad (4.36)$$

Next, we perform some calculations to simplify inequality (4.36),

$$\begin{aligned}2|x^T||Bg| &= 2|x^T||Bg|^{1/2}|Bg|^{1/2} \leq x^2|Bg| + |Bg|, \\ 2|x^T||A^T Bg| &= |x^T||A^T Bg|^{1/2}|A^T Bg|^{1/2} \leq x^2|A^T Bg| + |A^T Bg|, \\ 2 \int_0^t |x^T||B||C(t, s)||x(s)|\Delta s &\leq \int_0^t |B||C(t, s)|(x^2 + x^2(s))\Delta s, \\ \int_0^t |C(t, s)| 2|g^T B||x(s)|\Delta s &\leq \int_0^t |C(t, s)|(|g^T B|^2 + x^2(s))\Delta s, \\ 2 \int_0^t |x^T||A^T B||C(t, s)||x(s)|\Delta s &\leq \int_0^t |A^T B||C(t, s)|(x^2 + x^2(s))\Delta s.\end{aligned}\quad (4.37)$$

Finally,

$$\begin{aligned}
& \int_0^t x^T(s)C(t,s)\Delta s B \int_0^t C(t,s)x(s)\Delta s \\
& \leq |B| \left| \int_0^t x^T(s)C(t,s)\Delta s \right| \left| \int_0^t C(t,s)x(s)\Delta s \right| \\
& \leq \frac{|B| \left(\int_0^t x^T(s)C(t,s)\Delta s \right)^2}{2} + \frac{|B| \left(\int_0^t C(t,s)x(s)\Delta s \right)^2}{2} \\
& = |B| \left(\int_0^t C(t,s)x(s)\Delta s \right)^2 \\
& = |B| \left(\int_0^t |C(t,s)|^{1/2} |C(t,s)|^{1/2} |x(s)| \Delta s \right)^2 \\
& \leq |B| \int_0^t |C(t,s)| \Delta s \int_0^t |C(t,s)| x^2(s) \Delta s.
\end{aligned} \tag{4.38}$$

A substitution of the above inequalities into (4.36) yields

$$\begin{aligned}
\dot{V}(t,x) & \leq \left[-\xi + \mu(t)|A^T Bg| + |Bg| + \int_0^t |B| |C(t,s)| \Delta s \right. \\
& \quad \left. + \mu(t) \int_0^t |A^T B| |C(t,s)| \Delta s + \nu \int_{\sigma(t)}^\infty |C(u,t)| \Delta u \right] x^2 \\
& \quad + \left[|B| - \nu + \mu(t) \left((g^T B)^2 + 1 + |A^T B| \right. \right. \\
& \quad \left. \left. + |B| \int_0^t |C(t,s)| \Delta s \right) \right] \int_0^t |C(t,s)| x^2(s) \Delta s \\
& \quad + \mu(t) (|A^T Bg| + |g^T Bg|) + |Bg|.
\end{aligned} \tag{4.39}$$

Multiplying and dividing the above inequality by $1 + \mu(t)\lambda_3$, and then applying conditions (4.30) and (4.31) $\dot{V}(t,x)$ reduces to

$$\dot{V}(t,x) \leq \frac{-\beta_1 x^2 - \beta_2 \int_0^t |C(t,s)| x^2(s) \Delta s + L}{1 + \mu(t)\lambda_3}, \tag{4.40}$$

where $L = (\mu(t)(|A^T Bg| + |g^T Bg|) + |Bg|)(1 + \mu(t)\lambda_3)$. By taking $W_1 = r_1 x^T x$, $W_2 = x^T Bx$, $W_4 = x^T x$, $W_3 = W_5 = x^2(s)$, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \min\{\beta_1, \beta_2\}$, $\phi_1(t,s) = \nu \int_t^\infty |C(u,s)| \Delta u$, and $\phi_2(t,s) = |C(t,s)|$, we see that conditions (3.1) and (3.2) of Theorem 3.1 are satisfied. Next we make sure that condition (3.3) holds. Using (4.29) and (4.32), we obtain

$$\begin{aligned}
& W_2(|x|) - W_4(|x|) + \int_0^t (\phi_1(t,s)W_3(|x(s)|) - \phi_2(t,s)W_5(|x(s)|)) \Delta s \\
& = x^T Bx - x^T x + \int_0^t \left(\nu \int_t^\infty |C(u,s)| \Delta u - |C(t,s)| \right) x^2(s) \Delta s \leq 0.
\end{aligned} \tag{4.41}$$

Thus condition (3.3) is satisfied with $\gamma = 0$. An application of Theorem 3.1 yields the results. \square

Remark 4.7. It is worth mentioning that Theorem 4.6 is new when $\mathbb{T} = \mathbb{R}$.

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