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Research Article

Finding the Roots of System of Nonlinear Equations by a Novel Filled Function Method

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We present a novel filled function approach to solve box-constrained system of nonlinear equations. The system is first transformed into an equivalent nonsmooth global minimization problem, and then a new filled function method is proposed to solve this global optimization problem. Numerical experiments on several test problems are conducted and the computational results are also reported.

1. Introduction

Consider the system of nonlinear equations of the form: (SNE) : $F(x) = 0,\ x \in X$, where the mapping $F : R^n \to R^m$ is continuous, and $X \subset R^n$ is a box set.

Denote $F(x) = (f_1(x), f_2(x), \ldots, f_m(x))^T$, and let $f(x) = \Sigma_{k=1}^{m} |f_k(x)|$. To obtain the roots of (SNE), we may solve the following global optimization problem: (P) : $\min_{x \in X} f(x)$. Obviously, if (SNE), has at least one root, then the global optimizers of the problem (P) with the zero function value correspond to the roots of (SNE).

There are many theories and algorithms devoted to global optimization. Among of them, filled function method is a particularly useful tool (see [1–7]). The filled function method is originally introduced to solve smooth global optimization problem. In this paper, it is extended to solve nonsmooth global optimization problem (P). The filled function method for nonsmooth global optimization contains two phases: local minimization and filling. The local minimization phase is intended to search for a local optimizer x^* of the problem (P), and the filling phase aims to identify an improved initial point by minimizing the following filled function constructed at x^* : (FFP) $\min_{x\in X} F(x, x^*)$. The two phases are performed alternatively until no better optimizer can be found, and then the current optimizer is regarded as a global minimizer.

Comparing with the filled functions proposed in [1–7], our filled function has the following features.

(1) Since the functions f_i $(i = 1, \ldots, m)$ in this paper are assumed to be Lipschitz continuous only, and not necessarily to be continuous differentiable, so the problem (P) is a nonsmooth global optimization problem. Our filled function approach is proposed to tackle this kind of nonsmooth global optimization problem. Whereas these filled function methods proposed in the literature are mainly used for solving smooth global optimization problem.

(2) Although the filled function presented in [2] can be extended to solve nonsmooth global optimization problem, it has some disadvantages, such as it has two dependent parameters which are difficult to determine and hence make its algorithmic realization be fairly complicated. However, our proposed filled function contains only one parameter, which is easier to be appropriately chosen, and thus overcomes the disadvantages mentioned above in a certain extent.

(3) If each f_i is continuous differentiable and $f(x) = \Sigma_{k=1}^{m} f_k^2(x)$, then problem (P) is a smooth optimization problem. In this case, our proposed filled function method can also be used to solve this smooth problem, just as the above-mentioned filled function methods can do. Therefore, our filled function method is a natural extension of these filled function methods in the literature, and it has more scope of applications in the field of global optimization.

The paper is organized as follows: in Section 2, we make several assumptions on the problem. In Section 3, we present a novel filled function and investigate its properties. In Section 4, we describe the filled function algorithm. In Section 5, we make a numerical test. And, at last, in Section 6, we give our conclusion.

2. Basic Knowledge

In this section, we give a summary of main results on nonsmooth analysis, and define filled function for the problem (P). For more details, we refer the reader to [8].

Definition 2.1. Let f be Lipschitz with constant L at the point x, the generalized gradient of f at x is defined as

$$\partial f(x) = \left\{\xi \in R^n : \langle \xi, d\rangle \le f^0(x; d), \quad \forall d \in R^n\right\}, \tag{2.1}$$

where $f^0(x; d) = \limsup_{y\to x, t\downarrow 0}(f(y+td) - f(y))/t$ is the generalized directional derivative of $f(x)$ in the direction d at x.

Lemma 2.2. *Let f be Lipschitz with constant L at the point x, then*

(a) *$f^0(x; d)$ is finite, sublinear and satisfies $|f^0(x; d)| \le L\|d\|$,*

(b) *as a function of (x, d), $f^0(x; d)$ is super-semicontinuous; as a function of d, it is Lipschitz with constant L,*

(c) $\partial \Sigma s_i f_i(x) \subseteq \Sigma s_i \partial f_i(x)$, $\forall s_i \in R$,

(d) $\partial f(x)$ *is a nonempty compact convex set, and to any* $\xi \in \partial f(x)$, *one has* $\|\xi\| \leq L$,

(e) $\forall d \in X$, $f^0(x; d) = \max\{\langle \xi, d\rangle : \xi \in \partial f(x)\}$.

Consider the problem (P). To extend the filled function for smooth global optimization to nonsmooth case, throughout this paper, we make the following assumptions.

Assumption 1. The function $f_i(x)$ $(i = 1, 2, \ldots, m)$ is Lipschitz continuous on X with a rank $L_i > 0$. Then $f(x)$ is Lipschitz continuous with a rank $L = \sum_{i=1}^{m} L_i$.

Assumption 2. The value of $f(x)$ for x on the boundary of X is greater than that of $f(x)$ for any x within X.

Note that Assumption 2 implies that all optimizers of (P) lie in the interior of X.

Assumption 3. The problem (SNE) has at least one root.

Definition 2.3. A function $P(x, x^*)$ is called a filled function of $f(x)$ at a local minimizer x^*, if it has the following properties:

(1) $f(x)$ is a strictly maximizer of $P(x, x^*)$,

(2) for any $x \in S_1 = \{x \in X \setminus x^* : f(x) \geq f(x^*)\}$, one has $0 \notin \partial P(x, x^*)$,

(3) if x^* is not a global minimizer of (P), then $P(x, x^*)$ has at least one minimizer in the region $S_2 = \{x \in X : f(x) < f(x^*)\}$.

3. A New Filled Function and Its Properties

Let x^* be a local minimizer of (P), L the Lipschitz rank of $f(x)$, and $D = \max_{x_1, x_2 \in X} \|x_1 - x_2\|$. Define

$$F(x, x^*, r) = \left(\arctan f(x) - \frac{1}{5}\arctan f(x^*)\right)\exp\left(\frac{r}{\|x - x^*\| + 1}\right), \tag{3.1}$$

where $r > 0$ is a parameter.

Next, we will prove that $F(x, x^*, r)$ is a filled function under mild conditions.

Lemma 3.1. *Let* $g(t) = (Lt + (4/5)\arctan f(x^*))\exp(r/(t+1))$, *where* $0 \leq t \leq D$. *If* r *is an appropriately large such that* $r > (5L(D+1)^2/4\arctan f(x^*))$, *then* $g(t)$ *is strictly monotone decreasing on* $[0, D]$.

Proof. Since $r > (5L(D+1)^2/4\arctan f(x^*))$, then we have

$$\begin{aligned} g'(t) &= \exp\left(\frac{r}{t+1}\right)\left[L - \left(Lt + \frac{4}{5}\arctan f(x^*)\right)\frac{r}{(t+1)^2}\right] \\ &\le \exp\left(\frac{r}{t+1}\right)\left[L - \frac{4}{5}\arctan f(x^*)\frac{r}{(t+1)^2}\right] \\ &\le \exp\left(\frac{r}{t+1}\right)\left[L - \frac{4}{5}\arctan f(x^*)\frac{r}{(D+1)^2}\right] < 0, \end{aligned} \tag{3.2}$$

which implies that $g'(t) < 0$, and $g(t)$ is strictly monotone decreasing on $[0, D]$. □

Theorem 3.2. *Let x^* be a local minimizer of $f(x)$ with $f(x^*) > 0$. Suppose that Assumption 2 holds and $r > 0$ is appropriately large such that $r > (5L(D+1)^2/4\arctan f(x^*))$, then x^* is a strict local maximizer of $F(x, x^*, r)$.*

Proof. Since x^* is a local minimizer of $f(x)$ over X, there exists a neighborhood $O(x^*, \delta)$ of x^* with $\delta > 0$ such that $f(x) \ge f(x^*)$ for all $x \in O(x^*, \delta) \bigcap X$.

For any $x \in O(x^*, \delta) \bigcap X$ with $x \neq x^*$, if $r > 0$ is appropriately large such that $r > (5L(D+1)^2/4\arctan f(x^*))$, then, by Lemma 3.1 and the fact that the following inequality holds for $b \ge a$,

$$\arctan b - \arctan a \le b - a, \tag{3.3}$$

we have that

$$\begin{aligned} F(x, x^*, r) &= \exp\left(\frac{r}{\|x - x^*\| + 1}\right)\left(\arctan f(x) - \frac{1}{5}\arctan f(x^*)\right) \\ &\le \exp\left(\frac{r}{\|x - x^*\| + 1}\right)\left(f(x) - f(x^*) + \frac{4}{5}\arctan f(x^*)\right) \\ &\le \exp\left(\frac{r}{\|x - x^*\| + 1}\right)\left(L\|x - x^*\| + \frac{4}{5}\arctan f(x^*)\right) \\ &< \exp(r)\frac{4}{5}\arctan f(x^*) = F(x, x^*, r). \end{aligned} \tag{3.4}$$

Therefore, x^* is a strict local maximizer of $F(x, x^*, r)$. □

Theorem 3.3. *Let x^* be a local minimizer of $f(x)$ with $f(x^*) > 0$. Suppose that $r > 0$ is suitably large such that $r > (5L(D+1)^2/4\arctan f(x^*))$, then $0 \notin \partial F(x, x^*, r)$, for all $x \in S_1$.*

Proof. For any $x \in S_1$, that is, $f(x) \ge f(x^*)$ and $x \neq x^*$, we have

$$\begin{aligned}
&\partial F(x,x^*,r) \\
&\quad\subset \exp\left(\frac{r}{\|x-x^*\|+1}\right)\left[\frac{1}{1+f^2(x)}\partial f(x) \right.\\
&\qquad\qquad\left. -\frac{\arctan f(x)-(1/5)\arctan f(x^*)}{(\|x-x^*\|+1)^2}\frac{r(x-x^*)}{\|x-x^*\|}\right], \\
&\left\langle \partial F(x,x^*,r), \frac{x-x^*}{\|x-x^*\|}\right\rangle \\
&\quad\subset \exp\left(\frac{r}{\|x-x^*\|+1}\right)\left[\left\langle\frac{1}{1+f^2(x)}\partial f(x), \frac{x-x^*}{\|x-x^*\|}\right\rangle \right.\\
&\qquad\qquad\left. -r\frac{\arctan f(x)-(1/5)\arctan f(x^*)}{(\|x-x^*\|+1)^2}\right] \\
&\quad\le \exp\left(\frac{r}{\|x-x^*\|+1}\right)\left[L - r\frac{(4/5)\arctan f(x^*)}{(D+1)^2}\right] < 0,
\end{aligned} \tag{3.5}$$

since $\|\partial f(x)\| \le L$, and $r > (5L(D+1)^2/4\arctan f(x^*))$.

It follows that $0 \notin \partial F(x,x^*,r)$ for all $x \in S_1$. □

Theorem 3.4. *Assume that Assumptions 2 and 3 hold, x^* is a local optimizer with $f(x^*) > 0$ and is not a global optimizer of $f(x)$ over X. Then there exists a point $x_0 \in S_2$ such that x_0 is a minimizer of $F(x,x^*,r)$ over X.*

Proof. Since Assumption 3 holds and x^* is not a global minimizer of $f(x)$, there exists one point x_1^* with $f(x_1^*) = 0$ such that $f(x^*) > f(x_1^*)$. Denote the boundary of X by ∂X. Then, by Assumption 2, for any $x \in \partial X$, it holds $F(x,x^*,r) > F(x_1^*,x^*,r)$. Suppose that $F(x,x^*,r)$ attains its minimum at the point $x_0 \in X$, then we have

$$\min_{x\in X} F(x,x^*,r) = \min_{x\in X\setminus\partial X} F(x,x^*,r) = F(x_0,x^*,r) \le F\left(x_1^*,x^*,r\right) < 0. \tag{3.6}$$

Since $X \setminus \partial X$ is an open set, we have $x_0 \in S_2$. This completes the proof. □

4. Solution Algorithm

Based on the discussion of the theoretical properties of the above-proposed filled function, we describe a filled function algorithm for solving (SNE) as follows.

Filled function algorithm:

Initialization Step

Let $r_U = 10^5$ be a termination scalar. Choose an initial point x_1. Let $e_1, e_2, \ldots, e_{2n}$ be the coordinate directions. Set $k = 1$, and go to the main step.

Main Step

(1) Start from x_1, solve the problem (P) by using any nonsmooth local minimization procedure. Denote x_1^* the obtained optimizer, and go to (2).

(2) Let $r = 1$, construct the filled function $F(x, x^*, r)$ and go to (3).

(3) If $k \leq 2n$, then set $x = x_1^* + 0.1e_k$, and use x as an initial point for a nonsmooth local minimization method to find a local minimizer x_k of the following problem: $\min_{x \in X} F(x, x^*, r)$. If x attains the boundary of X during minimization, then set $k = k + 1$, and repeat (3); otherwise, go to (4).

(4) If all the following conditions hold, then set $x = x_k,\ k = 1$, use x as an initial point to get another local solution x_2^* of problem (P), and go to (5); otherwise go to (6),

  (a) $f(x_k) < f(x_1^*)$;
  (b) $F(x_k, x_1^*, r) > F(x_{k-1}, x_1^*, r)$;
  (c) $(x_k - x_1^*)^T \xi \geq 0,\ \forall \xi \in \partial F(x_k, x_1^*, r)$;
  (d) $\|\xi\| < 10^{-4},\ \xi \in \partial F(x_k, x_1^*, r)$.

(5) If $f(x) < f(x_1^*)$, then set $x_1^* = x_2^*$, and go to (2); otherwise, go to (6).

(6) Increase r by setting $r = 10r$. If $r \leq r_U$, then set $k = 1$, and go to (2); otherwise, the algorithm is incapable of finding a better local minimizer, the algorithm stops, and x_1^* is taken as a global minimizer.

Remarks. The above filled function algorithm consists of two phases: local minimization and filling. The local minimization phase is intended to minimize problem (P) by applying nonsmooth local optimization algorithms, such as hybrid Hooke and Jeeves-Direct method for nonsmooth Optimization [9], mesh adaptive direct search algorithms for constrained optimization [10], bundle methods, and Powell's method. The filling phase aims to search for a point x_k with $f(x_k) < f(x_1^*)$ by minimizing the filled function $F(x, x^*, r)$ over X. Once such kind of point is found in the process of minimization, the filling phase stops and the algorithm returns to the phase of local minimization. The two phases perform repeatedly until the certain conditions are satisfied.

5. Numerical Experiment

In this section, we perform the numerical experiments for four test problems by using the proposed filled function algorithm. All the numerical experiments are implemented in the FORTARN 95 environment. The hybrid Hooke and Jeeves-Direct method is used to search for the local minimizers of both the objective function and the filled function. Numerical results demonstrate that the proposed filled function approach is promising.

Problem 1. It holds that

$$\begin{aligned} &4x_1^3 + 4x_1x_2 + 2x_2^2 - 42x_1 - 14 = 0, \\ &4x_2^3 + 4x_1x_2 + 2x_1^2 - 26x_2 - 22 = 0, \\ &\text{s.t.} -5 \leq x_1, x_2 \leq 5. \end{aligned} \tag{5.1}$$

Table 1: Computational results for Problem 1.

k	x_k^0	x_k^*	$f(x_k^*)$	$F(x_k^*)$
1	$\begin{pmatrix} 4.0000 \\ 4.0000 \end{pmatrix}$	$\begin{pmatrix} 3.0000 \\ 1.9999 \end{pmatrix}$	5.3020×10^{-3}	$\begin{pmatrix} -2.0006 \times 10^{-3} \\ -3.3013 \times 10^{-3} \end{pmatrix}$
2	$\begin{pmatrix} 0.0866 \\ 2.8843 \end{pmatrix}$	$\begin{pmatrix} 0.0866 \\ 2.8842 \end{pmatrix}$	2.7777×10^{-3}	$\begin{pmatrix} 2.6285 \times 10^{-3} \\ 1.4689 \times 10^{-4} \end{pmatrix}$
3	$\begin{pmatrix} 3.3852 \\ 0.0738 \end{pmatrix}$	$\begin{pmatrix} 0.0866 \\ 2.8843 \end{pmatrix}$	3.7337×10^{-3}	$\begin{pmatrix} 2.9451 \times 10^{-4} \\ 2.4391 \times 10^{-3} \end{pmatrix}$

Table 2: Computational results for Problem 2.

k	x_k^0	x_k^*	$f(x_k^*)$	$F(x_k^*)$
1	$\begin{pmatrix} 2.0000 \\ 6.0000 \end{pmatrix}$	$\begin{pmatrix} 0.0000 \\ 6.8753 \end{pmatrix}$	3.4105×10^{-4}	$\begin{pmatrix} -1.0000 \\ -3.3252 \end{pmatrix}$
2	$\begin{pmatrix} 9.9999 \times 10^{-6} \\ 6.9667 \end{pmatrix}$	$\begin{pmatrix} 1.4354 \times 10^{-5} \\ 6.9667 \end{pmatrix}$	7.1718×10^{-5}	$\begin{pmatrix} 1.1803 \times 10^{-7} \\ -7.1597 \times 10^{-5} \end{pmatrix}$

There are nine known roots as shown in [11]. Table 1 records the numerical results obtained by our algorithm for Problem 1.

Problem 2. It holds that

$$\begin{aligned} & 10^4 x_1 x_2 - 1 = 0, \\ & e^{-x_1} + e^{-x_2} - 1.001 = 0, \\ & \text{s.t. } 5.49 \times 10^{-6} \leq x_1 \leq 4.553, \ \ 2.196 \times 10^{-3} \leq x_2 \leq 18.21. \end{aligned} \tag{5.2}$$

The algorithm successfully finds a root of the Problem 2: $(1.4564 \times 10^{-5}, 6.8753)$. Table 2 records its numerical results.

Problem 3. It holds that

$$\begin{aligned} & 1 - 2x_2 + 0.2\sin(4\pi x_2) - x_1 = 0, \\ & x_2 - 0.5\sin(2\pi x_1) = 0, \\ & \text{s.t.} - 10 \leq x_1, x_2 \leq 10. \end{aligned} \tag{5.3}$$

The solution is $(1.8784, -0.3458)$. Table 3 records the numerical results of Problem 3.

Table 3: Computational results for Problem 3.

k	x_k^0	x_k^*	$f(x_k^*)$	$F(x_k^*)$
1	$\begin{pmatrix}5.0000\\-3.0000\end{pmatrix}$	$\begin{pmatrix}4.7386\\-1.7417\end{pmatrix}$	1.5191	$\begin{pmatrix}-0.2762\\-1.2428\end{pmatrix}$
2	$\begin{pmatrix}2.5975\\-0.3301\end{pmatrix}$	$\begin{pmatrix}2.5762\\-0.4315\end{pmatrix}$	0.7627	$\begin{pmatrix}-0.5615\\-0.2012\end{pmatrix}$
3	$\begin{pmatrix}1.7394\\-0.4335\end{pmatrix}$	$\begin{pmatrix}1.8784\\-0.3458\end{pmatrix}$	2.2341×10^{-4}	$\begin{pmatrix}-1.1137 \times 10^{-4}\\1.1203 \times 10^{-4}\end{pmatrix}$

Table 4: Computational results for Problem 4.

k	x_k^0	x_k^*	$f(x_k^*)$	$F(x_k^*)$
1	$\begin{pmatrix}-1.0000\\1.0000\\-1.00000\\1.0000\\-1.0000\end{pmatrix}$	$\begin{pmatrix}1.0054\\1.0055\\1.0053\\1.0055\\0.9739\end{pmatrix}$	8.8951×10^{-3}	$\begin{pmatrix}1.0008 \times 10^{-3}\\1.1007 \times 10^{-3}\\9.0007 \times 10^{-4}\\1.1003 \times 10^{-3}\\-4.7932 \times 10^{-3}\end{pmatrix}$
2	$\begin{pmatrix}0.9935\\0.9933\\1.0016\\1.0133\\1.0127\end{pmatrix}$	$\begin{pmatrix}0.9990\\0.9989\\0.9991\\0.9996\\1.0042\end{pmatrix}$	1.7952×10^{-3}	$\begin{pmatrix}-2.0011 \times 10^{-4}\\-3.0009 \times 10^{-4}\\-1.0004 \times 10^{-4}\\4.0007 \times 10^{-4}\\7.9011 \times 10^{-4}\end{pmatrix}$

Problem 4. It holds that

$$\begin{aligned}
&2x_1 + x_2 + x_3 + x_4 + x_5 - 6 = 0,\\
&x_1 + 2x_2 + x_3 + x_4 + x_5 - 6 = 0,\\
&x_1 + x_2 + 2x_3 + x_4 + x_5 - 6 = 0,\\
&x_1 + x_2 + x_3 + 2x_4 + x_5 - 6 = 0,\\
&x_1 x_2 x_3 x_4 x_5 - 1 = 0,\\
&\text{s.t. } |x_i| \le 2, \quad i = 1, 2, \ldots, 5.
\end{aligned} \tag{5.4}$$

The known solutions of Problem 4 in [11] are $(1, 1, 1, 1, 1)$ and $(0.916, 0.916, 0.916, 0.916, 0.916)$. Table 4 records the numerical results of Problem 4.

The symbols used in the tables are given below:

k: the iteration number in finding the kth local minimizer,
x_k: the kth initial point to find the kth local minimizer,
x_k^*: the kth local minimizer,
$f(x_k^*)$: the function value of $f(x)$ at the kth local minimizer,
$F(x_k^*)$: the function value of $F(x)$ at the kth local minimizer.

6. Conclusions

In this paper, we developed a global minimization method based on filled function to identify the roots of the system of nonlinear equations. By converting a system nonlinear equations into an equivalent nonsmooth global minimization problem, we manage to search for a root or an appropriate root of the system of nonlinear equations by solving the formulated global minimization problem. A new filled function algorithm is proposed to solve the formulated global minimization problem. Numerical experiments for several test problems verify that our proposed algorithm is promising.

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