

# PERIODIC SOLUTIONS OF NONLINEAR VECTOR DIFFERENCE EQUATIONS

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Essentially nonlinear difference equations in a Euclidean space are considered. Conditions for the existence of periodic solutions and solution estimates are derived. Our main tool is a combined usage of the recent estimates for matrix-valued functions with the method of majorants.

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## 1. Introduction and notation

Periodic solutions of difference equations in Euclidean and Banach spaces have been considered by many authors, see, for example, [1–3, 5–10, 12] and the references therein. Mainly equations with separated linear parts and scalar equations were investigated. In this paper, we consider essentially nonlinear systems in a Euclidean space. We prove the existence of periodic solutions and derive the estimates for their norms.

Let  $\mathbb{C}^n$  be the set of all complex  $n$ -vectors with an arbitrary norm  $\|\cdot\|$ ,  $I$  is the unit matrix,  $R_s(A)$  denotes the spectral radius of a matrix  $A$ , and

$$\Omega(r) = \{z \in \mathbb{C}^n : \|z\| \leq r\}. \quad (1.1)$$

Consider in  $\mathbb{C}^n$  the equation

$$x(t+1) = B(x(t), t)x(t) + F(x(t), t) \quad (t = 0, 1, 2, \dots), \quad (1.2)$$

where  $F(\cdot, t)$  continuously maps  $\Omega(r)$  into  $\mathbb{C}^n$ , and  $B(z, t)$  are  $n \times n$ -matrices continuous in  $z \in \Omega(r)$  and dependent on  $t = 0, 1, \dots$ . In addition,  $F(v, t)$  and  $B(v, t)$  are periodic in  $t$ :

$$\begin{aligned} F(z, t) &= F(z, t + T) \quad (z \in \Omega(r); t = 0, 1, \dots), \\ B(z, t) &= B(z, t + T) \quad (z \in \Omega(r); t = 0, 1, \dots) \end{aligned} \quad (1.3)$$

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for some positive integer  $T$ . It is also assumed that there are nonnegative constants  $\nu$  and  $\mu$ , such that

$$\|F(z, t)\| \leq \nu\|z\| + \mu \quad (z \in \Omega(r), t = 0, 1, 2, \dots, T-1). \quad (1.4)$$

Denote by  $\omega(r, T)$  the set of the finite sequences  $h = \{v(k)\}_{k=0}^{T-1}$  whose elements  $v(k)$  belong to  $\Omega(r)$ .

For an  $h = \{v(k)\}_{k=0}^T \in \omega(r, T)$ , put

$$\begin{aligned} U_h(t, s) &= B(v(t-1), t-1)B(v(t-2), t-2) \cdots B(v(s), s), \\ U_h(t, t) &= I \quad (0 \leq s < t \leq T) \end{aligned} \quad (1.5)$$

and assume that

$$I - U_h(T, 0) \text{ is invertible} \quad \forall h \in \omega(r, T). \quad (1.6)$$

### 2. Statement of the main result

**THEOREM 2.1.** *Under conditions (1.3)–(1.6), with the notation*

$$\begin{aligned} M(r, T) := & \sup_{h \in \omega(r, T), k=0, \dots, T-1} \sum_{j=0}^{T-1} \|U_h(k, 0)(I - U_h(T, 0))^{-1}U_h(T, j+1)\| \\ & + \sum_{j=0}^{k-1} \|U_h(k, j+1)\| \end{aligned} \quad (2.1)$$

suppose that

$$M(r, T)(\nu r + \mu) < r. \quad (2.2)$$

Then system (1.2) has a  $T$ -periodic solution. Moreover, that periodic solution satisfies the estimates

$$\max_{j=0, 1, \dots, T-1} \|x(j)\| \leq \frac{\mu M(r, T)}{1 - \nu M(r, T)} < r. \quad (2.3)$$

We remark that if  $F(0, t) \neq 0$  for some  $t$  in  $\{0, 1, \dots, T-1\}$ , then the solution found in the above theorem cannot be trivial.

For instance, let

$$\|B(z, t)\| \leq q < 1 \quad (z \in \Omega(r), t = 0, \dots, T-1). \quad (2.4)$$

Then  $\|U_h(k, j)\| \leq q^{k-j}$  and

$$\|(I - U_h(T, 0))^{-1}\| \leq \frac{1}{1 - q^T}. \quad (2.5)$$

Therefore

$$M(r, T) \leq \sum_{j=0}^{T-1} \frac{1}{1-q^T} q^{T-j-1} + \max_k \sum_{j=0}^{k-1} q^{k-j-1} \leq \sum_{j=0}^{T-1} q^j \left( \frac{1}{1-q^T} + 1 \right) = \frac{2-q^T}{1-q^T} \sum_{j=0}^{T-1} q^j. \quad (2.6)$$

But

$$\sum_{j=0}^{T-1} q^j = \frac{1-q^T}{1-q}. \quad (2.7)$$

Thus

$$M(r, T) \leq \frac{2-q^T}{1-q}. \quad (2.8)$$

Now Theorem 2.1 implies the following corollary.

**COROLLARY 2.2.** *Under conditions (1.3)–(1.4) and (2.4), suppose that*

$$(r\nu + \mu) \frac{2-q^T}{1-q} < r. \quad (2.9)$$

*Then system (1.2) has a  $T$ -periodic solution. Moreover that periodic solution satisfies the estimates*

$$\max_{j=0,1,\dots,T-1} \|x(j)\| \leq \frac{\mu(2-q^T)}{1-q-\nu(2-q^T)} \leq r. \quad (2.10)$$

### 3. Proof of Theorem 2.1

To achieve our goal, let us first consider the nonhomogeneous periodic problem

$$y(t+1) = B(\nu(t), t) y(t) + f(t), \quad t = 0, 1, \dots, T-1 \quad (3.1)$$

$$y(0) = y(T), \quad (3.2)$$

where  $\{f(t)\}_{k=0}^{T-1}$  is a given sequence in  $\mathbb{C}^n$  and  $h = \{\nu(t)\} \in \omega(r, T)$ . Thanks to the Variation of constants formula, solution of (3.1) is given by

$$y(k) = U_h(k, 0)y(0) + \sum_{j=0}^{k-1} U_h(k-1, j+1)f(j), \quad k = 1, \dots, T. \quad (3.3)$$

Thus, the periodic boundary value problem (3.1), (3.2) has a solution provided

$$y(0) = y(T) = U_h(T, 0)y(0) + \sum_{j=0}^{T-1} U_h(T, j+1)f(j), \quad (3.4)$$

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or

$$y(0) = (I - U_h(T, 0))^{-1} \sum_{j=0}^{T-1} U_h(T, j+1) f(j), \quad (3.5)$$

and in such a case, this solution is given by

$$y(k) = U_h(k, 0) (I - U_h(T, 0))^{-1} \sum_{j=0}^{T-1} U_h(T, j+1) f(j) + \sum_{j=0}^{k-1} U_h(k, j+1) f(j), \quad k = 1, \dots, T, \quad (3.6)$$

and thus its maximum norm satisfies the inequality

$$\max_{j=0,1,\dots,T-1} \|y(j)\| \leq M(r, T) \max_{j=0,1,\dots,T-1} \|f(j)\|. \quad (3.7)$$

Let us consider the nonlinear periodic problem (1.2), (3.2).

**LEMMA 3.1.** *Under conditions (1.4), (1.6), and (2.2), the periodic problem (1.2), (3.2) has at least one solution  $\{x(t)\}_{t=0}^T \in \omega(r, T)$ . Moreover, that solution satisfies estimates (2.3).*

*Proof.* For an arbitrary  $h = \{v(t)\} \in \omega(r, T)$ , define a mapping  $Z$  by

$$\begin{aligned} (Zh)(k) &= U_h(k, 0) (I - U_h(T, 0))^{-1} \sum_{j=0}^{T-1} U_h(T, j+1) F(v(j), j) \\ &\quad + \sum_{j=0}^{k-1} U_h(k, j+1) F(v(j), j), \quad k = 0, \dots, T-1. \end{aligned} \quad (3.8)$$

Due to (2.2),

$$\begin{aligned} \max_{j=0,1,\dots,T-1} \|(Zh)(j)\| &\leq \max_{t=0,\dots,T-1} \|F(v(t), t)\| M(r, T) \\ &\leq \left( \nu \max_{j=0,\dots,T-1} \|v(j)\| + \mu \right) M(r, T) \leq \nu r + \mu. \end{aligned} \quad (3.9)$$

So  $Z$  continuously maps  $\omega(r, T)$  into itself. By Browder's fixed point theorem,  $Z$  has a fixed point  $x \in \omega(r, T)$ , cf. [11]. It is easily checked that the point is the desired solution of problem (1.2), (3.2).

Furthermore, if  $\{x(t)\}_{t=0}^T \in \omega(r, T)$  is a solution of (1.2), (3.2), then in view of (3.7) and (1.4), we will have the relations

$$\max_{j=0,1,\dots,T-1} \|x(j)\| \leq \max_{t=0,1,\dots,T-1} \|F(x(t), t)\| M(r, T) \leq \left( \nu \max_{j=0,\dots,T} \|x(j)\| + \mu \right) M(r, T), \quad (3.10)$$

which implies (2.3), since under (2.2)  $\nu M(r, T) < 1$ . The proof is complete.  $\square$

*Assertion of Theorem 2.1* follows from the previous lemma and the periodicity of  $F(\cdot, t)$  and  $B(\cdot, t)$  in  $t$ .

#### 4. Systems with linear majorants

In this section and the next one it is assumed that the norm is *ideal*. That is the vectors  $z = (z_k)_{k=1}^n$  and  $|z| = (|z_k|)_{k=1}^n$  have the same norm. For example,

$$\|z\| = \|z\|_p = \left[ \sum_{k=1}^n |z_k|^p \right]^{1/p} \quad (1 \leq p < \infty). \quad (4.1)$$

Let there be a variable matrix  $W(t) = (w_{jk}(t))_{j,k=1}^n$   $t = 0, \dots, T$  independent of  $z$  with nonnegative entries, such that the relation

$$|B(z, t)| \leq W(t) \quad (z \in \Omega(r), t = 0, \dots, T-1) \quad (4.2)$$

is valid with a positive  $r < \infty$ . Then we will say that  $B(\cdot, t) = (b_{\{jk\}}(\cdot, t))_{j,k=1}^n$  has in  $\Omega(r)$  the linear majorant  $W(t)$ .

Inequality (4.2) means that

$$|b_{jk}(z, t)| \leq w_{jk}(t) \quad (j, k = 1, \dots, n; z \in \Omega(r), t = 1, 2, \dots, T). \quad (4.3)$$

Let us introduce the equation

$$y(t+1) = W(t)y(t) \quad (t = 1, 2, \dots). \quad (4.4)$$

LEMMA 4.1. *Let  $B(\cdot, t)$  have a linear majorant  $W(t)$  in the ball  $\Omega(r)$ . Then*

$$\|U_h(t, s)\| \leq \|V(t, s)\| \quad (h \in \omega(r, T), 0 \leq s < t \leq T-1), \quad (4.5)$$

where  $V(t, s) = W(t-1)W(t-2) \cdots W(s)$ .

*Proof.* Clearly,

$$\|U_h(t, s)\| = \|B(v(t-1), t-1) \cdots B(v(s), s)\| \leq \|W(t-1) \cdots W(s)\|. \quad (4.6)$$

This proves the result.  $\square$

Furthermore, assume that the spectral radius of  $V(T, 0)$  is less than one. Then the matrix  $I - V(T, 0)$  is positively invertible. Put

$$m(W, T) := \sup_{k=0, \dots, T-1} \sum_{j=0}^{T-1} \|V(k, 0)(I - V(T, 0))^{-1}V(T, j+1)\| + \sum_{j=0}^{k-1} \|V(k, j+1)\|. \quad (4.7)$$

Now Theorem 2.1 implies the following theorem.

THEOREM 4.2. *Under conditions (1.3)–(1.4) and (4.2) assume that the evolution operator of (4.4) satisfy the inequality  $R_s(V(T, 0)) < 1$ . In addition, suppose that*

$$(r\nu + \mu)m(W, T) < r. \quad (4.8)$$

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Then system (1.2) has a  $T$ -periodic solution. Moreover, that periodic solution satisfies the estimates

$$\max_{j=0,1,\dots,T-1} \|x(j)\| \leq \frac{\mu m(W, T)}{1 - \nu m(W, T)} \leq r. \quad (4.9)$$

### 5. Systems with constant majorants

Assume that in (4.2)  $W(t) \equiv W_0$  is a constant matrix. Then we will say that  $B(h, t)$  has in set  $\Omega(r)$  the constant majorant  $W(t)$ . In this case  $V(t, s) = W_0^{t-s}$ . Set

$$m(W_0, T) = \max_{k=0,\dots,T-1} \{ \|W_0^k (I - W_0^T)^{-1}\| + 1 \} \sum_{j=0}^{T-1} \|W_0^j\|. \quad (5.1)$$

Now Theorem 4.2 yields the following theorem.

**THEOREM 5.1.** *Under conditions (1.3)–(1.4) assume that  $B(\cdot, s)$  has in  $\Omega(r)$  a constant majorant  $W_0$ , and  $R_s(W_0) < 1$ . In addition, suppose that*

$$(\mu + r\nu)m(W_0, T) < r. \quad (5.2)$$

*Then system (1.2) has a  $T$ -periodic solution. Moreover, that periodic solution satisfies the estimates*

$$\max_{j=0,1,\dots,T-1} \|x(j)\| \leq \frac{\mu m(W_0, T)}{1 - \nu m(W_0, T)} < r. \quad (5.3)$$

Let us derive an estimate for  $m(W_0; T)$  in terms of the eigenvalues and the Frobenius norm of  $W_0$  as follows. Let  $\|\cdot\|_2$  be the Euclidean norm in  $\mathbb{C}^n$ , and  $A$  be an  $n \times n$ -matrix. Let  $\lambda_1(A), \dots, \lambda_n(A)$  be the eigenvalues of  $A$  including their multiplicities. We will make use of the following quantity

$$g(A) = \left\{ N^2(A) - \sum_{i=1}^n |\lambda_i(A)|^2 \right\}^{1/2}, \quad (5.4)$$

where  $N(A)$  is the Frobenius (Hilbert-Schmidt) norm of  $A$ , that is,  $N^2(A) = \text{Trace}(AA^*)$ . Below we give simple estimates for  $g(A)$ .

Next, we recall that the following estimates are valid:

$$\|A^m\|_2 \leq \sum_{k=0}^{n-1} R_s^{m-k}(A) g^k(A) \frac{C_m^k}{\sqrt{k!}} \quad (m = 0, 1, \dots), \quad (5.5)$$

$$\|(A - \lambda I)^{-1}\|_2 \leq \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!} \rho^{k+1}(A, \lambda)}, \quad (5.6)$$

where

$$C_m^k = \frac{m!}{(m-k)!k!} \quad (5.7)$$

and  $\rho(A, \lambda)$  is the distance between  $\lambda \in \mathbb{C}$  and the spectrum of  $A$ . Estimates (5.5) and (5.6) are proved in [4, pages 12 and 21]. Thus,

$$\|W_0^m\|_2 \leq \theta_m(W_0), \quad m = 0, 1, 2, \dots, \quad (5.8)$$

where

$$\theta_m(W_0) = \sum_{k=0}^{n-1} R_s^{m-k}(W_0) g^k(W_0) \frac{C_m^k}{\sqrt{k!}}. \quad (5.9)$$

Furthermore, due to (5.6)

$$\|(W_0^T - I)^{-1}\|_2 \leq \nu(T, W_0), \quad (5.10)$$

where

$$\nu(T, W_0) = \sum_{k=0}^{n-1} \frac{g^k(W_0^T)}{\sqrt{k!} (1 - R_s^T(W_0))^{k+1}}. \quad (5.11)$$

Then

$$m(W_0; T) \leq \widetilde{M}(W_0; T), \quad (5.12)$$

where

$$\widetilde{M}(W_0; T) := \left\{ \nu(T, W_0) \max_{k=0, \dots, T-1} \theta_k(W_0) + 1 \right\} \sum_{j=0}^{T-1} \theta_j(W_0). \quad (5.13)$$

Under the condition,  $R_s(W_0) < 1$  we have

$$\max_{k=0, \dots, T-1} \theta_k(W_0) \leq 2^{T-1} \sum_{k=0}^{n-1} \frac{g^k(W_0)}{\sqrt{k!}}. \quad (5.14)$$

Note also that  $g(W_0^T) \leq N^T(W_0)$ . Moreover, if  $A$  is a normal matrix:  $AA^* = A^*A$ , then  $g(A) = 0$ . The following inequalities are also true

$$\begin{aligned} g^2(A) &\leq N^2(A) - |\text{Trace } A^2|, \\ g^2(A) &\leq \frac{1}{2} N^2(A^* - A), \end{aligned} \quad (5.15)$$

cf. [4, Section 2.1].

Now Theorem 5.1 implies the following theorem.

**THEOREM 5.2.** *Under conditions (1.3)–(1.4), assume that  $B(\cdot, t)$  has in  $\Omega(r)$  a constant majorant  $W_0$  and  $R_s(W_0) < 1$ . In addition, let*

$$(\mu + r\nu) \widetilde{M}(W_0; T) < r. \quad (5.16)$$

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Then system (1.2) has a  $T$ -periodic solution. Moreover, that periodic solution satisfies the estimates

$$\max_{j=0,1,\dots,T-1} \|x(j)\| \leq \frac{\mu \widetilde{M}(W_0, T)}{1 - \nu \widetilde{M}(W_0, T)} \leq r. \quad (5.17)$$

As an example, let  $W_0$  be a normal matrix, then  $g(W_0) = 0$ ,  $\theta_m(W_0) = R_s^m(W_0) \leq 1$  and

$$\widetilde{M}(W_0, T) = \frac{1}{1 - R_s^T(W_0)}. \quad (5.18)$$

Now we can directly apply the previous theorem.

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