

# EXISTENCE PROBLEMS FOR HOMOCLINIC SOLUTIONS

CEZAR AVRAMESCU

Received 9 November 2001

The problem  $\dot{x} = f(t, x)$ ,  $x(-\infty) = x(+\infty)$ , where  $x(\pm\infty) := \lim_{t \rightarrow \pm\infty} x(t) \in \mathbb{R}^n$ , is considered. Some existence results for this problem are established using the fixed point method and topological degree theory.

## 1. Introduction

Let  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous function; consider the boundary value problem

$$\dot{x} = f(t, x), \quad x(-\infty) = x(+\infty), \quad (1.1)$$

where

$$x(\pm\infty) := \lim_{t \rightarrow \pm\infty} x(t) \in \mathbb{R}^n. \quad (1.2)$$

The solutions of problem (1.1) are often called, by Poincaré, *homoclinic solutions*. They appear in certain celeste mechanics and cosmogony problems.

Problem (1.1) can be considered as a generalization of the boundary value problem

$$\dot{x} = f(t, x), \quad x(a) = x(b), \quad (1.3)$$

when  $a \rightarrow -\infty$  and  $b \rightarrow +\infty$ .

The boundary value problems on compact intervals have been studied in numerous papers but the boundary value problems on noncompact intervals have been less studied. A first substantial approach of these problems, using functional methods are due to Kartsatos [8]. Last time, this type of results has been published in [2, 3, 4, 5, 6].

For problem (1.3), Mawhin obtained many existence results through topological degree theory; in [9, 10, 11] the reader can find the fundamental ideas of the

## 2 Existence problems for homoclinic solutions

method developed by Mawhin, the main results, and a rich bibliography in this field. Some approaches of Mawhin dedicated to problem (1.3) can be adjusted for problem (1.1).

The present paper is dedicated to the existence of solutions for problem (1.1); the used method will be the reduction of problem (1.1) to a fixed point problem for a convenient operator defined in a suitable functional space. Such a space is

$$C_l := \{x : \mathbb{R} \longrightarrow \mathbb{R}^n, \exists x(\pm\infty) \in \mathbb{R}^n\}. \quad (1.4)$$

[Section 2](#) deals especially to praise the main properties of the space  $C_l$ . The specified isomorphism between  $C_l$  and  $C([a, b], \mathbb{R}^n)$  permits to obtain a compactness criterion in  $C_l$  (see [1]). We define in  $C_l$  the notion of an associated operator to problem (1.1) and indicate the construction method of such operator together with its main properties. An associated operator for problem (1.1) is an operator whose fixed points are solutions for (1.1).

In [Section 3](#), assuming the existence and uniqueness on  $\mathbb{R}$  of the solutions for the problem

$$\dot{x} = f(t, x), \quad x(0) = y, \quad (1.5)$$

one builds up associated operators mapping in  $\mathbb{R}^n$ ; consequently, their topological degree will be a Brouwer one.

In [Section 4](#), the continuation method is presented (see [Proposition 4.1](#)). Through this method we obtain existence results for perturbed equations. The starting equation is chosen such that the topological degree of its associated operator is easy to be evaluated, and the perturbation is done through homogeneous or “small” functions.

For further details about the construction of the associated operators, the reader can consult [12]. For the topological degree theory we recommend the delightful book [13].

### 2. General hypotheses and preliminary results

**2.1. Introduction.** Let  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous mapping; consider problem (1.1) where  $x(\pm\infty) := \lim_{t \rightarrow \pm\infty} x(t) \in \mathbb{R}^n$  (notation used throughout this paper).

It is clear from the introduction that the aim of this paper is to find sufficient conditions to assure the existence of solutions for problem (1.1). The method will be the reduction of the existence solutions for problem (1.1) to the existence of fixed points for an adequate operator which maps in an adequate functional space.

In this section, we present the principal function spaces, their main properties, the notations, and the principal theoretical results needed in what follows.

**2.2. Function spaces.** Denote by  $|\cdot|$  an arbitrary norm in  $\mathbb{R}^n$  and

$$C_c := \{x : \mathbb{R} \longrightarrow \mathbb{R}^n, x \text{ continuous}\}. \quad (2.1)$$

As is well known,  $C_c$  is a Fréchet space endowed with the uniform convergence on compact subsets of  $\mathbb{R}$  with the usual topology. Let  $C_c^1$  denote the linear subspace of  $C^1$  functions in  $C_c$ .

The principal function spaces are

$$\begin{aligned} C_l &:= \{x \in C_c, \exists x(\pm\infty) \in \mathbb{R}^n\}, \\ C_{ll} &:= \{x \in C_l, x(-\infty) = x(+\infty)\}, \end{aligned} \quad (2.2)$$

where  $C_l$  and  $C_{ll}$  are Banach spaces with respect to the norm

$$\|x\|_\infty := \sup_{t \in \mathbb{R}} \{|x(t)|\}, \quad (2.3)$$

where  $\mathbb{R}^n$  will be identified naturally with the constant functions subspace. Consider  $C_l^1 := C_l \cap C_c^1$ ,  $C_{ll}^1 := C_{ll} \cap C_c^1$ .

Another function space, interesting only as linear space, is the space of all Riemann integrable functions on  $\mathbb{R}$ ,

$$C_R := \left\{ x \in C_c; \int_{-\infty}^{+\infty} x(t) dt < +\infty \right\}, \quad (2.4)$$

where

$$\int_{-\infty}^{+\infty} x(t) dt := \lim_{A \rightarrow -\infty} \int_A^0 x(t) dt + \lim_{A \rightarrow +\infty} \int_0^A x(t) dt. \quad (2.5)$$

*Remark 2.1.* A function  $x$  of class  $C^1$  belongs to  $C_l$  if and only if  $\dot{x}$  belongs to  $C_R$ .

Finally, we use the spaces

$$\begin{aligned} C_{(a,b)} &:= \{x : [a,b] \longrightarrow \mathbb{R}^N, x \text{ continuous}\}, \\ C_{[a,b]} &:= \{x \in C_{(a,b)}, x(a) = x(b)\}, \end{aligned} \quad (2.6)$$

endowed with the usual norm

$$\|x\| := \sup_{t \in [a,b]} \{|x(t)|\}. \quad (2.7)$$

In the case of a Banach space  $X$ , where  $X = C_l$  or  $X = C_{ll}$ , set

$$B(\rho) := \{x \in X, \|x\|_\infty < \rho\}, \quad \Sigma(\rho) := \{x \in \mathbb{R}^n, |x| < \rho\}. \quad (2.8)$$

## 4 Existence problems for homoclinic solutions

**2.3. Properties of the space  $C_l$ .** We state certain properties of the space  $C_l$ .

PROPOSITION 2.2. *The spaces  $C_l$  and  $C_{(a,b)}$  are isomorphic.*

*Proof.* Indeed, consider  $\varphi : (a, b) \rightarrow \mathbb{R}$  a continuous and bijective mapping; define the mapping  $\Phi : C_l \rightarrow C_{(a,b)}$  by the equality

$$(\Phi x)(t) := \begin{cases} x(\varphi(t)), & \text{if } t \in (a, b), \\ x(-\infty), & \text{if } t = a, \\ x(+\infty), & \text{if } t = b. \end{cases} \quad (2.9)$$

It is clear that  $\Phi$  is an isometric isomorphism and the proof ends.  $\square$

*Remark 2.3.* The same mapping  $\Phi$  is an isomorphism between  $C_{ll}$  and  $C_{[a,b]}$ .

The property in [Proposition 2.2](#) allows us to obtain a compactness criterion in  $C_l$ ; obviously, it will work in  $C_{ll}$  too, since  $C_{ll}$  is a closed subspace of  $C_l$ .

*Definition 2.4.* A family  $A \subset C_l$  is called equiconvergent if and only if

$$\begin{aligned} \forall \varepsilon > 0, \quad \exists T = T(\varepsilon) > 0, \quad \forall x \in A, \forall t_1, t_2 \in \mathbb{R}, t_1 t_2 > 0, \\ |t_i| > T(\varepsilon), \quad |x(t_1) - x(t_2)| < \varepsilon. \end{aligned} \quad (2.10)$$

PROPOSITION 2.5. *A family  $A \subset C_l$  is relatively compact if and only if the following three conditions are fulfilled:*

- (i)  *$A$  is uniformly bounded on  $\mathbb{R}$ ;*
- (ii)  *$A$  is equicontinuous on every compact interval of  $\mathbb{R}$ ;*
- (iii)  *$A$  is equiconvergent.*

[Proposition 2.5](#) results immediately from the fact that the isomorphism  $\Phi$  given by (2.9) transforms a set  $A$ , satisfying conditions (i), (ii), and (iii), into an equicontinuous and uniformly bounded set in  $C_{(a,b)}$ .

*Definition 2.6.* A family  $A \subset C_c$  is called  $C_R$ -bounded if and only if there exists a function  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\alpha(t) \geq 0$  for every  $t \in \mathbb{R}$ ,  $\alpha \in C_R$ , such that

$$\forall x \in A, t \in \mathbb{R}, \quad |x(t)| \leq \alpha(t). \quad (2.11)$$

COROLLARY 2.7. *A family  $A \subset C_l \cap C_c^1$ , uniformly bounded on  $\mathbb{R}$  having the family of derivatives  $C_R$ -bounded, is relatively compact in  $C_l$ .*

**2.4. Operators.** The first operator is the Nemitzky operator,  $F : C_c \rightarrow C_c$  generated by the continuous function  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and defined by

$$(Fx)(t) := f(t, x(t)). \quad (2.12)$$

Taking into account [Remark 2.1](#), it results that for every solution  $x$  of [\(1.1\)](#) it holds

$$x \in C_l \iff Fx \in C_R. \quad (2.13)$$

Similarly, for every solution  $x$  of [\(1.1\)](#),

$$x \in C_{ll} \iff \int_{-\infty}^{+\infty} (Fx)(s) ds = 0. \quad (2.14)$$

In what follows,  $X \subseteq C_l$  denotes a closed subspace of  $C_l$  and  $D \subset X$  is a void set. Define on  $D$  an important category of operators called *associated*.

*Definition 2.8.* The operator  $U : D \subset X \rightarrow X$  is *associated* to problem [\(1.1\)](#) on the set  $D$  if and only if every fixed point of  $U$  is a solution for problem [\(1.1\)](#).

By using the formula of a solution for [\(1.1\)](#), it is naturally, in the building of the operator  $U$ , to admit

$$FD \subset C_R. \quad (2.15)$$

Remark, in addition, that if  $U$  maps in  $C_l$ , then only the fixed points satisfy condition [\(1.1\)](#) and if  $D \subset C_{ll}$  then we have  $UD \subset C_{ll}$ .

By [Remark 2.1](#), we can easily obtain associated operators to problem [\(1.1\)](#). Such an operator is, for example,

$$(Ux)(t) := x(b) + \alpha(t) \int_{-\infty}^{+\infty} (Fx)(s) ds + \int_b^{+\infty} (Fx)(s) ds, \quad (2.16)$$

where  $b \in \bar{\mathbb{R}}$ , and  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  is an arbitrary continuous function with  $\alpha(b) \neq 0$ , and  $\alpha \in C_l$ ; this operator maps in  $C_l$ . If, in addition,  $\alpha(-\infty) = \alpha(+\infty)$ , then  $UC_l \subset C_{ll}$ .

Another possibility to construct associated operators in  $C_{ll}$  is the next: we search a linear and continuous operator  $T : C_{ll} \rightarrow C_{ll}$  such that the operator  $Lx := \dot{x} + Tx$  is invertible; then

$$U = L^{-1}(F + T). \quad (2.17)$$

Examples of such operators  $T$  are  $Tx = \theta(\cdot)x(0)$  or  $T = \theta(\cdot)x$ , where  $\theta : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous and strictly positive mapping with  $\int_{-\infty}^{+\infty} \theta(t) dt = 1$ .

There exist general procedures to build up the associated operators, like the one from below having a pure algebraic character.

Let  $X$  and  $Z$  be two linear spaces,  $L : D(L) \subset X \rightarrow Z$  a linear operator, and  $N : D(N) \subset X \rightarrow Z$  an arbitrary operator.

## 6 Existence problems for homoclinic solutions

If  $\dim N(L) = \text{codim } R(L) < \infty$  and  $P : X \rightarrow X$ ,  $Q : Z \rightarrow Z$  are two projectors such that  $R(P) = N(L)$ ,  $N(Q) = R(L)$ , then  $x \in X$  is a solution for the equation

$$Lx = Nx \quad (2.18)$$

if and only if  $x$  is a fixed point for the operator

$$U = P + aQN + K(I - Q)N, \quad (2.19)$$

where  $I$  is the identity operator in  $Z$ ,  $a \in \mathbb{R}$ ,  $a \neq 0$ , and  $K$  is the right inverse of  $L$  (more precisely  $K = (L|_{D(L) \cap N(P)})^{-1}$ ).

This result has been successfully used by Mawhin for the building of associated operators to the boundary value problem (closely related to the periodic solutions problem [9, 10, 11]).

$$\dot{x} = f(t, x), \quad x(0) = x(T). \quad (2.20)$$

By this model in the next subsection, we briefly describe how can we construct associated operators to problem (1.1) and their properties.

**2.5. Construction of associated operators.** The form of associated operators depends firstly on the fundamental space  $X$  and next on the space  $Z$  and on the choice of the operators  $L, N$  and the choice of the projectors  $P, Q$ ; only after this  $K$  can be determined and also the final form of the operator  $U$ . Having so many arbitrary elements we can find many associated operators.

In what follows, we sketch the building of associated operators in two important cases:  $X = C_l$  and  $X = C_{ll}$ ; further details about the construction can be found in [4, 12].

In the case  $X = C_l$  we distinguish three subcases related to  $L$  and  $N$ ; this choice must be made such that the equation  $(L, N)$  does contain (1.1). The expression of projectors  $P$  and  $Q$  depends on the considered case.

In all three cases, we have  $Z = C_R \times \mathbb{R}^n$ ,  $D(L) = C_l^1$ .

### 2.5.1. The case $L_1$ .

$$Lx = (\dot{x}, x(+\infty)), \quad Nx = (Fx, x(-\infty)). \quad (2.21)$$

In this case  $P = Q = 0$ , so the operator  $L$  is invertible and therefore  $U = L^{-1}N$ . This case gives us the easiest associated operators,

$$Ux = x(+\infty) + \int_{-\infty}^{(\cdot)} (Fx)(s) ds \quad (2.22)$$

and the symmetric form

$$Ux = x(-\infty) + \int_{+\infty}^{(\cdot)} (Fx)(s) ds. \quad (2.23)$$

2.5.2. The case  $L_2$ .

$$Lx = (\dot{x}, 0), \quad Nx = (Fx, x(+\infty) - x(-\infty)). \quad (2.24)$$

In this case, since  $R(L) = C_R \times \{0\}$  the projector  $Q$  may be

$$Q(y, c) = (0, c). \quad (2.25)$$

For  $P$ , we take

$$Px = x(b), \quad x \in \bar{\mathbb{R}}, \quad (2.26)$$

or

$$Px = \int_{-\infty}^{+\infty} e(t)x(t) dt, \quad (2.27)$$

where

$$e : \mathbb{R} \longrightarrow \mathbb{R}, \quad e \text{ continuous,} \quad \int_{-\infty}^{+\infty} e(s) ds = 1. \quad (2.28)$$

For  $U$ , we can construct

$$Ux = x(b) + a[x(+\infty) - x(-\infty)] + \int_b^{(\cdot)} (Fx)(s) ds \quad (2.29)$$

or

$$\begin{aligned} Ux &= a[x(+\infty) - x(-\infty)] \\ &+ \int_{-\infty}^{+\infty} \left[ x(s) - \frac{1}{2}(Fx)(s) \right] e^{-2|s|} ds + \int_0^{(\cdot)} (Fx)(s) ds. \end{aligned} \quad (2.30)$$

2.5.3. The case  $L_3$ .

$$Lx = (\dot{x}, x(+\infty) - x(-\infty)), \quad Nx = (Fx, 0). \quad (2.31)$$

In this case,

$$R(L) = \left\{ (y, c) \in C_R \times \mathbb{R}^n \mid c = \int_{-\infty}^{+\infty} y(s) ds \right\} \quad (2.32)$$

and hence the projector  $Q$  must be changed; we can take for example

$$Q(y, c) = \left( 0, c - \int_{-\infty}^{+\infty} y(s) ds \right) \quad (2.33)$$

and therefore,

$$Ux = x(b) + a \int_{-\infty}^{+\infty} (Fx)(s) ds + \int_b^{(\cdot)} (Fx)(s) ds \quad (2.34)$$

and other more complicated forms.

## 8 Existence problems for homoclinic solutions

In the case  $X = C_{ll}$ , we have  $D(L) = C_{ll}^1$ ,  $Z = C_R$ ,  $Lx = \dot{x}$  and consequently,

$$R(L) = \left\{ y \in C_R, \int_{-\infty}^{+\infty} y(s) ds = 0 \right\}. \quad (2.35)$$

We can take, for example,

$$(Qy)(t) = e(t) \int_{-\infty}^{+\infty} y(s) ds. \quad (2.36)$$

In general, in this case the expression of  $U$  is more complicated since all its values must be in  $C_{ll}$ . For example, for (2.26) we get

$$(Ux)(t) = x(b) + \left[ ae^{-2|t|} - \int_b^t e^{-2|s|} ds \right] \cdot \int_{-\infty}^{+\infty} (Fx)(s) ds + \int_b^t (Fx)(s) ds \quad (2.37)$$

and with (2.27), where  $e(t) = e^{-2|t|}$ ,

$$\begin{aligned} (Ux)(t) &= \int_0^t e^{-2|s|} ds + ae^{-2|t|} - \int_0^t e^{-2|s|} \left( 1 - \frac{1}{2} e^{-2|s|} \right) ds \\ &\quad - \frac{1}{2} \int_{-\infty}^{+\infty} e^{-2|s|} (Fx)(s) ds \int_{-\infty}^{+\infty} (Fx)(s) ds + \int_0^t (Fx)(s) ds. \end{aligned} \quad (2.38)$$

**2.6. Admissible operators.** It is obvious that this construction of the associated operators has an algebraic character; the condition

$$FD \subset C_R \quad (2.39)$$

is sufficient for the existence of these operators, but it is not sufficient to confer their important topological properties.

*Definition 2.9.* An associated operator on the set  $D \subset X$  to problem (1.1), constructed as in [Section 2.5](#), is called *admissible* if and only if  $U : D \subset X \rightarrow X$  is compact.

**PROPOSITION 2.10.** *Let  $X$  be a subspace of  $C_l$  and  $D \subset C_l$  be a bounded subset. If  $FD$  is  $C_R$ -bounded, then every associated operator constructed as in [Section 2.5](#) is compact.*

The proof of this proposition is complicated in calculus, but it is basically an easy application of the elementary known properties of uniform convergence, which allows to establish immediately the continuity of the operator  $U$  which contains finite rank projectors and application of type

$$x \longrightarrow \int_{-\infty}^{+\infty} (Fx)(s) ds, \quad x \longrightarrow \int_b^{(\cdot)} (Fx)(s) ds, \quad b \in \bar{\mathbb{R}}. \quad (2.40)$$

The compactness of the operator  $U$  is an immediate consequence of [Corollary 2.7](#). At least for the operators  $U$  given by (2.22), (2.23), (2.29), (2.30), (2.34), (2.37), and (2.38) the verification of compactness is immediate.

Remark that if  $f$  satisfies the condition

$$|f(t, x)| \leq \theta(t) \cdot \beta(|x|), \quad (2.41)$$

where  $\theta \in C_R$ ,  $\theta \geq 0$ ,  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, then  $FD$  is  $C_R$ -bounded for every bounded set  $D \subset X$ ; indeed, we have

$$|(Fx)(t)| \leq \rho \theta(t), \quad (2.42)$$

where

$$\rho := \sup \{ \beta(u), |u| \leq r \}, \quad r := \sup \{ \|x\|_\infty, x \in D \}. \quad (2.43)$$

The situation is more complicated in the case when (1.1) proceeds from a second-order equation

$$\ddot{y} = h(t, y, \dot{y}), \quad (2.44)$$

where  $h : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$  is a continuous function. Substituting (2.44) in (1.1), where

$$x = (x_1, x_2), \quad x_1 = y, \quad x_2 = \dot{y}, \quad f(t, x) = (x_2, h(t, x_1, x_2)), \quad (2.45)$$

then

$$x \in C_l \iff y \in C_l, \quad \dot{y} \in C_R. \quad (2.46)$$

Since

$$y \in C_l \iff \dot{y} \in C_R, \quad (2.47)$$

it results that

$$\dot{y} \in C_l \cap C_R \quad (2.48)$$

and hence

$$\lim_{t \rightarrow \pm\infty} \dot{y}(t) = 0. \quad (2.49)$$

Therefore, the boundary value problem defining the homoclinic solutions for (2.34) has the form

$$\ddot{y} = h(t, y, \dot{y}), \quad y(-\infty) = y(+\infty), \quad \dot{y}(-\infty) = \dot{y}(+\infty) = 0. \quad (2.50)$$

We give an example to obtain the  $C_R$ -boundedness of  $F(D)$  in this case.

## 10 Existence problems for homoclinic solutions

Let  $\alpha_1, \alpha_2 \in C_R$ ,  $\alpha_1, \alpha_2$  be positive; in addition, suppose that  $\alpha_2(\pm\infty) = 0$ . Let  $\gamma, \beta : \mathbb{R} \rightarrow \mathbb{R}$  be two continuous and positive functions. We take as fundamental space  $X = C_l \times C_{Rl}$ , where  $C_{Rl} := C_R \cap C_l = \{x \in C_R, x(\pm\infty) = 0\}$ .

Let  $D_1$  be a bounded set in  $C_R$  and let  $D = D_1 \times D_2$ , where

$$D_2 = \{x \in C_R, |x(t)| \leq \alpha_2(t), t \in \mathbb{R}\}. \quad (2.51)$$

It is easy to check that if

$$|h(t, x_1, x_2)| \leq k_1 \alpha_1(t) \gamma |x_1| + k_2 |x_2| \beta(|x_2|), \quad (2.52)$$

then  $F(D)$  is  $C_R$ -bounded (more precisely,  $C_R \times C_R$ -bounded).

The case of second-order equation is different from the first-order equation; this is why it will not be treated here, but it will make the object of a future note.

**2.7. Remarks on the topological degree of the admissible operators.** Let  $\Omega \subset X$  be an open and bounded set, where  $X$  is  $C_l$  or  $C_{ll}$ .

Suppose that  $F(\Omega)$  is  $C_R$ -bounded, for an admissible operator  $U$ , if

$$x \neq Ux, \quad x \in \partial\Omega, \quad (2.53)$$

where  $\partial\Omega$  is the boundary of  $\Omega$ , we can consider its topological degree

$$\deg(I - U, \Omega, 0). \quad (2.54)$$

If this degree is nonzero, then  $U$  admits fixed points and so problem (1.1) has solutions.

As we said, the results contained in this section are based on the ones by Mawhin related to the boundary value problem

$$\dot{x} = f(t, x), \quad x(0) = x(T). \quad (2.55)$$

This author proves that the associated operators to problem (2.55) in the space  $C_{(0,T)}$  or  $C_{[0,T]}$  are compact on the bounded sets without supplementary conditions on the mapping  $f$  as it was to be expected. Moreover, these operators have the same topological degree which does not depend on the choice of  $L, N, P, Q$ . In addition, if in particular  $f(t, x) = g(x)$ , then for each associated operator  $U$  to problem (2.55) on the bounded and open set  $\Omega$  from  $C_{(0,T)}$  (or  $C_{[0,T]}$ ), we have

$$\deg(I - U, \Omega, 0) = (-1)^n \deg_B(g, \Omega \cap \mathbb{R}^n, 0), \quad (2.56)$$

where  $\deg_B$  denotes the Brouwer degree.

The associated operators to problem (1.1) on  $\Omega$  from  $C_l$  or  $C_{ll}$  have the degrees invariant with respect to  $L, N, P, Q$ ; the proof, based on the invariance of topological degree to homeomorphisms, is essentially simple but complicated to achieve. As we do not use this property in the present paper, we renounce to its proof.

Finally we make only a remark on the isomorphism  $\Phi$  given by (2.9).

Let  $\Omega \subset X$  be an open and bounded set in  $X$  ( $X = C_l$  or  $C_{ll}$ ) and let  $U$  be an admissible operator on  $\Omega$  for problem (1.1) fulfilling (2.56).

Set

$$\Omega_\Phi := \Omega(\Phi), \quad U_\Phi := \Phi U \Phi^{-1}, \quad (2.57)$$

where  $\Phi$  is given by (2.9) with  $a = 0$ ,  $b = T$ .

Then  $\Omega_\Phi$  is open and bounded,  $\Phi(\partial\Omega) = \partial\Omega_\Phi$  and  $\Omega_\Phi \subset C_{(0,T)}$  (resp.,  $\Omega_\Phi \subset C_{[0,T]}$ ).

Furthermore,  $U_\Phi$  is compact and since  $\partial\Omega_\Phi = \Phi(\partial\Omega)$ , we have

$$x \neq Ux, x \in \partial\Omega \iff x \neq U_\Phi x, x \in \partial\Omega_\Phi. \quad (2.58)$$

Hence

$$\deg(I - U, \Omega, 0) = \deg(I - U_\Phi, \Omega_\Phi, 0). \quad (2.59)$$

### 3. Existence results in the hypothesis of uniqueness of solutions

**3.1. Introduction.** Let  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous function; consider again problem (1.1).

We research the existence of solutions for problem (1.1) in the hypothesis that the Cauchy problem

$$\dot{x} = f(t, x), \quad x(0) = y \quad (3.1)$$

has a unique solution defined on the whole real axis  $\mathbb{R}$ , for every  $G$  a bounded set in  $\mathbb{R}^n$  and for every  $y \in G$ ; denote the solution of (3.1) by

$$x(t; y), \quad y \in G. \quad (3.2)$$

The uniqueness condition is fulfilled in particular if  $f(t, x)$  is locally Lipschitz with respect to  $x$ . Condition (2.41) is sufficient to assure the existence on  $\mathbb{R}$  of the solution (3.2), it is in particular fulfilled in conditions of type (2.41) and even more general.

It is known that the uniqueness condition assures the continuous dependence of the function  $x(t; \cdot)$ ; this property would be stated as: for every  $[a, b] \subset \mathbb{R}$  and for every  $y_n \in G$ ,  $y_n \rightarrow y \in G$ , the sequence  $x(t; y_n)$  converges uniformly on  $[a, b]$  to  $x(t; y)$ .

In this section, we present certain existence results for problem (1.1), exploiting this continuous dependence with respect to initial data.

**3.2. Generalized Poincaré operator.** Let  $\Omega \subset C_l$  be a bounded and open set; let

$$G := \{y \in \mathbb{R}^n, x(\cdot; y) \in \Omega\}. \quad (3.3)$$

Obviously,  $G$  is a bounded and open set.

## 12 Existence problems for homoclinic solutions

THEOREM 3.1. *Suppose that*

- (i)  $F\bar{\Omega}$  is  $C_R$ -bounded;
- (ii) for every  $y \in \partial G$  and for every  $t > 0$

$$x(t; y) \neq x(-t; y); \quad (3.4)$$

(iii) for every  $y \in \partial G$

$$f(0, y) \neq 0; \quad (3.5)$$

(iv) for every  $y \in \partial G$

$$\deg_B(f(0, y), G, 0) \neq 0. \quad (3.6)$$

Then problem (1.1) has solutions in  $\bar{\Omega}$ .

*Proof.* By hypothesis (i) it results that  $x(\cdot; y) \in C_l$ , for every  $y \in \bar{G}$ ; set

$$Py = \frac{1}{2} [x(+\infty; y) - x(-\infty; y)] \quad (3.7)$$

(we call  $P$  the *generalized Poincaré operator*). It is easy to check that the solution  $x(\cdot; y) \in C_{ll}$  if and only if  $Py = 0$ .

We want to show that  $P : \bar{G} \rightarrow \bar{G}$  is continuous; for this aim we remark that

$$Py = \frac{1}{2} \int_{-\infty}^{+\infty} f(s, x(s; y)) ds. \quad (3.8)$$

By hypothesis (i) it results that the integral in (3.8) is uniformly convergent with respect to  $y \in \bar{G}$ ; on the other hand, since the mapping  $y \rightarrow x(\cdot; y)$  is continuous (as mentioned in the previous paragraph) we conclude the continuity of the mapping  $(t, y) \rightarrow f(t, x(t; y))$  on every set of type  $[-A, A] \times \bar{G}$ . Hence the mapping  $y \rightarrow Py$  is continuous on  $\bar{G}$ .

Define the application  $h : \bar{G} \times [0, 1] \rightarrow \mathbb{R}^n$  by

$$h(y, \lambda) := \begin{cases} \frac{1}{2\lambda} \left[ x\left(\frac{\lambda}{1-\lambda}; y\right) - x\left(\frac{\lambda}{\lambda-1}; y\right) \right], & \lambda \in (0, 1), y \in \bar{G}, \\ Py, & \lambda = 1, \\ f(0, y), & \lambda = 0. \end{cases} \quad (3.9)$$

By L'Hospital rule,

$$\lim_{\lambda \downarrow 0} h(y, \lambda) = f(0, y). \quad (3.10)$$

Since

$$\lim_{\lambda \uparrow 1} h(y, \lambda) = Py, \quad (3.11)$$

it follows that  $h$  is continuous.

If for  $y \in \partial G$  we have  $Py = 0$ , then  $x(\cdot; y)$  is a solution for (1.1).

Suppose then  $Py \neq 0$ , for every  $y \in \partial G$ ; by hypotheses (ii) and (iii) it results that

$$h(y, \lambda) \neq 0, \quad \forall \lambda \in [0, 1], \forall y \in \partial G. \quad (3.12)$$

By homotopic invariance property of the topological degree it results that  $\deg_B(h(\cdot, \lambda), G, 0)$  is constant for  $\lambda \in [0, 1]$ ; in particular,

$$\deg_B(h(\cdot, 0), G, 0) = \deg_B(h(\cdot, 1), G, 0), \quad (3.13)$$

that is,

$$\deg_B(P, G, 0) = \deg_B(f(\cdot, y), G, 0) \quad (3.14)$$

and hence, by (3.6)

$$\deg_B(P, G, 0) \neq 0, \quad (3.15)$$

which assures the existence of  $y \in G$  with  $Py = 0$ . The theorem is proved.  $\square$

**3.3. The case  $\Omega$  connected.** The advantage of the previous result is that the topological degrees appearing are Brouwer degrees; the drawback is that condition (3.4) is not easy to be checked. We state now another existence result.

As usual, suppose that  $\Omega \subset C_l$  is a bounded and open set; define on  $\bar{\Omega}$  the operators

$$Hx = \int_0^{(\cdot)} (Fx)(s) ds, \quad S = I - H. \quad (3.16)$$

LEMMA 3.2. *If*

- (i)  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is locally Lipschitz with respect to the second variable;
- (ii)  $F(\bar{\Omega})$  is  $C_R$ -bounded,

*then  $S : \bar{\Omega} \rightarrow C_l$  is injective.*

*Proof.* Let  $x, z \in \bar{\Omega}$  such that

$$S(x) = S(z). \quad (3.17)$$

If  $x \neq z$ , then there exists  $t_0 \in \mathbb{R}$  such that  $x(t_0) = z(t_0)$ ; we can assume that  $t_0 > 0$ . Let  $A > 0$  be such that  $t_0 \in [0, A]$  and  $r = \max\{\|x\|_\infty, \|z\|_\infty\}$ .

Since

$$\begin{aligned} \exists L_r > 0, \quad \forall u, v \in \Sigma(r), \quad |f(t, u) - f(t, v)| &\leq L_r |u - v|, \\ |x(t) - z(t)| &\leq L_r \int_0^t |u(s) - v(s)| ds, \quad t \in [0, A], \end{aligned} \quad (3.18)$$

## 14 Existence problems for homoclinic solutions

we obtain by using Gronwall's lemma

$$x(t) = z(t), \quad \forall t \in [0, A]. \quad (3.19)$$

□

*Remark 3.3.* The mapping  $S : \bar{\Omega} \rightarrow S(\bar{\Omega})$  is a homeomorphism. In addition, since  $H$  is a compact operator,  $S^{-1}$  is a compact perturbation of identity, too.

*Remark 3.4.* If  $y \in \mathbb{R}^n \cap S(\bar{\Omega})$ , then

$$S^{-1}y = x(\cdot; y). \quad (3.20)$$

Set

$$\Pi x := x(0) - Px. \quad (3.21)$$

Observe that the operator

$$U := \Pi + H \quad (3.22)$$

is just the admissible operator (2.22), where  $b = 0$  and  $a = 1/2$ .

*Remark 3.5.* The following identity holds:

$$I - U = (I - \Pi S^{-1})S. \quad (3.23)$$

**THEOREM 3.6.** *Assume that the following hypotheses are fulfilled:*

- (i)  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is locally Lipschitz;
- (ii)  $\Omega \subset C_l$  is a connected, open, and bounded set;
- (iii)  $F(\bar{\Omega})$  is  $C_R$ -bounded;
- (iv) the following relations hold:

$$\begin{aligned} x &\neq Ux, \quad x \in \partial\Omega, \\ y &\neq Py, \quad y \in \partial(S(\bar{\Omega}) \cap \mathbb{R}^n). \end{aligned} \quad (3.24)$$

Then

$$\deg(I - U, \Omega, 0) = \pm \deg_B(P, S(\Omega) \cap \mathbb{R}^n, 0). \quad (3.25)$$

In addition, if

$$\deg_B(P, S(\Omega) \cap \mathbb{R}^n, 0) \neq 0, \quad (3.26)$$

then problem (1.1) admits solutions.

*Proof.* By identity (3.22) and applying the Leray-Schauder result for topological degree of product operators, we get

$$\deg(I - U, \Omega, 0) = \deg(S - y, \Omega, 0) \cdot \deg(I - \Pi S^{-1}, S(\Omega), 0), \quad (3.27)$$

where  $y \in S(\Omega)$  is arbitrary. Since  $S(\Omega)$  is connected and  $S : \bar{\Omega} \rightarrow S(\bar{\Omega})$  is a homeomorphism, then

$$|\deg(S-y, \Omega, 0)| = 1, \quad \forall y \in S(\Omega). \quad (3.28)$$

Since  $\Pi S^{-1}$  takes values in  $\mathbb{R}^n$ , we have

$$\deg(I - \Pi S^{-1}, S(\Omega), 0) = \deg_B(I - \Pi S^{-1}, S(\Omega) \cap \mathbb{R}^n, 0) \quad (3.29)$$

and since

$$\Pi S^{-1} = I - P, \quad (3.30)$$

it results (3.25).

Finally, if (3.26) is satisfied, then  $U$  admits fixed points and since  $U$  is associated to (1.1), every fixed point is a solution for problem (1.1).  $\square$

*Remark 3.7.* If condition (3.4) is fulfilled for every  $t \in \bar{\mathbb{R}}$  and  $y \in \partial G$ , then

$$\deg(I - U, \Omega, 0) = \pm \deg_B(f(\cdot, y), G, 0). \quad (3.31)$$

*Remark 3.8.* Formula (3.31) is available for every operator  $U : \bar{\Omega} \subset C_l \rightarrow C_l$  admissible for problem (1.1).

**3.4. Existence results using Miranda's theorem.** Let  $K = \Pi_{i=1}^n [-l, l] \subset \mathbb{R}^n$  and  $\Phi : K \rightarrow \mathbb{R}^n$  be a continuous function; denote by  $\Phi_i$  the  $i$ th component of  $\Phi$  and by  $y_i$  the  $i$ th component of  $y \in \mathbb{R}^n$ . Define  $L_i^+, L_i^- \subset \mathbb{R}^n$  by

$$\begin{aligned} L_i^+ &:= (y_1, \dots, y_{i-1}, l, y_{i+1}, \dots, y_n), \\ L_i^- &:= (y_1, \dots, y_{i-1}, -l, y_{i+1}, \dots, y_n), \quad i \in \overline{1, n}. \end{aligned} \quad (3.32)$$

Remark that if we take in  $\mathbb{R}^n$  the norm

$$|y| = \max_{1 \leq i \leq n} \{ |y_i| \}, \quad (3.33)$$

then, if  $|y_j| \leq l$ ,  $j \neq i$ , it results that  $K = \overline{\Sigma(l)}$  and  $L_i^+, L_i^-$  are on two contrary faces of a hypercube  $K$  (so  $L_i^+, L_i^- \in \partial K$ ).

Miranda's theorem states that, if

$$\begin{aligned} \Phi_i(L_i^+) &\leq 0, \quad \Phi_i(L_i^-) \geq 0, \quad i \in \overline{1, n}, \\ |y_j| &\leq l, \quad j \neq i, \end{aligned} \quad (3.34)$$

then the equation

$$\Phi(y) = 0 \quad (3.35)$$

admits solutions in  $K$ .

Suppose that  $f$  satisfies the following hypotheses:

## 16 Existence problems for homoclinic solutions

(H<sub>1</sub>) for every  $l > 0$ , problem (3.1) has a unique solution defined on the whole  $\mathbb{R}$ , for every  $y \in K$ ;

(H<sub>2</sub>) the functions  $\alpha_i(t) := \inf_{x \in \mathbb{R}^n} \{f_i(t, x)\}$ ,  $\beta_i(t) := \sup_{x \in \mathbb{R}^n} \{f_i(t, x)\}$  (where  $f = (f_i)_{i \in \overline{1, n}}$ ) are defined on  $\mathbb{R}$  and

$$\alpha_i, \beta_i \in L^1(\mathbb{R}), \quad i \in \overline{1, n}; \quad (3.36)$$

(H<sub>3</sub>) there exists a constant  $c > 0$  such that for every  $i \in \overline{1, n}$  and for every  $(t, y) \in \mathbb{R} \times \mathbb{R}^n$  with  $|y_i| > c$ , we have

$$y_i \cdot f_i(t, y) \geq 0. \quad (3.37)$$

**THEOREM 3.9.** *Assume that the hypotheses (H<sub>1</sub>), (H<sub>2</sub>), and (H<sub>3</sub>) are fulfilled. Then problem (1.1) admits solutions.*

*Proof.* Consider the operator  $P$  on  $K$  given by (3.8), that is,

$$Py = \frac{1}{2} \int_{-\infty}^{+\infty} f(s; x(s, y)) ds. \quad (3.38)$$

The operator  $P$  is well defined since hypotheses (H<sub>1</sub>), (H<sub>2</sub>), and (H<sub>3</sub>) are assumed; in addition, as remarked, it is continuous on  $\mathbb{R}^n$ .

Set

$$\begin{aligned} a_i &:= \inf_{t \in \mathbb{R}} \int_0^t \alpha_i(s) ds, & b_i &:= \sup_{t \in \mathbb{R}} \int_0^t \beta_i(s) ds, \\ a &:= \max_{1 \leq i \leq n} \{a_i\}, & b &:= \min_{1 \leq i \leq n} \{b_i\}. \end{aligned} \quad (3.39)$$

Then we have for every solution  $x(t; y) = (x_i(t; y))_{i \in \overline{1, n}}$ ,

$$y_i + a \leq x_i(t; y) \leq y_i + b, \quad i \in \overline{1, n}. \quad (3.40)$$

Considering  $l \geq 0$  such that

$$l \geq \max\{c - a, c + b\}, \quad (3.41)$$

we obtain

$$x_i(t; L_i^+) \geq c, \quad x_i(t; L_i^-) \leq -c, \quad \forall i \in \overline{1, n}, \quad \forall L_i^+, L_i^- \in \partial K. \quad (3.42)$$

If relation (3.41) is fulfilled, it follows from (H<sub>3</sub>),

$$P_i(L_i^+) \leq 0, \quad P_i(L_i^-) \geq 0, \quad \forall i \in \overline{1, n}, \quad \forall L_i^+, L_i^- \in \partial K, \quad (3.43)$$

where  $P = (P_i)_{i \in \overline{1, n}}$ .

Applying Miranda's theorem, it results that  $P$  has a zero in  $K$ . The proof is now complete.  $\square$

#### 4. Continuation method

**4.1. Introduction.** In this section  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function,  $X$  is the space  $C_l$  or  $C_{ll}$  and  $\Omega \subset X$  is an open and bounded set. If  $F\bar{\Omega}$  is  $C_R$ -bounded, as remarked in [Section 2](#), one can associate to problem (1.1) operators  $U : \bar{\Omega} \rightarrow X$  which are compact and whose fixed points coincide with the solutions of (1.1).

In particular, if

$$x \neq Ux, \quad x \in \partial\Omega, \quad (4.1)$$

then we can define the topological degree of  $U$  and if

$$\deg(I - U, \Omega, 0) \neq 0, \quad (4.2)$$

then  $U$  admits fixed points in  $\Omega$ .

However, when we face to check condition (4.2), then we can use the so-called *continuation method*, which is based on the well-known homotopic invariance property of the topological degree (used in [Section 3](#)).

One of the most used forms of this method is the following. Let  $h : \mathbb{R} \times \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^n$  be a continuous and  $C_R$ -bounded on  $\bar{\Omega}$  function in the sense that there exists  $\theta \in C_R$ ,  $\theta > 0$ , such that for every  $x \in \bar{\Omega}$  and for every  $\lambda \in [0, 1]$  we have  $|h(t, x(t), \lambda)| \leq \theta(t)$ ,  $t \in \mathbb{R}$ .

Consider the problem

$$\dot{x} = h(t, x, \lambda), \quad x(+\infty) = x(-\infty). \quad (4.3)$$

We can associate to problem (4.3) an operator  $U_\lambda$  which in addition is compact for every  $\lambda$ .

If the condition

$$x \neq U_\lambda x, \quad x \in \partial\Omega, \quad \lambda \in [0, 1] \quad (4.4)$$

is fulfilled, then we can define the degree  $\deg(I - U_\lambda, \Omega, 0)$ ; but a homotopic invariance property tells us that this degree is constant with respect to  $\lambda$ . In particular,

$$\deg(I - U_0, \Omega, 0) = \deg(I - U_1, \Omega, 0). \quad (4.5)$$

Equality (4.5) is useful if  $U_0$  is an associated operator to problem (1.1) ( $h(t, x, 0) = f(t, x)$ ) and the degree of  $I - U_1$  is easier to be computed, for example, when it is a Brouwer degree.

Condition (4.4) can be formulated under the following form: *for every*  $\lambda \in [0, 1]$  problem (4.3) *has no solutions*  $x(\cdot; \lambda)$  *with*  $x \in \partial\Omega$ . If this condition is fulfilled, *every* associated operator  $U_\lambda$  satisfies (4.4) because the fixed points of an

## 18 Existence problems for homoclinic solutions

associated operator coincide with the set of solutions for the problem whose it is associated.

We get therefore the following proposition.

**PROPOSITION 4.1.** *Assume that*

- (i) *there exists  $\theta \in C_R$ ,  $\theta(t) \geq 0$ , such that  $|h(t, x(t), \lambda)| \leq \theta(t)$ , for every  $x \in \bar{\Omega}$ , for every  $\lambda \in [0, 1]$ ;*
- (ii) *for every  $\lambda \in [0, 1]$ , problem (4.3) does not admit solutions  $x(\cdot)$  with  $x \in \partial\Omega$ ;*
- (iii)  *$h(t, x, 1) = f(t, x)$ ;*
- (iv)  *$\deg(I - U_0, \Omega, 0) \neq 0$ .*

*Then problem (1.1) admits solutions.*

The question that problem (4.3) has no solutions in  $\partial\Omega$  can be formulated under the following form.

“A priori estimates”: for every possible solution  $x(\cdot)$  of problem (4.3) with  $x \in \bar{\Omega}$  we have  $x \in \Omega$ .

Another form of the same condition is the next.

“A priori bound”: there exists a number  $r > 0$  such that problem (4.3) does not admit solutions  $x(\cdot)$  with  $\|x\|_\infty = r$ .

In this case we set  $\Omega := \{x \in X, \|x\| < r\}$ .

Another variant of the same condition is the following.

“Bounded set condition”: for every  $\lambda \in [0, 1]$  for which problem (4.3) has solutions  $x(\cdot)$  with  $x(t) \in \bar{D}$ ,  $t \in \bar{\mathbb{R}}$ , we have  $x(t) \in D$ , for every  $t \in \bar{\mathbb{R}}$ .

In this case when  $D \subset \mathbb{R}^n$  is an open and bounded set we take  $\Omega := \{x \in X, x(t) \in D\}$ .

In this section, we indicate certain simple functions candidates to be homotopic linked through  $h$  with  $f$ , functions for which the computation of their topological degree is more advantageously.

The most difficult problem remains to establish the fact that problem (4.3) has no solutions in  $\partial\Omega$ ; in what follows we consider certain cases when this thing is easy to be checked.

**4.2. Homotopy with a linear equation.** In this paragraph consider  $X = C_{ll}$ .

Let  $A : \mathbb{R} \rightarrow M_n(\mathbb{R})$  be a continuous quadratic matrix; denote by  $|\cdot|$  an arbitrary norm for the constant matrices.

Consider the system

$$\dot{x} = A(t)x \quad (4.6)$$

and denote by  $X = X(t)$  its fundamental matrix with  $X(0) = I$ . In [5], the following result is proved.

**PROPOSITION 4.2.** *Assume that*

$$\int_{-\infty}^{+\infty} |A(t)| dt < \infty, \quad (4.7)$$

then there exists  $X(\pm\infty) = \lim_{t \rightarrow \pm\infty} X(t)$ . If in addition

$$\text{rank}[X(+\infty) - X(-\infty)] = n, \quad (4.8)$$

then the problem

$$\dot{x} = A(t)x, \quad x(+\infty) = x(-\infty) \quad (4.9)$$

admits only the zero solution.

**THEOREM 4.3.** Assume that

- (i)  $A : \mathbb{R} \rightarrow M_n(\mathbb{R})$  is a continuous matrix such that conditions (4.7), (4.8) are fulfilled;
- (ii)  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function fulfilling the condition

$$|f(t, x)| \leq \theta(t) \cdot \omega(|x|), \quad (4.10)$$

where  $\theta : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\omega : \mathbb{R} \rightarrow \mathbb{R}$  are continuous and positive functions,  $\theta \in C_R$ ;

- (iii) there exists  $r > 0$  such that for every  $\lambda \in [0, 1]$  the problem

$$\dot{x} = (1 - \lambda)A(t)x + \lambda f(t, x), \quad x(+\infty) = x(-\infty) \quad (4.11)$$

has no solution  $x(\cdot)$  such that  $\|x\|_\infty = r$ .

Then problem (1.1) admits solutions.

*Proof.* Set

$$\begin{aligned} h(t, x, \lambda) &= (1 - \lambda)A(t)x + \lambda f(t, x), \\ \Omega &= B(r) := \{x \in C_{ll}, \|x\|_\infty < r\}. \end{aligned} \quad (4.12)$$

For every  $x \in \bar{\Omega}$ , we have

$$|h(t, x(t), \lambda)| \leq \rho |A(t)| + \omega(\rho) \theta(t), \quad (4.13)$$

where

$$\rho = \sup_{|u| \leq r} \omega(u) \quad (4.14)$$

and so hypothesis (i) of [Proposition 4.1](#) is satisfied; obviously (iii) is satisfied, too.

For  $\lambda = 0$ , problem (4.3) becomes

$$\dot{x} = A(t)x, \quad x(+\infty) = x(-\infty) \quad (4.15)$$

which, by [Proposition 4.2](#), admits only the zero solution; that means every operator  $U_0$  attached to problem (4.15) is injective. Since  $U_0$  is linear and compact, then after a known property,

$$\deg(I - U_0, \Omega, 0) = \pm 1. \quad (4.16)$$

This ends the proof.  $\square$

Hypothesis (ii) of [Proposition 4.1](#) is difficult to be checked in practice. In the next theorems it will be fulfilled.

Consider the problem

$$\dot{x} = A(t)x + g(t, x) + p(t), \quad x(-\infty) = x(+\infty), \quad (4.17)$$

where  $g : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $p : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are continuous functions.

**THEOREM 4.4.** *Assume that*

- (i) *conditions (4.7), (4.8) are fulfilled;*
- (ii)  $p \in L^1(\mathbb{R}) \cap C_c$ ;
- (iii) *there exists  $\alpha \in (0, 1)$  such that*

$$g(t, x) = k^\alpha g(t, x), \quad \forall k > 0, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n; \quad (4.18)$$

- (iv) *the following inequality holds:*

$$|g(t, x)| \leq \theta(t), \quad \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n, |x| \leq 1, \quad (4.19)$$

where  $\theta \in C_R$ ,  $\theta(t) \geq 0$ ,  $(\forall t \in \mathbb{R})$ .

Then problem (4.17) admits solutions.

*Proof.* Set in [Proposition 4.1](#)

$$\begin{aligned} h(t, x, \lambda) &:= (1 - \lambda)A(t)x + \lambda((A(t)x) + p(t) + h(t, x)), \\ \Omega &= B(\rho) := \{x \in C_{ll}, \|x\|_\infty < \rho\}. \end{aligned} \quad (4.20)$$

By (4.18) it results that

$$(|x| \leq \rho) \implies (|g(t, x)| \leq \rho^\alpha \theta(t)), \quad (4.21)$$

which shows that for  $x \in \bar{\Omega}$ ,

$$|h(t, x, \lambda)| \leq 2\rho|A(t)| + \rho^\alpha \theta(t) + |p(t)| \in C_R. \quad (4.22)$$

Obviously, to apply [Proposition 4.1](#), it remains to check only hypothesis (ii). For this aim, we will show that there exists  $\rho_0 > 0$  such that for every  $\lambda \in [0, 1]$  and for every  $\rho > \rho_0$  problem (4.3) has no solution  $x(\cdot)$  with  $\|x\|_\infty = \rho$ .

Indeed, if not, then we could find a sequence  $\lambda_k \in [0, 1]$ , a sequence  $\rho_k \rightarrow \infty$ , such that the problem

$$\dot{x} = h(t, x, \lambda_k), \quad x(-\infty) = x(+\infty) \quad (4.23)$$

admits solutions  $x_k(\cdot)$  with

$$\|x_k\|_\infty = \rho_k. \quad (4.24)$$

Setting

$$u_k := \frac{1}{\|x_k\|_\infty} x_k = \frac{1}{\rho_k} x_k, \quad (4.25)$$

we have

$$\begin{aligned} \|u_k\|_\infty &= 1, \\ \dot{u}_k &= (1 - \lambda_k)A(t)u_k + \lambda_k[A(t)u_k + \rho_k^{\alpha-1}g(t, u_k) + \rho_k^{-1}p(t)]. \end{aligned} \quad (4.26)$$

By [Corollary 2.7](#), we get the compactness of the sequence  $(u_k)_k$  in  $C_{ll}$ .

Let  $u \in \overline{(u_k)_k}$ ,  $\lambda \in \overline{(\lambda_k)_k}$ ; by using the classical properties of uniform convergence we obtain, after computations,

$$\dot{u} = A(t)u, \quad u(-\infty) = u(+\infty), \quad \|u\|_\infty = 1, \quad (4.27)$$

which contradicts [Proposition 4.2](#).  $\square$

**4.3. Auxiliary results.** In [Section 4.2](#), the homotopy has been achieved through a linear mapping for which it was easy to evaluate its topological degree. We give rise to another case when the topological degree computation is not too difficult in the sense that it becomes a Brouwer degree. This result will be a consequence of a more general result which links the existence of solutions for problem (1.1) to the existence of solutions for the problems of the type

$$\dot{y} = g(t, y), \quad y(0) = y(T), \quad 0 < T < \infty. \quad (4.28)$$

Let  $\theta : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\theta \in C_R$ ,  $\theta(t) > 0$ , for every  $t \in \mathbb{R}$ ; set

$$\psi(t) := \int_{-\infty}^t \theta(s) ds, \quad \varphi := \psi^{-1}, \quad T := \int_{-\infty}^{+\infty} \theta(s) ds. \quad (4.29)$$

Obviously, through (2.9),  $\varphi : (0, T) \rightarrow \mathbb{R}$  determines by (2.9) an isomorphism between  $C_l$  and  $C_{(0, T)}$  (or between  $C_{ll}$  and  $C_{[0, T]}$ ).

**PROPOSITION 4.5.** *Suppose that  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function such that*

$$\lim_{t \rightarrow \pm\infty} \frac{1}{\theta(t)} f(t, y) = \gamma_\pm(y), \quad y \in \mathbb{R}^n, \quad (4.30)$$

*the convergence being uniform with respect to  $y$  on every compact subset of  $\mathbb{R}^n$ .*

*Let*

$$g(t, y) := \begin{cases} \dot{\varphi}(t)f(\varphi(t), y), & \text{if } t \in (0, T), y \in \mathbb{R}^n, \\ \gamma_-(y), & \text{if } t = 0, y \in \mathbb{R}^n, \\ \gamma_+(y), & \text{if } t = 1, y \in \mathbb{R}^n. \end{cases} \quad (4.31)$$

*Then problem (1.1) admits solutions if and only if (4.28) admits solutions.*

## 22 Existence problems for homoclinic solutions

*Proof.* Remark that if  $D \subset C_l$  is a bounded set, then  $FD \subset C_R$ ; indeed, for  $|t| \geq A$ , we have

$$|f(t, y)| \leq (a+m)\theta(t), \quad (4.32)$$

where

$$m := \max \{ \sup_{|y| \leq r} \gamma_+(u), \gamma_-(u), u \in D \cap \mathbb{R}^n \}. \quad (4.33)$$

Set

$$\begin{aligned} r &:= \sup \{ \|x\|_\infty, x \in D \}, & \alpha(t) &:= \sup_{|y| \leq r} \{ |f(t, y)| \}, \quad t \in [-A, A], \\ \beta(t) &:= \begin{cases} \alpha(t), & |t| < A, \\ (a+m)\theta(t), & |t| \geq A. \end{cases} \end{aligned} \quad (4.34)$$

We obtain  $\beta \in C_R$  and

$$|f(t, x)| \leq \beta(t), \quad \forall t \in \mathbb{R}, \forall x \in D. \quad (4.35)$$

Let  $x(t)$  be a solution for (1.1); then  $y = \Phi(x)$  is a solution for the differential equation appearing in (4.28) on the interval  $(0, T)$ . Since  $y(t)$  has limits in 0 and  $T$ , it can be prolonged as solution on  $[0, T]$ ; but by definition of  $y(t)$  it follows that

$$y(0) = \varphi(x(-\infty)) = \varphi(x(+\infty)) = y(T). \quad (4.36)$$

The converse is proved by using the isomorphism  $\Phi^{-1}$ .

Let  $\Omega \subset C_l$  be an open and bounded set. Hypothesis (4.30) allows us, as remarked, to associate to problem (1.1) the operator

$$U : \bar{\Omega} \subset C_l \longrightarrow C_l, \quad Ux = x(+\infty) + \int_{-\infty}^{(\cdot)} (Fx)(s) ds, \quad (4.37)$$

which, from (4.35), is compact.

The operator  $U_\Phi$  defined in (2.15) is

$$U_\Phi : \overline{\Omega_\Phi} \subset C_{(0, T)} \longrightarrow C_{(0, T)}, \quad U_\Phi y := y(T) + \int_0^{(\cdot)} g(\tau, y(\tau)) d\tau. \quad (4.38)$$

But the operator  $U_\Phi$  is associated to problem (4.28). By using the remarks from 2.4 we obtain the following result.

**COROLLARY 4.6.** *If  $x \neq Ux$ , for every  $x \in \partial\Omega$ , then*

$$\deg(I - U, \Omega, 0) = \deg(I - U_\Phi, \Omega_\Phi, 0). \quad (4.39)$$

*(Obviously, the first degree is computed in  $C_l$ , the second in  $C_{(0, T)}$ .)*

An important particular case is

$$f(t, x) = \theta(t) \cdot g(x). \quad (4.40)$$

In this case, when (4.30) is fulfilled, (4.24) becomes

$$\dot{y} = g(y), \quad y(0) = y(T). \quad (4.41)$$

As it is proved in [7], for every associated operator to problem (4.41) in  $C_{(0, T)}$  or  $C_{[0, T]}$  (so for  $U_\Phi$ , too) we have

$$\deg(I - U_\Phi, \Omega_\Phi, 0) = \pm \deg_B(g, \Omega \cap \mathbb{R}^n, 0). \quad (4.42)$$

We obtain therefore the following proposition.

**PROPOSITION 4.7.** *Suppose that*

- (i)  $\theta : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\theta(t) \geq 0$ , for every  $t \in \mathbb{R}$ ;
- (ii)  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $g$  continuous.

*Consider the problem*

$$\dot{x} = \theta(t)g(x), \quad x(-\infty) = x(+\infty). \quad (4.43)$$

*Let  $\Omega \subset C_l$  be an open and bounded set. If for the operator  $U$  associated to problem (4.43), we have*

$$x \neq Ux, \quad x \in \partial\Omega, \quad (4.44)$$

*then*

$$\deg(I - U, \Omega, 0) = \pm \deg_B(g, \Omega \cap \mathbb{R}^n, 0). \quad (4.45)$$

*Furthermore, if*

$$\deg_B(g, \Omega \cap \mathbb{R}^n, 0) \neq 0, \quad (4.46)$$

*it results that (4.43) admits solutions.* □

**4.4. Homotopies with nonlinear equations.** We consider the problem

$$\dot{x} = f(t, x) + p(t), \quad x(-\infty) = x(+\infty). \quad (4.47)$$

Suppose that the following hypotheses are fulfilled:

- (a<sub>1</sub>)  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function;
- (a<sub>2</sub>)  $|f(t, x)| \leq \beta(t)$ ,  $x \in \mathbb{R}^n$ ,  $|x| \leq 1$ ,  $t \in \mathbb{R}$ ,  $\beta \in C_R \cap C_c$ ;
- (a<sub>3</sub>) there exists  $\alpha \in (0, 1)$ ,  $f(t, kx) = k^\alpha \cdot f(t, x)$ ,  $k > 0$ ,  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^n$ ;
- (a<sub>4</sub>)  $|p| \in C_R$ .

## 24 Existence problems for homoclinic solutions

In addition, let  $\theta \in C_R$ ,  $\theta(t) > 0$ , with  $\int_{-\infty}^{+\infty} \theta(t) dt = 1$ .  
Set

$$g(x) := \int_{-\infty}^{+\infty} f(s, x) ds. \quad (4.48)$$

**THEOREM 4.8.** *Assume that hypotheses  $(a_1)$ ,  $(a_2)$ ,  $(a_3)$ , and  $(a_4)$  are fulfilled. Consider the problem*

$$\dot{x} = (1 - \lambda) \theta(t) g(x) + \lambda [f(t, x) + p(t)], \quad x(-\infty) = x(+\infty). \quad (4.49)$$

Then

(1) if

$$g(y) \neq 0, \quad y \in \mathbb{R}^n, \|y\| = 1, \quad (4.50)$$

it results that there exists  $\rho > 0$  such that for every  $\lambda \in [0, 1]$ , problem (4.49) has no solution  $x(\cdot)$  with  $\|x\|_\infty = \rho_0$ ;

(2) if for this  $\rho_0$

$$\deg_B(g, \Sigma(\rho_0), 0) \neq 0, \quad (4.51)$$

then (4.47) admits solutions.

*Proof.* If conclusion (1) is not true, then there would exist the sequences  $(\rho_k)_k \subset (0, \infty)$ ,  $(x_k)_k \subset X$ , with  $\|x_k\|_\infty = \rho_k$ ,  $\lambda_k \in [0, 1]$ ,  $\rho_k > k$  and

$$\dot{x}_k = (1 - \lambda_k) \theta(t) g(x_k) + \lambda_k [f(t, x_k) + p(t)], \quad x(-\infty) = x(+\infty). \quad (4.52)$$

Setting

$$u_k = \frac{x_k}{\rho_k}, \quad (4.53)$$

we get

$$\begin{aligned} \dot{u}_k &= \rho_k^{-1} [(1 - \lambda_k) \theta(t) g(u_k) + \lambda_k f(t, u_k)] + \rho_k^{-1} \lambda_k p(t), \\ \|u_k\| &= 1, \quad u_k(-\infty) = u_k(+\infty). \end{aligned} \quad (4.54)$$

Based on [Corollary 2.7](#), it results that  $(u_k)_k$  is relatively compact in  $X$ . We can assume, up to subsequences, that  $u_k \rightarrow u$ ,  $\lambda_k \rightarrow \lambda$ ; we have

$$\|u\|_\infty = 1. \quad (4.55)$$

By (4.50) it results that  $\dot{u}_k \rightarrow 0$ , in  $X$ . Therefore  $u \in \mathbb{R}^n$ .

On the other hand, since

$$\int_{-\infty}^{+\infty} \dot{u}_k(s) ds = 0 \quad (4.56)$$

and (4.54), it follows that

$$0 = (1 - \lambda_k) \int_{-\infty}^{+\infty} \theta(t)g(u_k(t)) dt \\ + \lambda_k \int_{-\infty}^{+\infty} f(s, u_k(s)) ds + \rho_k^{-1} \lambda_k p(t); \quad (4.57)$$

hence, for  $k \rightarrow \infty$ , we get

$$g(u) = 0, \quad u \in \mathbb{R}^n, \|u\| = 1, \quad (4.58)$$

which contradicts (4.50).  $\square$

The second part of the theorem follows then by [Proposition 4.1](#) for  $\Omega = B(\rho_0)$  and [Proposition 4.7](#).

A similar result can be obtained in the case  $\alpha > 1$ , if  $\int_{-\infty}^{+\infty} |p(t)| dt < 1$ .

**4.5. Small perturbations.** This paragraph deals with the problem

$$\dot{x} = \theta(t)g(x) + e(t, x), \quad x(-\infty) = x(+\infty), \quad (4.59)$$

where  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\theta : \mathbb{R} \rightarrow \mathbb{R}$ ,  $e : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  are continuous functions; in addition  $\theta(t) > 0$ ,  $t \in \mathbb{R}$ ,  $\theta \in C_R$ .

Consider problem (4.43). Let  $D \subset \mathbb{R}^n$  be an open and bounded set.

Assume the following hypotheses:

- (b<sub>1</sub>) for every solution  $x(\cdot)$  of problem (4.43) for which  $x(t) \in \bar{D}$ ,  $t \in \mathbb{R}$ , it results that  $x(t) \in D$ ,  $t \in \bar{\mathbb{R}}$ ;
- (b<sub>2</sub>)  $e(t, x)$  is  $C_R$ -bounded on  $\bar{\Omega}$ , where

$$\Omega := \{x \in X, x(t) \in D, t \in \mathbb{R}\}. \quad (4.60)$$

Finally, consider the problem

$$\dot{x} = \theta(t)g(x) + \lambda e(t, x), \quad x(-\infty) = x(+\infty), \lambda \in [0, 1]. \quad (4.61)$$

**THEOREM 4.9.** *If hypotheses (b<sub>1</sub>) and (b<sub>2</sub>) are fulfilled, there exists  $\varepsilon_0 > 0$  such that if*

$$\|e(\cdot, y)\|_{\infty} < \varepsilon_0, \quad \forall y \in \partial D, \quad (4.62)$$

*then for every solution  $x(\cdot)$  of problem (4.61) for which  $x(t) \in \bar{D}$ , for every  $t \in \mathbb{R}$ , it results that  $x(t) \in D$ , for every  $t \in \bar{\mathbb{R}}$ .*

*If, in addition,*

$$\deg_B(g, D, 0) \neq 0, \quad (4.63)$$

*then for every  $e(\cdot, \cdot)$  satisfying (4.62), problem (4.59) admits solutions.*

## 26 Existence problems for homoclinic solutions

*Proof.* We prove the first part. If the conclusion is not true, then for every  $k \in \mathbb{N}^*$  there exists a function  $e_k(\cdot, \cdot)$  with  $\|e_k(\cdot, y)\|_\infty < 1/k$ , for every  $y \in \bar{D}$  and a function  $x_k(\cdot)$  such that

$$\dot{x}_k = \theta(t)g(x_k) + \lambda_k e(t, x_k), \quad x_k(-\infty) = x_k(+\infty), \quad \lambda_k \in [0, 1], \quad (4.64)$$

with  $x_k(t) \in \bar{D}$ , for every  $t \in \bar{\mathbb{R}}$ , and  $x_k(t_k) \notin \partial D$ , for an  $t_k \in \mathbb{R}$ .

**Corollary 2.7** assures the compactness of the sequence  $(x_k)_k$  in  $C_l$ . If  $x_k \rightarrow x$  in  $C_l$ ,  $\lambda_k \rightarrow \lambda$ , and  $t_k \rightarrow t \in \bar{\mathbb{R}}$ , one contradicts hypothesis (b<sub>1</sub>).

The second part follows by Propositions 4.1 and 4.7 for  $\Omega := \{x \in C_l, x(t) \in D, \text{ for every } t \in \mathbb{R}\}$ .  $\square$

**4.6. Asymptotically homogeneous systems.** Consider again the problems

$$\dot{x} = \theta(t)g(x) + e(t, x), \quad x(-\infty) = x(+\infty), \quad (4.65)$$

$$\dot{x} = \theta(t)g(x), \quad x(-\infty) = x(+\infty), \quad (4.66)$$

$$\dot{x} = \theta(t)g(x) + \lambda e(t, x), \quad x(-\infty) = x(+\infty), \quad \lambda \in [0, 1]. \quad (4.67)$$

Assume the following hypotheses:

(c<sub>1</sub>)  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function such that

$$g(kx) = g(x), \quad x \in \mathbb{R}^n, \quad k > 0; \quad (4.68)$$

(c<sub>2</sub>)  $\lim_{|x| \rightarrow \infty} e(t, x)/|x| = 0$ ;

(c<sub>3</sub>) for every  $\rho > 0$ , there exists  $\alpha_\rho \in C_R$ ,  $\alpha_\rho > 0$ , for every  $x \in \Sigma(\rho)$ ,

$$|e(t, x)| \leq \alpha_\rho(t), \quad t \in \mathbb{R}; \quad (4.69)$$

(c<sub>4</sub>)  $\theta \in C_R$ ,  $\theta(t) > 0$ , for every  $t \in \mathbb{R}$ ;

(c<sub>5</sub>) problem (4.43) admits only the zero solution.

**THEOREM 4.10.** *Assuming that hypotheses (c<sub>1</sub>), (c<sub>2</sub>), (c<sub>3</sub>), (c<sub>4</sub>), and (c<sub>5</sub>) are fulfilled. Then there exists  $\rho_0 > 0$ , such that for every solution  $x(\cdot)$  for problem (4.67) and for every  $\lambda \in [0, 1]$*

$$\|x\|_\infty < \rho_0. \quad (4.70)$$

*If, in addition,*

$$\deg_B(g, \Sigma(\rho_0), 0) \neq 0, \quad (4.71)$$

*then problem (4.65) admits solutions.*

The proof is analogous with the proofs of Theorems 4.8 and 4.9. If the first conclusion is not true, then one finds  $\lambda_k \in [0, 1]$  and  $x_k \in X$  satisfying

$$\dot{x}_k = \theta(t)g(x_k) + \lambda_k e(t, x_k), \quad x_k(-\infty) = x_k(+\infty), \quad \|x_k\|_\infty \rightarrow \infty. \quad (4.72)$$

Setting again  $u_k = x_k/\|x_k\|$ , we deduce that  $(u_k)_k$  is compact in  $X$ ; if  $u \in \overline{\{x_k\}_k}$ , then  $u$  satisfies (4.64) and  $\|u\|_\infty = 1$ , which contradicts hypothesis  $(c_5)$ .

The second part is an immediate consequence of Propositions 4.1 and 4.7 and of condition (4.71).

## References

- [1] C. Avramescu, *Sur l'existence des solutions convergentes des systèmes d'équations différentielles non linéaires*, Ann. Mat. Pura Appl. (4) **81** (1969), 147–168 (French).
- [2] ———, *Quelques remarques concernant l'existence et l'approximation des solutions d'une équation non linéaire dans un espace de Banach*, Ann. Univ. Craiova Ser. Mat. Inform. **22** (1995), 14–23 (French).
- [3] ———, *Sur un problème bilocal sur un intervalle infini*, Ann. Univ. of the West, Timișoara **37** (1999), no. 1, 8–16 (French).
- [4] ———, *Continuation theorems and closed orbits for ordinary differential equations*, preprint no. 131, Ann. Univ. of the West, Timișoara, 2001.
- [5] C. Avramescu and C. Vladimirescu, *Homoclinic solutions for linear and linearizable ordinary differential equations*, Abstr. Appl. Anal. **5** (2000), no. 2, 65–85.
- [6] ———, *Homoclinic solutions for second order differential equations*, Ann. Univ. Craiova **27** (2000), 1–8.
- [7] A. Capietto, J. Mawhin, and F. Zanolin, *Continuation theorems for periodic perturbations of autonomous systems*, Trans. Amer. Math. Soc. **329** (1992), no. 1, 41–72.
- [8] A. G. Kartsatos, *Advanced Ordinary Differential Equations*, Mariner, Florida, 1980.
- [9] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, Regional Conference Series in Mathematics, vol. 40, American Mathematical Society, Rhode Island, 1979.
- [10] ———, *Topological degree and boundary value problems for nonlinear differential equations*, Topological Methods for Ordinary Differential Equations (Montecatini Terme, 1991) (M. Furi et al., eds.), Lect. Notes Math., vol. 1537, Springer, Berlin, 1993, pp. 74–142.
- [11] ———, *Continuation Theorems and Ordinary Differential Equations*, edited by A. Gramas and M. Frigon, Kluwer, Dordrecht, 1995.
- [12] C. Vladimirescu, *Associated operators to Homoclinic solutions*, Ann. Univ. Craiova **27** (2000), 61–73.
- [13] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. II/B*, Springer-Verlag, New York, 1990.

CEZAR AVRAMESCU: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CRAIOVA, 13 AI CUZA STREET, CRAIOVA 1100, ROMANIA

E-mail address: [cezaravramescu@hotmail.com](mailto:cezaravramescu@hotmail.com)

# Special Issue on Singular Boundary Value Problems for Ordinary Differential Equations

### Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/bvp/guidelines.html>. Authors should follow the Boundary Value Problems manuscript format described at the journal site <http://www.hindawi.com/journals/bvp/>. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	May 1, 2009
First Round of Reviews	August 1, 2009
Publication Date	November 1, 2009

### Lead Guest Editor

**Juan J. Nieto**, Departamento de Análisis Matemático,  
Facultad de Matemáticas, Universidad de Santiago de

Compostela, Santiago de Compostela 15782, Spain;  
[juanjose.nieto.roig@usc.es](mailto:juanjose.nieto.roig@usc.es)

### Guest Editor

**Donal O'Regan**, Department of Mathematics, National  
University of Ireland, Galway, Ireland;  
[donal.oregan@nuigalway.ie](mailto:donal.oregan@nuigalway.ie)