

ON THE SYSTEM OF RATIONAL DIFFERENCE EQUATIONS $x_{n+1} = f(x_n, y_{n-k}), y_{n+1} = f(y_n, x_{n-k})$

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We study the global asymptotic behavior of the positive solutions of the system of rational difference equations $x_{n+1} = f(x_n, y_{n-k}), y_{n+1} = f(y_n, x_{n-k}), n = 0, 1, 2, \dots$, under appropriate assumptions, where $k \in \{1, 2, \dots\}$ and the initial values $x_{-k}, x_{-k+1}, \dots, x_0, y_{-k}, y_{-k+1}, \dots, y_0 \in (0, +\infty)$. We give sufficient conditions under which every positive solution of this equation converges to a positive equilibrium. The main theorem in [1] is included in our result.

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1. Introduction

Recently there has been published quite a lot of works concerning the behavior of positive solutions of systems of rational difference equations [2–7]. These results are not only valuable in their own right, but they can provide insight into their differential counterparts.

In [1], Camouzis and Papaschinopoulos studied the global asymptotic behavior of the positive solutions of the system of rational difference equations

$$\begin{aligned} x_{n+1} &= 1 + \frac{x_n}{y_{n-k}}, \\ y_{n+1} &= 1 + \frac{y_n}{x_{n-k}}, \end{aligned} \quad n = 0, 1, 2, \dots, \quad (1.1)$$

where $k \in \{1, 2, \dots\}$ and the initial values $x_{-k}, x_{-k+1}, \dots, x_0, y_{-k}, y_{-k+1}, \dots, y_0 \in (0, +\infty)$.

To be motivated by the above studies, in this paper, we consider the more general equation

$$\begin{aligned} x_{n+1} &= f(x_n, y_{n-k}), \\ y_{n+1} &= f(y_n, x_{n-k}), \end{aligned} \quad n = 0, 1, 2, \dots, \quad (1.2)$$

2 System of rational difference equations

where $k \in \{1, 2, \dots\}$, the initial values $x_{-k}, x_{-k+1}, \dots, x_0, y_{-k}, y_{-k+1}, \dots, y_0 \in (0, +\infty)$ and f satisfies the following hypotheses.

(H₁) $f \in C(E \times E, (0, +\infty))$ with $a = \inf_{(u,v) \in E \times E} f(u, v) \in E$, where $E \in \{(0, +\infty), [0, +\infty)\}$.

(H₂) $f(u, v)$ is increasing in u and decreasing in v .

(H₃) There exists a decreasing function $g \in C((a, +\infty), (a, +\infty))$ such that

- (i) For any $x > a$, $g(g(x)) = x$ and $x = f(x, g(x))$;
- (ii) $\lim_{x \rightarrow a^+} g(x) = +\infty$ and $\lim_{x \rightarrow +\infty} g(x) = a$.

A pair of sequences of positive real numbers $\{(x_n, y_n)\}_{n=-k}^{\infty}$ that satisfies (1.2) is a positive solution of (1.2). If a positive solution of (1.2) is a pair of positive constants (x, y) , then (x, y) is called a positive equilibrium of (1.2). In this paper, our main result is the following theorem.

THEOREM 1.1. *Assume that (H₁)–(H₃) hold. Then the following statements are true.*

- (i) *Every pair of positive constant $(x, y) \in (a, +\infty) \times (a, +\infty)$ satisfying the equation*

$$y = g(x) \quad (1.3)$$

is a positive equilibrium of (1.2).

- (ii) *Every positive solution of (1.2) converges to a positive equilibrium (x, y) of (1.2) satisfying (1.3) as $n \rightarrow \infty$.*

2. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. To do this we need the following lemma.

LEMMA 2.1. *Let $\{(x_n, y_n)\}_{n=-k}^{\infty}$ be a positive solution of (1.2). Then there exists a real number $L \in (a, +\infty)$ with $L < g(L)$ such that $x_n, y_n \in [L, g(L)]$ for all $n \geq 1$. Furthermore, if $\limsup x_n = M$, $\liminf x_n = m$, $\limsup y_n = P$, $\liminf y_n = p$, then $M = g(p)$ and $P = g(m)$.*

Proof. From (H₁) and (H₂), we have

$$\begin{aligned} x_i &= f(x_{i-1}, y_{i-1-k}) > f(x_{i-1}, y_{i-1-k} + 1) \geq a, \\ y_i &= f(y_{i-1}, x_{i-1-k}) > f(y_{i-1}, x_{i-1-k} + 1) \geq a, \end{aligned} \quad \text{for every } 1 \leq i \leq k+1. \quad (2.1)$$

Since $\lim_{x \rightarrow a^+} g(x) = +\infty$, there exists $L \in (a, +\infty)$ with $L < g(L)$ such that

$$x_i, y_i \in [L, g(L)] \quad \text{for every } 1 \leq i \leq k+1. \quad (2.2)$$

It follows from (2.2) and (H₃) that

$$\begin{aligned} g(L) &= f(g(L), L) \geq x_{k+2} = f(x_{k+1}, y_1) \geq f(L, g(L)) = L, \\ g(L) &= f(g(L), L) \geq y_{k+2} = f(y_{k+1}, x_1) \geq f(L, g(L)) = L. \end{aligned} \quad (2.3)$$

Inductively, we have that $x_n, y_n \in [L, g(L)]$ for all $n \geq 1$.

Let $\limsup x_n = M$, $\liminf x_n = m$, $\limsup y_n = P$, $\liminf y_n = p$, then there exist sequences $l_n \geq 1$ and $s_n \geq 1$ such that

$$\lim_{n \rightarrow \infty} x_{l_n} = M, \quad \lim_{n \rightarrow \infty} y_{s_n} = p. \quad (2.4)$$

Without loss of generality, we may assume (by taking a subsequence) that there exist $A, D \in [m, M]$ and $B, C \in [p, P]$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} x_{l_n-1} &= A, \\ \lim_{n \rightarrow \infty} y_{l_n-k-1} &= B, \\ \lim_{n \rightarrow \infty} y_{s_n-1} &= C, \\ \lim_{n \rightarrow \infty} x_{s_n-k-1} &= D. \end{aligned} \quad (2.5)$$

Thus, from (1.2), (H₂) and (H₃), we have

$$\begin{aligned} f(M, g(M)) &= M = f(A, B) \leq f(M, p), \\ f(p, g(p)) &= p = f(C, D) \geq f(p, M), \end{aligned} \quad (2.6)$$

from which it follows that

$$g(M) \geq p, \quad g(p) \leq M. \quad (2.7)$$

By (H₃), we obtain

$$p = g(g(p)) \geq g(M). \quad (2.8)$$

Therefore, $M = g(p)$. By the symmetry, we have also $P = g(m)$. Lemma 2.1 is proven. \square

Proof of Theorem 1.1.

(i) Is obvious.

(ii) Let $\{(x_n, y_n)\}_{n=-k}^{\infty}$ be a positive solution of (1.2) with the initial conditions $x_0, x_{-1}, \dots, x_{-k}, y_0, y_{-1}, \dots, y_{-k} \in (0, +\infty)$. By Lemma 2.1, we have that

$$\begin{aligned} a < \liminf x_n = g(P) &\leq \limsup x_n = M < +\infty, \\ a < \liminf y_n = g(M) &\leq \limsup y_n = P < +\infty. \end{aligned} \quad (2.9)$$

Without loss of generality, we may assume (by taking a subsequence) that there exists a sequence $l_n \geq 4k$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} x_{l_n} &= M, \\ \lim_{n \rightarrow \infty} x_{l_n-j} &= M_j \in [g(P), M], \quad \text{for } j \in \{1, 2, \dots, 3k+1\}, \\ \lim_{n \rightarrow \infty} y_{l_n-j} &= P_j \in [g(M), P], \quad \text{for } j \in \{1, 2, \dots, 3k+1\}. \end{aligned} \quad (2.10)$$

4 System of rational difference equations

From (1.2), (2.10) and (H₃), we have

$$f(M, g(M)) = M = f(M_1, P_{k+1}) \leq f(M_1, g(M)) \leq f(M, g(M)), \quad (2.11)$$

from which it follows that

$$M_1 = M, \quad P_{k+1} = g(M). \quad (2.12)$$

In a similar fashion, we may obtain that

$$f(M, g(M)) = M = M_1 = f(M_2, P_{k+2}) \leq f(M_2, g(M)) \leq f(M, g(M)), \quad (2.13)$$

from which it follows that

$$M_2 = M, \quad P_{k+2} = g(M). \quad (2.14)$$

Inductively, we have that

$$\begin{aligned} M_j &= M, \\ P_{k+j} &= g(M), \quad \text{for } j \in \{1, 2, \dots, 2k+1\}, \end{aligned} \quad (2.15)$$

from which it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} x_{l_n - j} &= M, \quad \text{for } j \in \{0, 1, \dots, 2k+1\}, \\ \lim_{n \rightarrow \infty} y_{l_n - j} &= g(M), \quad \text{for } j \in \{k+1, \dots, 3k+1\}. \end{aligned} \quad (2.16)$$

In view (2.16), for any $0 < \varepsilon < M - a$, there exists some $l_s \geq 4k$ such that

$$\begin{aligned} M - \varepsilon &< x_{l_s - j} < M + \varepsilon, \quad \text{if } j \in \{0, 1, \dots, 2k+1\}, \\ g(M + \varepsilon) &< y_{l_s - j} < g(M - \varepsilon), \quad \text{if } j \in \{k+1, \dots, 2k+1\}. \end{aligned} \quad (2.17)$$

From (1.2) and (2.17), we have

$$y_{l_s - k} = f(y_{l_s - k - 1}, x_{l_s - 2k - 1}) < f(g(M - \varepsilon), M - \varepsilon) = g(M - \varepsilon). \quad (2.18)$$

Also (1.2), (2.17) and (2.18) implies

$$x_{l_s + 1} = f(x_{l_s}, y_{l_s - k}) > f(M - \varepsilon, g(M - \varepsilon)) = M - \varepsilon. \quad (2.19)$$

Inductively, it follows that

$$\begin{aligned} y_{l_s + n - k} &< g(M - \varepsilon) \quad \forall n \geq 0, \\ x_{l_s + n} &> M - \varepsilon \quad \forall n \geq 0. \end{aligned} \quad (2.20)$$

Since $\limsup x_n = M$ and $\liminf y_n = g(M)$, we have

$$\lim_{n \rightarrow \infty} x_n = M, \quad \lim_{n \rightarrow \infty} y_n = g(M). \quad (2.21)$$

Thus $\lim_{n \rightarrow \infty} (x_n, y_n) = (M, P)$ with $P = g(M)$. Theorem 1.1 is proven. \square

3. Examples

To illustrate the applicability of Theorem 1.1, we present the following examples.

Example 3.1. Consider equation

$$\begin{aligned} x_{n+1} &= \frac{p + x_n}{1 + y_{n-k}}, & n = 0, 1, \dots, \\ y_{n+1} &= \frac{p + y_n}{1 + x_{n-k}}, \end{aligned} \quad (3.1)$$

where $k \in \{1, 2, \dots\}$, the initial conditions $x_{-k}, x_{-k+1}, \dots, x_0, y_{-k}, y_{-k+1}, \dots, y_0 \in (0, +\infty)$ and $p \in (0, +\infty)$. Let $E = [0, +\infty)$ and

$$f(x, y) = \frac{p + x}{1 + y} \quad (x \geq 0, y \geq 0), \quad g(x) = \frac{p}{x} \quad (x > 0). \quad (3.2)$$

It is easy to verify that (H₁)–(H₃) hold for (3.1). It follows from Theorem 1.1 that

- (i) every pair of positive constant $(x, y) \in (0, +\infty) \times (0, +\infty)$ satisfying the equation

$$xy = p \quad (3.3)$$

is a positive equilibrium of (3.1).

- (ii) every positive solution of (3.1) converges to a positive equilibrium (x, y) of (3.1) satisfying (3.3) as $n \rightarrow \infty$.

Example 3.2. Consider equation

$$\begin{aligned} x_{n+1} &= 1 + \frac{x_n}{y_{n-k}}, & n = 0, 1, \dots, \\ y_{n+1} &= 1 + \frac{y_n}{x_{n-k}}, \end{aligned} \quad (3.4)$$

where $k \in \{1, 2, \dots\}$ and the initial conditions $x_{-k}, x_{-k+1}, \dots, x_0, y_{-k}, y_{-k+1}, \dots, y_0 \in (0, +\infty)$. Let $E = (0, +\infty)$ and

$$f(x, y) = 1 + \frac{x}{y} \quad (x > 0, y > 0), \quad g(x) = \frac{x}{x - 1} \quad (x > 1). \quad (3.5)$$

It is easy to verify that (H₁)–(H₃) hold for (3.4). It follows from Theorem 1.1 that

- (i) every pair of positive constant $(x, y) \in (1, +\infty) \times (1, +\infty)$ satisfying the equation

$$xy = x + y \quad (3.6)$$

is a positive equilibrium of (3.4);

6 System of rational difference equations

- (ii) every positive solution of (3.4) converges to a positive equilibrium (x, y) of (3.4) satisfying (3.6) as $n \rightarrow \infty$.

Example 3.3. Consider equation

$$\begin{aligned} x_{n+1} &= p + \frac{A + x_n}{q + y_{n-k}}, \\ y_{n+1} &= p + \frac{A + y_n}{q + x_{n-k}}, \end{aligned} \quad n = 0, 1, \dots, \quad (3.7)$$

where $k \in \{1, 2, \dots\}$, the initial conditions $x_{-k}, x_{-k+1}, \dots, x_0, y_{-k}, y_{-k+1}, \dots, y_0 \in (0, +\infty)$, $A \in (0, +\infty)$ and $p, q \in [0, 1]$ with $p + q = 1$. Let $E = (0, +\infty)$ if $p > 0$ and $E = [0, +\infty)$ if $p = 0$ and

$$f(x, y) = p + \frac{A + x}{q + y}, \quad (3.8)$$

then $a = \inf_{(x, y) \in E \times E} f(x, y) = p$. Let $g(x) = (pq + px + A)/(x - p)$ ($x > p$). It is easy to verify that (H_1) – (H_3) hold for (3.7). It follows from Theorem 1.1 that

- (i) every pair of positive constant $(x, y) \in (p, +\infty) \times (p, +\infty)$ satisfying the equation

$$xy = pq + px + py + A \quad (3.9)$$

is a positive equilibrium of (3.7);

- (ii) every positive solution of (3.7) converges to a positive equilibrium (x, y) of (3.7) satisfying (3.9) as $n \rightarrow \infty$

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