GLOBAL BEHAVIOR OF A HIGHER-ORDER RATIONAL DIFFERENCE EQUATION

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We investigate in this paper the global behavior of the following difference equation: $x_{n+1} = (P_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) + b)/(Q_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) + b)$, $n = 0, 1, \ldots$, under appropriate assumptions, where $b \in [0, \infty)$, $k \geq 1$, $i_0, i_1, \ldots, i_{2k} \in \{0, 1, \ldots\}$ with $i_0 < i_1 < \cdots < i_{2k}$, the initial conditions $x_{i_{-2k}}, x_{i_{-2k+1}}, \ldots, x_0 \in (0, \infty)$. We prove that unique equilibrium $\bar{x} = 1$ of that equation is globally asymptotically stable.

1. Introduction

For some difference equations, although their forms (or expressions) look very simple, it is extremely difficult to understand thoroughly the global behaviors of their solutions. Accordingly, one is often motivated to investigate the qualitative behaviors of difference equations (e.g., see [2, 3, 6, 9, 10]).

In [6], Ladas investigated the global asymptotic stability of the following rational difference equation:

(E1)

$$x_{n+1} = \frac{x_n + x_{n-1}x_{n-2}}{x_n x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the initial values $x_{-2}, x_{-1}, x_0 \in \mathbb{R}_+ \equiv (0, +\infty)$.

In [9], Nesemann utilized the strong negative feedback property of [1] to study the following difference equation:

(E2)

$$x_{n+1} = \frac{x_{n-1} + x_n x_{n-2}}{x_n x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where the initial values $x_{-2}, x_{-1}, x_0 \in \mathbb{R}_+$.

In [10], Papaschinopoulos and Schinas investigated the global asymptotic stability of the following nonlinear difference equation:

(E3)

$$x_{n+1} = \frac{\sum_{i \in \mathbb{Z}_k - \{j-1, j\}} x_{n-i} + x_{n-j} x_{n-j+1} + 1}{\sum_{i \in \mathbb{Z}_k} x_{n-i}}, \quad n = 0, 1, \dots, \tag{1.3}$$

where $k \in \{1, 2, 3, \dots\}$, $\{j, j-1\} \subset \mathbb{Z}_k \equiv \{0, 1, \dots, k\}$, and the initial values $x_{-k}, x_{-k+1}, \dots, x_0 \in \mathbb{R}_+$.

Recently, Li [7, 8] studied the global asymptotic stability of the following two nonlinear difference equations:

(E4)

$$x_{n+1} = \frac{x_{n-1} x_{n-2} x_{n-3} + x_{n-1} + x_{n-2} + x_{n-3} + a}{x_{n-1} x_{n-2} + x_{n-1} x_{n-3} + x_{n-2} x_{n-3} + 1 + a}, \quad n = 0, 1, \dots \tag{1.4}$$

(E5)

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-3} + x_n + x_{n-1} + x_{n-3} + a}{x_n x_{n-1} + x_n x_{n-3} + x_{n-1} x_{n-3} + 1 + a}, \quad n = 0, 1, \dots, \tag{1.5}$$

where $a \in [0, +\infty)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in \mathbb{R}_+$.

Let $k \geq 1$ and $i_0, i_1, \dots, i_{2k} \in \{0, 1, \dots\}$ with $i_0 < i_1 < \cdots < i_{2k}$. Let $P_0(x_{n-i_0}) = x_{n-i_0}$ and $Q_0(x_{n-i_0}) = 1$, for any $1 \leq j \leq k$, let

$$
\begin{aligned}
P_j(x_{n-i_0}, \dots, x_{n-i_{2j}}) &= (x_{n-i_{2j}} x_{n-i_{2j-1}} + 1) P_{j-1}(x_{n-i_0}, \dots, x_{n-i_{2j-2}}) \\
&\quad + (x_{n-i_{2j}} + x_{n-i_{2j-1}}) Q_{j-1}(x_{n-i_0}, \dots, x_{n-i_{2j-2}}), \\
Q_j(x_{n-i_0}, \dots, x_{n-i_{2j}}) &= (x_{n-i_{2j}} x_{n-i_{2j-1}} + 1) Q_{j-1}(x_{n-i_0}, \dots, x_{n-i_{2j-2}}) \\
&\quad + (x_{n-i_{2j}} + x_{n-i_{2j-1}}) P_{j-1}(x_{n-i_0}, \dots, x_{n-i_{2j-2}}).
\end{aligned}
\tag{1.6}
$$

In this paper, we consider the following difference equation:

$$x_{n+1} = \frac{P_k(x_{n-i_0}, x_{n-i_1}, \dots, x_{n-i_{2k}}) + b}{Q_k(x_{n-i_0}, x_{n-i_1}, \dots, x_{n-i_{2k}}) + b}, \quad n = 0, 1, \dots, \tag{1.7}$$

where $b \in [0, \infty)$ and the initial conditions $x_{-i_{2k}}, x_{-i_{2k}+1}, \dots, x_0 \in (0, \infty)$.

It is easy to see that the positive equilibrium $\bar{x}$ of (1.7) satisfies

$$
\begin{aligned}
\bar{x} &= \frac{P_k(\bar{x}, \bar{x}, \dots, \bar{x}) + b}{Q_k(\bar{x}, \bar{x}, \dots, \bar{x}) + b} \\
&= \frac{(\bar{x}^2 + 1) P_{k-1}(\bar{x}, \bar{x}, \dots, \bar{x}) + 2\bar{x} Q_{k-1}(\bar{x}, \bar{x}, \dots, \bar{x}) + b}{(\bar{x}^2 + 1) Q_{k-1}(\bar{x}, \bar{x}, \dots, \bar{x}) + 2\bar{x} P_{k-1}(\bar{x}, \bar{x}, \dots, \bar{x}) + b}.
\end{aligned}
\tag{1.8}
$$

Thus, we have

$$(\bar{x} - 1)\big[(\bar{x}^2 + \bar{x})Q_{k-1}(\bar{x}, \bar{x}, \ldots, \bar{x}) + (\bar{x} + 1)P_{k-1}(\bar{x}, \bar{x}, \ldots, \bar{x}) + b\big] = 0, \tag{1.9}$$

from which one can see that (1.7) has the unique positive equilibrium $\bar{x} = 1$.

Remark 1.1. Let $k = 1$, then (1.7) is (1.4) when $(i_0, i_1, i_2) = (1, 2, 3)$ and is (1.5) when $(i_0, i_1, i_2) = (0, 1, 3)$.

2. Properties of positive solutions of (1.7)

In this section, we will study properties of positive solutions of (1.7). Since

$$
\begin{aligned}
P_k(x_{n-i_0}, &x_{n-i_1}, \ldots, x_{n-i_{2k}}) - Q_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) \\
&= (x_{n-i_{2k}} - 1)(x_{n-i_{2k-1}} - 1)\big[P_{k-1}(x_{n-i_0}, \ldots, x_{n-i_{2k-2}}) - Q_{k-1}(x_{n-i_0}, \ldots, x_{n-i_{2k-2}})\big] \\
&= \cdots \\
&= (x_{n-i_{2k}} - 1)(x_{n-i_{2k-1}} - 1) \cdots (x_{n-i_2} - 1)(x_{n-i_1} - 1)\big[P_0(x_{n-i_0}) - Q_0(x_{n-i_0})\big] \\
&= (x_{n-i_0} - 1)(x_{n-i_1} - 1) \cdots (x_{n-i_{2k}} - 1),
\end{aligned}
\tag{2.1}
$$

it follows from (1.7) that for any $n \geq 0$,

$$x_{n+1} - 1 = \frac{(x_{n-i_0} - 1)(x_{n-i_1} - 1) \cdots (x_{n-i_{2k}} - 1)}{Q_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) + b}. \tag{2.2}$$

Definition 2.1. Let $\{x_n\}_{n=-i_{2k}}^{\infty}$ be a solution of (1.7) and $\{a_n\}_{n=-i_{2k}}^{\infty}$ a sequence with $a_n \in \{-1, 0, 1\}$ for every $n \geq -i_{2k}$. $\{a_n\}_{n=-i_{2k}}^{\infty}$ is called itinerary of $\{x_n\}_{n=-i_{2k}}^{\infty}$ if $a_n = -1$ when $x_n < 1$, $a_n = 0$ when $x_n = 1$, and $a_n = 1$ when $x_n > 1$.

From (2.2), we get the following.

PROPOSITION 2.2. *Let $\{x_n\}_{n=-i_{2k}}^{\infty}$ be a solution of (1.7) whose itinerary is $\{a_n\}_{n=-i_{2k}}^{\infty}$, then $a_{n+1} = a_{n-i_0}a_{n-i_1} \cdots a_{n-i_{2k}}$ for any $n \geq 0$.*

PROPOSITION 2.3. *Let $\{x_n\}_{n=-i_{2k}}^{\infty}$ be a solution of (1.7), then it follows that $x_n \neq 1$ for any $n \geq 1 \Leftrightarrow \prod_{j=0}^{i_{2k}}(x_{-j} - 1) \neq 0$.*

Proof. Let itinerary of $\{x_n\}_{n=-i_{2k}}^{\infty}$ be $\{a_n\}_{n=-i_{2k}}^{\infty}$, then it follows from Proposition 2.2 that $x_n \neq 1$ for any $n \geq 1 \Leftrightarrow a_n \neq 0$ for any $n \geq 1 \Leftrightarrow \prod_{j=0}^{i_{2k}} a_{-j} \neq 0 \Leftrightarrow \prod_{j=0}^{i_{2k}}(x_{-j} - 1) \neq 0$.   □

PROPOSITION 2.4. *If $\gcd(i_s + 1, i_{2k} + 1) = 1$ for some $s \in \{0, 1, \ldots, 2k - 1\}$, then a positive solution $\{x_n\}_{n=-i_{2k}}^{\infty}$ of (1.7) is eventually equal to $1 \Leftrightarrow x_p = 1$ for some $p \geq -i_{2k}$.*

Proof. "If" part is obvious.

"Only if" part. If $x_p = 1$ for some $p \geq -i_{2k}$, then $a_p = 0$, where $\{a_n\}_{n=-i_{2k}}^{\infty}$ is itinerary of $\{x_n\}_{n=-i_{2k}}^{\infty}$. By Proposition 2.2, we have $a_{j(i_{2k}+1)+p} = a_{j(i_s+1)+p} = 0$ for any $j \geq 0$. Since $\gcd(i_s + 1, i_{2k} + 1) = 1$, we see that for any $t \in \{0, 1, \ldots, i_{2k}\}$, there exist $j_t \in \{1, 2, \ldots, i_{2k} + 1\}$

and $m_t \in \{0, 1, \ldots, i_s + 1\}$ such that

$$j_t(i_s + 1) = m_t(i_{2k} + 1) + t. \tag{2.3}$$

Together with Proposition 2.2, it follows that

$$a_{(i_s+1)(i_{2k}+1)+t+p} = 0. \tag{2.4}$$

Again by Proposition 2.2, we have $a_n = 0$ for any $n \geq (i_s + 1)(i_{2k} + 1) + p$, which implies $x_n = 1$ for any $n \geq (i_s + 1)(i_{2k} + 1) + p$. $\qquad \square$

Example 2.5. Consider the equation

$$x_{n+1} = \frac{x_{n-i_0}x_{n-i_1}x_{n-3} + x_{n-i_0} + x_{n-i_1} + x_{n-3} + b}{x_{n-i_0}x_{n-i_1} + x_{n-i_0}x_{n-3} + x_{n-i_1}x_{n-3} + 1 + b}, \quad n = 0, 1, \ldots, \tag{2.5}$$

where $b \in [0, +\infty)$, $0 \leq i_0 < i_1 < 3$, and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in \mathbb{R}_+$. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of (2.5) whose itinerary is $\{a_n\}_{n=-3}^{\infty}$, then the following hold.

(1) If $(i_0, i_1) \in \{(0,1), (1,2)\}$ and $\{x_n\}_{n=-3}^{\infty}$ is not eventually equal to 1, then $\{a_n\}_{n=-3}^{\infty}$ is a periodic sequence of period 7.

(2) If $(i_0, i_1) = (0, 2)$ and $\{x_n\}_{n=-3}^{\infty}$ is not eventually equal to 1, then $\{a_n\}_{n=-3}^{\infty}$ is a periodic sequence of period 6.

(3) $x_n \neq 1$ for any $n \geq 1 \Leftrightarrow \prod_{j=-3}^{0}(x_j - 1) \neq 0$.

(4) $\{x_n\}_{n=-3}^{\infty}$ is eventually equal to 1 $\Leftrightarrow x_p = 1$ for some $p \geq -3$.

Proof. (1) If $(i_0, i_1) = (0, 1)$, then from Proposition 2.2, it follows that for any $n \geq 0$,

$$a_{n+4} = a_{n+3}a_{n+2}a_n = a_{n+2}a_{n+1}a_{n-1}a_{n+2}a_n$$

$$= a_{n+1}a_na_{n-1} = a_na_{n-1}a_{n-3}a_na_{n-1} \tag{2.6}$$

$$= a_{n-3}.$$

If $(i_0, i_1) = (1, 2)$, then in a similar fashion, it is true that $a_{n+4} = a_{n-3}$ for any $n \geq 0$.

(2) If $(i_0, i_1) = (0, 2)$, then from Proposition 2.2, it follows that for any $n \geq 0$,

$$a_{n+3} = a_{n+2}a_na_{n-1} = a_{n+1}a_{n-1}a_{n-2}a_na_{n-1}$$

$$= a_{n+1}a_na_{n-2} = a_na_{n-2}a_{n-3}a_na_{n-2} \tag{2.7}$$

$$= a_{n-3}.$$

(3) It follows from Proposition 2.3.

(4) It follows from Proposition 2.4 since either $\gcd(i_0 + 1, 4) = 1$ or $\gcd(i_1 + 1, 4) = 1$. $\qquad \square$

3. Global asymptotic stability of (1.7)

In this section, we will study global asymptotic stability of (1.7). To do this, we need the following lemmas.

LEMMA 3.1. *Let* $(y_0, y_1, \ldots, y_{i_{2k}}) \in \mathbb{R}_+^{i_{2k}+1} - \{(1, 1, \ldots, 1)\}$ *and* $M = \max\{y_j, 1/y_j \mid 0 \le j \le i_{2k}\}$, *then*

$$\frac{1}{M} < \frac{P_k(y_{i_0}, y_{i_1}, \ldots, y_{i_{2k}})}{Q_k(y_{i_0}, y_{i_1}, \ldots, y_{i_{2k}})} < M. \tag{3.1}$$

Proof. Since $(y_0, y_1, \ldots, y_{i_{2k}}) \in \mathbb{R}_+^{i_{2k}+1} - \{(1, 1, \ldots, 1)\}$ and $M = \max\{y_j, 1/y_j \mid 0 \le j \le i_{2k}\}$, we have $M > 1$ and either $M \ge a > 1/M$ or $M > a \ge 1/M$ for any $a \in \{y_j, 1/y_j \mid 0 \le j \le i_{2k}\}$.

It is easy to verify that

$$
\begin{aligned}
P_1(y_{i_0}, y_{i_1}, y_{i_2}) &= (y_{i_1} y_{i_2} + 1) y_{i_0} + (y_{i_1} + y_{i_2}) \\
&< (y_{i_1} y_{i_2} + 1) M + (y_{i_1} + y_{i_2}) y_{i_0} M \\
&= Q_1(y_{i_0}, y_{i_1}, y_{i_2}) M, \\
P_1(y_{i_0}, y_{i_1}, y_{i_2}) M &= [(y_{i_1} y_{i_2} + 1) y_{i_0} + (y_{i_1} + y_{i_2})] M \\
&> (y_{i_1} y_{i_2} + 1) + (y_{i_1} + y_{i_2}) y_{i_0} \\
&= Q_1(y_{i_0}, y_{i_1}, y_{i_2}).
\end{aligned}
\tag{3.2}
$$

From that we have

$$
\begin{aligned}
P_2(y_{i_0}, y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}) &= (y_{i_3} y_{i_4} + 1) P_1(y_{i_0}, y_{i_1}, y_{i_2}) + (y_{i_3} + y_{i_4}) Q_1(y_{i_0}, y_{i_1}, y_{i_2}) \\
&< (y_{i_3} y_{i_4} + 1) Q_1(y_{i_0}, y_{i_1}, y_{i_2}) M + (y_{i_3} + y_{i_4}) P_1(y_{i_0}, y_{i_1}, y_{i_2}) M \\
&= Q_2(y_{i_0}, y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}) M, \\
P_2(y_{i_0}, y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}) M &= [(y_{i_3} y_{i_4} + 1) P_1(y_{i_0}, y_{i_1}, y_{i_2}) + (y_{i_3} + y_{i_4}) Q_1(y_{i_0}, y_{i_1}, y_{i_2})] M \\
&> (y_{i_3} y_{i_4} + 1) Q_1(y_{i_0}, y_{i_1}, y_{i_2}) + (y_{i_3} + y_{i_4}) P_1(y_{i_0}, y_{i_1}, y_{i_2}) \\
&= Q_2(y_{i_0}, y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}).
\end{aligned}
\tag{3.3}
$$

By induction, we have that for any $1 \le j \le k$,

$$
\begin{aligned}
P_j(y_{i_0}, y_{i_1}, \ldots, y_{i_{2j}}) &< Q_j(y_{i_0}, y_{i_1}, \ldots, y_{i_{2j}}) M, \\
P_j(y_{i_0}, y_{i_1}, \ldots, y_{i_{2j}}) M &> Q_j(y_{i_0}, y_{i_1}, \ldots, y_{i_{2j}}).
\end{aligned}
\tag{3.4}
$$

Thus

$$\frac{1}{M} < \frac{P_k(y_{i_0}, y_{i_1}, \ldots, y_{i_{2k}})}{Q_k(y_{i_0}, y_{i_1}, \ldots, y_{i_{2k}})} < M. \tag{3.5}$$

$\square$

Let n be a positive integer and let ρ denote the part-metric on $\mathbb{R}^n_+$ (see [11]) which is defined by

$$\rho(x,y) = -\log\min\left\{\frac{x_i}{y_i}, \frac{y_i}{x_i} \mid 1 \le i \le n\right\} \quad \text{for } x = (x_1,\ldots,x_n),\ y = (y_1,\ldots,y_n) \in \mathbb{R}^n_+. \tag{3.6}$$

It was shown by Thompson [11] that $(\mathbb{R}^n_+,\rho)$ is a complete metric space. In [4], Krause and Nussbaum proved that the distances indicated by the part-metric and by the Euclidean norm are equivalent on $\mathbb{R}^n_+$.

LEMMA 3.2 [5]. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a continuous mapping with unique fixed point $x^* \in \mathbb{R}^n_+$. Suppose that there exists some $l \ge 1$ such that for the part-metric ρ,*

$$\rho(T^l x, x^*) < \rho(x, x^*) \quad \forall x \ne x^*. \tag{3.7}$$

Then x^* is globally asymptotically stable.

THEOREM 3.3. *The unique equilibrium $\overline{x} = 1$ of (1.7) is globally asymptotically stable.*

Proof. Let $\{x_n\}_{n=-i_{2k}}^{\infty}$ be a solution of (1.7) with initial conditions $x_{-i_{2k}}, x_{-i_{2k}+1}, \ldots, x_0 \in \mathbb{R}^{i_{2k}+1}_+$ such that $\{x_n\}_{n=-i_{2k}}^{\infty}$ is not eventually equal to 1 since otherwise there is nothing to show. Denoted by $T : \mathbb{R}^{i_{2k}+1}_+ \to \mathbb{R}^{i_{2k}+1}_+$ the mapping

$$T(x_{n-i_{2k}}, x_{n-i_{2k}+1}, \ldots, x_n) = \left(x_{n-i_{2k}+1}, x_{n-i_{2k}+2}, \ldots, x_n, \frac{P_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) + b}{Q_k(x_{n-i_0}, x_{n-i_1}, \ldots, x_{n-i_{2k}}) + b}\right). \tag{3.8}$$

Then solution $\{x_n\}_{n=-i_{2k}}^{\infty}$ of (1.7) is represented by the first component of the solution $\{y_n\}_{n=0}^{\infty}$ of the system $y_{n+1} = Ty_n$ with initial condition $y_0 = (x_{-i_{2k}}, x_{-i_{2k}+1}, \ldots, x_0)$. It follows from Lemma 3.1 that for all $n \ge 0$, the following inequalities hold:

$$\min\left\{x_{n-i}, \frac{1}{x_{n-i}} \mid 0 \le i \le i_{2k}\right\} < x_{n+1} < \max\left\{x_{n-i}, \frac{1}{x_{n-i}} \mid 0 \le i \le i_{2k}\right\}. \tag{3.9}$$

Inductively, we obtain that for all $n \ge 0$ and all $1 \le j \le i_{2k} + 1$,

$$\min\left\{x_{n-i}, \frac{1}{x_{n-i}} \mid 0 \le i \le i_{2k}\right\} < x_{n+j} < \max\left\{x_{n-i}, \frac{1}{x_{n-i}} \mid 0 \le i \le i_{2k}\right\}, \tag{3.10}$$

from which it follows that

$$\min\left\{x_{n-i}, \frac{1}{x_{n-i}} \mid 0 \le i \le i_{2k}\right\} < \min\left\{x_{n+i}, \frac{1}{x_{n+i}} \mid 1 \le i \le i_{2k} + 1\right\}. \tag{3.11}$$

Thus, for $x^* = (1,1,\dots,1)$ and the part-metric ρ of $\mathbb{R}_+^{i_{2k}+1}$, we have

$$
\begin{aligned}
\rho\big(T^{i_{2k}+1}(y_n), x^*\big) &= -\log\min\left\{x_{n+i}, \frac{1}{x_{n+i}} \mid\mid 1 \le i \le i_{2k}+1\right\} \\
&< -\log\min\left\{x_{n-i}, \frac{1}{x_{n-i}} \mid 0 \le i \le i_{2k}\right\} \\
&= \rho(y_n, x^*)
\end{aligned}
\tag{3.12}
$$

for all $n \ge 0$. It follows from Lemma 3.2 that the positive equilibrium $\overline{x} = 1$ of (1.7) is globally asymptotically stable. $\qquad\square$

Acknowledgments

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Publication Date	June 1, 2009

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