ON NONAUTONOMOUS DISCRETE DYNAMICAL SYSTEMS DRIVEN BY MEANS

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We investigate the asymptotic behavior of nonautonomous discrete dynamical systems governed by the system of difference equations (recursive equations): $y_j(n+1) = F_j(n, \mathbf{y}(n))$; $j = 1, \ldots, k$, $n = 0, 1, 2, \ldots$, where $\mathbf{y}(n) = (y_1(n), \ldots, y_k(n)) \in \mathbb{R}^k$, $\mathbf{y}(0) = \mathbf{x}$, and $F_j(n, \cdot)$ is a mean of k (≥ 2) positive real numbers, that is, a real-valued function satisfying the internality property $\min(\mathbf{x}) \leq F_j(n, \mathbf{x}) \leq \max(\mathbf{x})$.

1. Introduction

In this research, we are concerned with the long-term behavior of discrete dynamical systems driven by means, namely,

$$y_j(n+1) = F_j(n, \mathbf{y}(n)); \quad j = 1, \ldots, k, \; n = 0, 1, 2, \ldots, \tag{1.1}$$

where $\mathbf{y}(n) = (y_1(n), \ldots, y_k(n))$, $\mathbf{y}(0) = \mathbf{x}$, and for each j, $F_j(n, \cdot)$ is a mean of k (≥ 2) positive real numbers, that is, a real-valued function satisfying the internality property

$$\min(\mathbf{x}) \leq F_j(n, \mathbf{x}) \leq \max(\mathbf{x}). \tag{1.2}$$

A mean *M* is *strict* if

$$\min(\mathbf{x}) < \max(\mathbf{x}) \Longrightarrow \min(\mathbf{x}) < M(\mathbf{x}) < \max(\mathbf{x}), \tag{1.3}$$

and it is said to be *isotone* if the functions $M(\cdot, x_2, \ldots, x_k), \ldots, M(x_1, \ldots, x_{k-1}, \cdot)$ are nondecreasing.

There is a sizable number of means in the literature. For example
  (i) the (weighted) arithmetic means $A(\mathbf{x}) = \sum_{j=1}^{k} w_j x_j$, $w_j \geq 0$, $\sum_{j=1}^{k} w_j = 1$;
  (ii) the (weighted) geometric means $G(\mathbf{x}) = \prod_{j=1}^{k} x_j^{v_j}$, $v_j \geq 0$, $\sum_{j=1}^{k} v_j = 1$;

(iii) the Holder means $H_p(\mathbf{x}) = [\sum_{j=1}^{k} w_j x_j^p]^{1/p}$, $p > 0$, $w_j \geq 0$, $\sum_{j=1}^{k} w_j = 1$;

(iv) the Lehmer means $L_p(\mathbf{x}) = \sum_{j=1}^{k} x_j^p / \sum_{j=1}^{k} x_j^{p-1}$, $p > 1$,

and so forth. For more information on means and related results, we refer the reader to [2–4] and the references cited therein.

The dynamical system described in (1.1) is, also, known in literature as compounding [3] or iteration [9] of means. Markov chains, which are basically compounding of arithmetic means, illustrate, in a clear way, the importance of such problem in modeling and applications. Also, Gauss arithmetic-geometric mean, one of the jewels of the classical analysis, connects iteration of means to special functions and integrability of difference equations.

Most of the literature related to compounding of means deal with time-invariant (autonomous) dynamical systems. It is our objective in this research to extend these results to time-variant (nonautonomous) dynamical systems driven by means. The presence of time accounts for environmental fluctuations, seasonal change, and so forth [5]. Our main results are established in Section 2. We present some applications of these results in Section 3, and conclude in Section 4 by formulating a conjecture about the limit sets of the dynamical systems under study.

2. Main results

In this section, we state and prove our main results. The following lemma is important in its own right, and will play a key role in the sequel.

LEMMA 2.1. *Let $\mathcal{Y} = \{\mathbf{y}(n)\}_{n=0}^{\infty}$ be a solution of (1.1) with $\mathbf{y}(0) = \mathbf{x}$. Then, there exist real numbers L and U with $L \leq U$ such that limit set $w(\mathbf{x})$ is a subset of the intersection of the levels sets $\min(\tilde{\mathbf{x}}) = L$ and $\max(\tilde{\mathbf{x}}) = U$. In particular, if $k = 2$, then $\mathcal{Y}$ can have at most two limit points.*

Proof. First, let $L_n = \min(\mathbf{y}(n))$ and $U_n = \max(\mathbf{y}(n))$. Then for all $n \geq 0$,

$$
\begin{aligned}
L_{n+1} &= \min\left(\mathbf{y}(n+1)\right) = \min\left(F_1\left(n, \mathbf{y}(n)\right), \ldots, F_k\left(n, \mathbf{y}(n)\right)\right) \\
&\geq \min\left(L_n, \ldots, L_n\right) = L_n, \\
U_{n+1} &= \max\left(\mathbf{y}(n+1)\right) = \max\left(F_1\left(n, \mathbf{y}(n)\right), \ldots, F_k\left(n, \mathbf{y}(n)\right)\right) \\
&\leq \max\left(U_n, \ldots, U_n\right) = U_n.
\end{aligned}
\tag{2.1}
$$

Therefore, by monotonic convergence theorem, the sequences $\{L_n\}_{n=0}^{\infty}$ and $\{U_n\}_{n=0}^{\infty}$ converge say to L and U, respectively.

Now, if $\mathbf{c}$ is a limit point and $\{\mathbf{y}(n_i)\}_{i=0}^{\infty}$ is a subsequence that converges to $\mathbf{c}$, then by the continuity of the min and the max functions,

$$
L = \lim_{i \to \infty} \min\left(\mathbf{y}(n_i)\right) = \min(\mathbf{c}), \qquad U = \lim_{i \to \infty} \max\left(\mathbf{y}(n_i)\right) = \max(\mathbf{c}).
\tag{2.2}
$$

The last part of the assertion follows from the fact that the level curves $\min(x_1, x_2) = L$ and $\max(x_1, x_2) = U$ can intersect in at most two points. This completes the proof.   □

THEOREM 2.2. *Suppose that for each j, $F_j(n, \cdot)$ is continuous and converges to the continuous mean $M_j(\cdot)$. If at least $k - 1$ of the means M_j's are strict, then every solution $\{\mathbf{y}(n)\}_{n=0}^{\infty}$ converges to $\mu(\mathbf{x})\mathbf{e}$, where $\mathbf{e} = (1,\ldots,1)$. Furthermore, μ is a continuous mean.*

Proof. First, recall the definition of L and U from the proof of Lemma 2.1, and observe that by Weierstrass boundedness theorem, the convergence of $F_j(n, \cdot)$ to $M_j(\cdot)$ is uniform on any compact subset of $\mathbb{R}^k$, in particular on $[L_0, U_0]^k$. Therefore (see Theorem 2 and its proof in [8]), if $\{\mathbf{y}(n_i)\}_{i=0}^{\infty}$ is a subsequence that converges to a limit point $\mathbf{c}$, then $\{F_j(n_i, \mathbf{y}(n_i))\}_{i=0}^{\infty}$ converges $M_j(\mathbf{c})$.

Now, since

$$\min(\mathbf{c}) = L = \min\left(M_1(\mathbf{c}),\ldots,M_k(\mathbf{c})\right),$$
$$\max(\mathbf{c}) = U = \max\left(M_1(\mathbf{c}),\ldots,M_k(\mathbf{c})\right),$$

(2.3)

there exist i_1, i_2, j_1, and j_2 such that $c_{i_1} = M_{j_1}(\mathbf{c})$ and $c_{i_2} = M_{j_1}(\mathbf{c})$. Yet, either M_{j_1} or M_{j_2} is strict, so $c_1 = \cdots = c_k$. Hence, $\{\mathbf{y}(n)\}_{n=0}^{\infty}$ converges to $\mu(x)\mathbf{e}$. Furthermore, $\mu(\mathbf{x})$ is squeezed between two means, so it is a mean. On the other hand, by Dini's theorem [6, page 350], $U_n(\mathbf{x})$ converges uniformly to $\mu(\mathbf{x})$. Since, U_n is continuous for each n, μ is continuous. This completes the proof. $\square$

Remark 2.3. It is worth mentioning that
   (i) in the literature, $\mu(\mathbf{x})$, whenever it exists, is denoted by $\bigotimes_{j=1}^{k} F_j$, particularly, when F_j is time invariant;
   (ii) also, Theorem 2.2 can be viewed as a generalization of [8, Theorem 8.8].

Our next result is concerned with isotone means. If $M(x_1,\ldots,x_k)$ is an isotone mean of k real numbers, $L = \min(x_1,\ldots,x_k)$, and $U = \max(x_1,\ldots,x_k)$, then M can be bounded by two means $\phi_M(L, U)$ and $\psi_M(L, U)$ obtained from M by replacing those x_j's with $L < x_j < U$ by L and U, respectively. For example, if $k = 3$ and $x_1 \le x_2 \le x_3$, then

$$\phi_A(L, U) = \frac{2L + U}{3} \le A(x_1, x_2, x_3) = \frac{x_1 + x_2 + x_3}{3} \le \psi_A(L, U) = \frac{L + 2U}{3},$$

(2.4)

and so on. Now, we state and prove our second main result in this section.

THEOREM 2.4. *Suppose that F_j's are isotone means, and let $\phi_j(n, \cdot)$ and $\psi_j(n, \cdot)$ be the associated means with F_j as defined above. If there exists a continuous strict mean ϕ that is dominated by $\phi_j(n, \cdot)$ for all j, then every solution $\{\mathbf{y}(n)\}_{n=0}^{\infty}$ converges to $\mu(\mathbf{x})\mathbf{e}$, where μ is a continuous mean. A similar result holds whenever $\psi_j(n, \cdot)$ are dominated by a strict continuous mean ψ.*

Proof. Let L_n and U_n be defined as in the proof of Theorem 2.2. Then

$$L_{n+1} = \min\left(\mathbf{y}(n+1)\right) = \min\left(F_1(n, \mathbf{y}(n)),\ldots,F_k(n, \mathbf{y}(n))\right)$$
$$= F_{j_n}(n, \mathbf{y}(n)) \ge \phi_{j_n}(L_n, U_n) \ge \phi(L_n, U_n) \ge L_n, \quad \forall n.$$

(2.5)

Since $L_n \to L$, $U_n \to U$, and ϕ is continuous, we have $\phi(L, U) = L$. But ϕ is strict, therefore $L = U$. Continuity, once again, is a consequence of Dini's theorem.

The proof in the other case is similar and will be omitted. $\qquad\square$

Our last result in this section is concerned with the case when each F_j is periodic in the first argument, that is, there exists a positive integer p such that

$$F_j(n + p, \mathbf{x}) = F_j(n, \mathbf{x}), \quad \forall j, \text{ and all } \mathbf{x}. \tag{2.6}$$

To simplify notation, we will use

$$
\begin{aligned}
\mathbf{F}_n(\mathbf{x}) &= (F_1(n, \mathbf{x}), \dots, F_k(n, \mathbf{x})), \\
\mathbf{G}(\mathbf{x}) &= \mathbf{F}_{p-1} \circ \cdots \circ \mathbf{F}_0(\mathbf{x}).
\end{aligned}
\tag{2.7}
$$

THEOREM 2.5. *Suppose that there exists a positive integer p such that $F_j(n + p, \mathbf{x}) = F_j(n, \mathbf{x})$ for all j, and all $\mathbf{x}$. If the components of $\mathbf{G}$ are continuous and at least $k - 1$ of them are strict, or if the components of G are isotone and either ϕ or ψ (as defined in Theorem 2.4) is continuous and strict, then every solution $\{\mathbf{y}(n)\}_{n=0}^{\infty}$ converges to $\mu(\mathbf{x})\mathbf{e}$, where μ is a continuous mean. Furthermore, $\mu(\mathbf{x})$ is the unique mean that satisfies the functional equation*

$$\mu(\mathbf{G}(\mathbf{x})) = \mu(\mathbf{x}). \tag{2.8}$$

Proof. First, since

$$\mathbf{y}(np) = \mathbf{y}((n-1)p + p) = F_{p-1} \circ \cdots \circ F_0(\mathbf{y}((n-1)p)) = G(\mathbf{y}((n-1)p)), \tag{2.9}$$

by applying Theorems 2.2 and 2.4 to the subsequence $\{\mathbf{y}(np)\}_{n=0}^{\infty}$, in the two situations, respectively, we obtain $U = L$, hence the convergence.

Now,

$$\mu(\mathbf{x})\mathbf{e} = \lim_{n \to \infty} \mathbf{y}((n+1)p) = \lim_{n \to \infty} G^{n+1}(\mathbf{x}) = \lim_{n \to \infty} G^n(G(\mathbf{x})) = \mu(G(\mathbf{x}))\mathbf{e}. \tag{2.10}$$

This completes the proof. $\qquad\square$

Remark 2.6. The first part of Theorem 2.5 can, also, be viewed as a generalization of [3, Theorem 8.8], and the last part is a generalization of the invariance principle in [3, page 269].

3. Applications

To illustrate the applicability of Theorems 2.2, 2.4, and 2.5, we present the following example.

Example 3.1. Let $\mathbf{y}(n) = (y_1(n), \dots, y_k(n))$ such that $\mathbf{y}(0) = (x_1, \dots, x_k) \in \mathbb{R}_+^k$, and $F_j(n, \mathbf{y}(n)) = H_{p_j}(n, O(\mathbf{y}(n)))$, where O is the sorting operator that arranges $\mathbf{y}(n)$ in ascending order. Then, we have the iterative process

$$y_j(n + 1) = H_{p_j}(n, O(\mathbf{y}(n))), \quad j = 1, \dots, k. \tag{3.1}$$

Now, one may think of many possible scenarios where Theorems 2.2, 2.4, and 2.5 can be applied. In particular, if $H_{p_j}(n, \mathbf{x}) = (\sum_{i=1}^{k} w_{ij}(n) x_i^{p_j})^{1/p_j}$, and $c = \inf_n \min_{i,j} w_{ij}(n) > 0$, then with

$$Z = \{1, \ldots, k\},$$

$$L_n = \min_{i \in Z} y_i(n) = y_{k_n}(n) \quad \text{for some } k_n \in Z,$$

$$U_n = \max_{i \in Z} y_i(n) = y_{\ell_n}(n) \quad \text{for some } \ell_n \in Z,$$

$$I_n = \{i \in Z : y_i(n) = L_n\},$$

$$J_n = \{i \in Z : y_i(n) = U_n\},$$

$$K_n = Z \setminus (I_n \cup J_n),$$

(3.2)

we have

$$
\begin{aligned}
L_{n+1}^{p_{k_{n+1}}} &= \left[y_{k_{n+1}}(n+1)\right]^{p_{k_{n+1}}} = \sum_{j=1}^{k} w_{j,k_{n+1}}(n) \left[y_j(n)\right]^{p_{k_{n+1}}} \\
&= \left(\sum_{j \in I_n} w_{j,k_{n+1}}(n)\right) L_n^{p_{k_{n+1}}} + \left(\sum_{j \in J_n} w_{j,k_{n+1}}(n)\right) U_n^{p_{k_{n+1}}} \\
&\quad + \left(\sum_{j \in K_n} w_{j,k_{n+1}}(n) \left[y_j(n)\right]^{p_{k_{n+1}}}\right) \\
&\geq \left(\sum_{j \in I_n \cup K_n} w_{j,k_{n+1}}(n)\right) L_n^{p_{k_{n+1}}} + \left(\sum_{j \in J_n} w_{j,k_{n+1}}(n)\right) U_n^{p_{k_{n+1}}} \\
&= L_n^{p_{k_{n+1}}} + \left(\sum_{j \in J_n} w_{j,k_{n+1}}(n)\right) \left(U_n^{p_{k_{n+1}}} - L_n^{p_{k_{n+1}}}\right) \geq L_n^{p_{k_{n+1}}} + c\left(U_n^{p_{k_{n+1}}} - L_n^{p_{k_{n+1}}}\right) \\
&= (1 - c) L_n^{p_{k_{n+1}}} + c U_n^{p_{k_{n+1}}},
\end{aligned}
$$

(3.3)

and so

$$L_{n+1} \geq \left[(1 - c) L_n^{p_{k_{n+1}}} + c U_n^{p_{k_{n+1}}}\right]^{1/p_{k_{n+1}}}. \tag{3.4}$$

Thus, if $p_j = p$ for all j, then

$$L_{n+1} \geq \left[(1 - c) L_n^p + c U_n^p\right]^{1/p}, \tag{3.5}$$

and so Theorem 2.4 is applicable. Furthermore, if w_{ij}'s are independent of n, $W = [w_{ij}] \in \mathbb{R}_+^{k \times k}$, $\mathbf{x}^p = (x_1^p, \ldots, x_k^p)$, and $\mathbf{z}(n) = \mathbf{y}^p(n)$, then

$$\mathbf{z}(n+1) = W\mathbf{z}(n), \tag{3.6}$$

and so

$$\tilde{\mu}(W\mathbf{z}(0)) = \tilde{\mu}(\mathbf{z}(0)), \quad \tilde{\mu} = \mu^p. \tag{3.7}$$

Substituting

$$\tilde{\mu}(\mathbf{t}) = \sum_{i=1}^{k} \lambda_i t_i, \quad \lambda_i \geq 0, \ \sum_{i=1}^{k} \lambda_i = 1, \tag{3.8}$$

we reach the conclusion that $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)^T$ satisfies

$$W^T \boldsymbol{\lambda} = \boldsymbol{\lambda}, \tag{3.9}$$

that is, $\boldsymbol{\lambda}$ is a normalized nonnegative eigenvector associated with 1, if there is any. But the existence of such an eigenvector is assured by the celebrated Perron's theorem [7, page 503]. Hence,

$$\mu(\mathbf{x}) = \left[\sum_{i=1}^{k} \lambda_i \tilde{x}_i^p \right]^{1/p}, \qquad \tilde{\mathbf{x}} = O(\mathbf{x}). \tag{3.10}$$

Remark 3.2. It is worth mentioning that it is possible to formulate a weaker condition on the weights w_{ij}'s that ensures convergence. Also, by taking $p = 1$ and 0+, we obtain results about compounding arithmetic means and geometric means, respectively. Indeed, in the later case, by L'Hôpital's rule,

$$\mu(\mathbf{x}) = \lim_{p \to 0+} \exp\left(\frac{\ln\left(\sum_{i=1}^{k} \lambda_i \tilde{x}_i^p\right)}{p} \right) = \lim_{p \to 0+} \exp\left(\frac{\sum_{i=1}^{k} \lambda_i \tilde{x}_i^p \ln(\tilde{x}_i)}{\sum_{i=1}^{k} \lambda_i \tilde{x}_i^p} \right)$$

$$= \exp\left(\sum_{i=1}^{k} \lambda_i \ln(\tilde{x}_i) \right) = \prod_{i=1}^{k} \tilde{x}_i^{\lambda_i}. \tag{3.11}$$

4. Conclusion

In this research, we investigated the convergence of solutions of nonautonomous dynamical systems driven by means. In particular, when all the components converge to a common limit. In light of Lemma 2.1 and the results established in [1], we formulate the following conjecture.

CONJECTURE 4.1. *If* $\{\mathbf{y}(n)\}_{n=0}^{\infty}$ *with* $\mathbf{y}(0) = \mathbf{x}$ *a solution of (1.1), then the limit set* $w(\mathbf{x}) = \{\mathbf{z} \mid \lim_{i \to \infty} \mathbf{y}(n_i) = \mathbf{z}\}$ *is finite.*

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