

## Research Article

# Stability of Functional Inequalities with Cauchy-Jensen Additive Mappings

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We investigate the generalized Hyers-Ulam stability of the functional inequalities associated with Cauchy-Jensen additive mappings. As a result, we obtain that if a mapping satisfies the functional inequalities with perturbation which satisfies certain conditions, then there exists a Cauchy-Jensen additive mapping near the mapping.

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## 1. Introduction

In 1940, Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group  $G$  and a metric group  $G'$  with metric  $\rho(\cdot, \cdot)$ . Given  $\epsilon > 0$ , does there exist a  $\delta > 0$  such that if  $f : G \rightarrow G'$  satisfies  $\rho(f(xy), f(x)f(y)) < \delta$  for all  $x, y \in G$ , then a homomorphism  $h : G \rightarrow G'$  exists with  $\rho(f(x), h(x)) < \epsilon$  for all  $x \in G$ ?

In 1941, Hyers [2] considered the case of approximately additive mappings  $f : E \rightarrow E'$ , where  $E$  and  $E'$  are Banach spaces and  $f$  satisfies *Hyers' inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \quad (1.1)$$

for all  $x, y \in E$ . It was shown that the limit  $L(x) = \lim_{n \rightarrow \infty} (f(2^n x)/2^n)$  exists for all  $x \in E$  and that  $L : E \rightarrow E'$  is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon. \quad (1.2)$$

In 1978, Rassias [3] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded*.

Let  $f : E \rightarrow E'$  be a mapping from a normed vector space  $E$  into a Banach space  $E'$  subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \quad (1.3)$$

for all  $x, y \in E$ , where  $\epsilon$  and  $p$  are constants with  $\epsilon > 0$  and  $p < 1$ .

Then, the limit  $L(x) = \lim_{n \rightarrow \infty} (f(2^n x)/2^n)$  exists for all  $x \in E$  and  $L : E \rightarrow E'$  is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2-2^p} \|x\|^p \quad (1.4)$$

for all  $x \in E$ . If  $p < 0$ , then inequality (1.3) holds for  $x, y \neq 0$  and (1.4) for  $x \neq 0$ .

In 1991, Gajda [4], following the same approach as in Rassias [3], gave an affirmative solution to this question for  $p > 1$ . It was shown by Gajda [4] as well as by Rassias and Šemrl [5] that one cannot prove a Rassias-type theorem when  $p = 1$ . Inequality (1.3) that was introduced for the first time by Rassias [3] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept of stability is known as *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations (cf. the books of Czerwik [6], Hyers et al. [7]).

Găvruta [8] provided a further generalization of Rassias' theorem. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [9–14]).

Gilányi [15] and Rätz [16] showed that if  $f$  satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\|, \quad (1.5)$$

then  $f$  satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}). \quad (1.6)$$

Gilányi [17] and Fechner [18] proved the generalized Hyers-Ulam stability of the functional inequality (1.3).

Now, we consider the following functional inequalities:

$$\left\| f\left(\frac{x-y}{2} - z\right) + f(y) + 2f(z) \right\| \leq \left\| f\left(\frac{x+y}{2} + z\right) \right\| + \phi(x, y, z), \quad (1.7)$$

$$\|f(x) + f(y) + 2f(z)\| \leq \left\| 2f\left(\frac{x+y}{2} + z\right) \right\| + \phi(x, y, z), \quad (1.8)$$

which are associated with Jordan-von Neumann-type Cauchy-Jensen additive functional equations.

The purpose of this paper is to prove that if  $f$  satisfies one of the inequalities (1.7) and (1.8) which satisfies certain conditions, then we can find a Cauchy-Jensen additive mapping near  $f$ , and thus we prove the generalized Hyers-Ulam stability of the functional inequalities (1.7) and (1.8).

## 2. Stability of functional inequality (1.7)

We prove the generalized Hyers-Ulam stability of a functional inequality (1.7) associated with a Jordan-von Neumann-type 3-variable Cauchy-Jensen additive functional equation. Throughout this paper, let  $G$  be a normed vector space and  $Y$  a Banach space.

LEMMA 2.1. *Let  $f : G \rightarrow Y$  be a mapping such that*

$$\left\| f\left(\frac{x-y}{2} - z\right) + f(y) + 2f(z) \right\| \leq \left\| f\left(\frac{x+y}{2} + z\right) \right\| \quad (2.1)$$

*for all  $x, y, z \in G$ . Then,  $f$  is Cauchy-Jensen additive.*

*Proof.* Letting  $x, y, z := 0$  in (2.1), we get  $\|4f(0)\| \leq \|f(0)\|$ . So,  $f(0) = 0$ .

And by setting  $y := -x$  and  $z := 0$  in (2.1), we get  $\|f(x) + f(-x)\| \leq \|f(0)\| = 0$  for all  $x \in G$ . Hence,  $f(-x) = -f(x)$  for all  $x \in G$ .

Also by letting  $x := 0$ ,  $y := 2x$ , and  $z := -x$  in (2.1), we get  $\|f(2x) + 2f(-x)\| \leq \|2f(0)\| = 0$  for all  $x \in G$ . Thus,  $f(2x) = 2f(x)$  for all  $x \in G$ .

Letting  $z = (-x - y)/2$  in (2.1), we get

$$\left\| f\left(\frac{x-y}{2} + \frac{x+y}{2}\right) + f(y) + 2f\left(\frac{-x-y}{2}\right) \right\| \leq \|f(0)\| = 0 \quad (2.2)$$

for all  $x, y \in G$ . Thus,  $f(x+y) = f(x) + f(y)$  for all  $x, y \in G$ , as desired.  $\square$

THEOREM 2.2. *Assume that a mapping  $f : G \rightarrow Y$  satisfies the inequality*

$$\left\| f\left(\frac{x-y}{2} - z\right) + f(y) + 2f(z) \right\| \leq \left\| f\left(\frac{x+y}{2} + z\right) \right\| + \phi(x, y, z) \quad (2.3)$$

*and that the map  $\phi : G \times G \times G \rightarrow [0, \infty)$  satisfies the condition*

$$\Phi(x, y, z) := \sum_{j=0}^{\infty} 3^j \phi\left(\frac{x}{3^j}, \frac{y}{3^j}, \frac{z}{3^j}\right) < \infty \quad (2.4)$$

*for all  $x, y, z \in G$ . Then, there exists a unique Cauchy-Jensen additive mapping  $A : G \rightarrow Y$  such that*

$$\|A(x) - f(x)\| \leq \Phi\left(-\frac{x}{3}, x, -\frac{x}{3}\right) + \frac{3}{2}\Phi\left(\frac{x}{3}, \frac{x}{3}, -\frac{x}{3}\right) \quad (2.5)$$

*for all  $x \in G$ .*

*Proof.* Letting  $y := x$  and  $z := -x$  in (2.3), we get

$$\|2f(x) + 2f(-x)\| \leq \phi(x, x, -x) + \|f(0)\| \quad (2.6)$$

for all  $x \in G$ . And by letting  $x := -x$ ,  $y := 3x$ , and  $z := -x$  in (2.3), we get

$$\|3f(-x) + f(3x)\| \leq \phi(-x, 3x, -x) + \|f(0)\| \quad (2.7)$$

for all  $x \in G$ . It follows from (2.6) and (2.7) that

$$\|f(3x) - 3f(x)\| \leq \phi(-x, 3x, -x) + \frac{3}{2}\phi(x, x, -x) + \frac{5}{2}\|f(0)\|. \quad (2.8)$$

Also letting  $x, y, z := 0$  in (2.3), we get  $3\|f(0)\| \leq \phi(0, 0, 0) = 0$ . Hence, we have  $f(0) = 0$ .

Now, it follows from (2.8) that for all nonnegative integers  $m$  and  $l$  with  $m > l$

$$\begin{aligned} \left\| 3^l f\left(\frac{x}{3^l}\right) - 3^m f\left(\frac{x}{3^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 3^j f\left(\frac{x}{3^j}\right) - 3^{j+1} f\left(\frac{x}{3^{j+1}}\right) \right\| \\ &\leq \sum_{j=l}^{m-1} 3^j \left[ \phi\left(-\frac{x}{3^{j+1}}, \frac{x}{3^j}, -\frac{x}{3^{j+1}}\right) + \frac{3}{2}\phi\left(\frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}, -\frac{x}{3^{j+1}}\right) \right] \end{aligned} \quad (2.9)$$

for all  $x \in G$ . It means that a sequence  $\{3^n f(x/3^n)\}$  is a Cauchy sequence for all  $x \in G$ . Since  $Y$  is complete, the sequence  $\{3^n f(x/3^n)\}$  converges. So, one can define a mapping  $A : G \rightarrow Y$  by  $A(x) := \lim_{n \rightarrow \infty} 3^n f(x/3^n)$  for all  $x \in G$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (2.9), we get the approximation (2.5) of  $f$  by  $A$ .

Next, we claim that the mapping  $A : G \rightarrow Y$  is Cauchy-Jensen additive. In fact, it follows easily from (2.3) and condition of  $\phi$  that

$$\begin{aligned} \left\| A\left(\frac{x-y}{2} - z\right) + A(y) + 2A(z) \right\| &= \lim_{n \rightarrow \infty} 3^n \left\| f\left(\frac{1}{3^n}\left(\frac{x-y}{2} - z\right)\right) + f\left(\frac{y}{3^n}\right) + 2f\left(\frac{z}{3^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} 3^n \left[ \left\| f\left(\frac{1}{3^n}\left(\frac{x+y}{2} + z\right)\right) \right\| + \phi\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) \right] \\ &= A\left(\frac{x+y}{2} + z\right). \end{aligned} \quad (2.10)$$

Thus, the mapping  $A : G \rightarrow Y$  is Cauchy-Jensen additive by Lemma 2.1.

Now, let  $T : G \rightarrow Y$  be another Cauchy-Jensen additive mapping satisfying (2.5). Then we obtain

$$\begin{aligned} &\|A(x) - T(x)\| \\ &= 3^n \left\| A\left(\frac{x}{3^n}\right) - T\left(\frac{x}{3^n}\right) \right\| \\ &\leq 3^n \left( \left\| A\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right) \right\| + \left\| T\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right) \right\| \right) \\ &\leq 2 \sum_{j=0}^{\infty} 3^j \left[ \phi\left(-\frac{x}{3^{n+j+1}}, \frac{x}{3^{n+j}}, -\frac{x}{3^{n+j+1}}\right) + \frac{3}{2}\phi\left(\frac{x}{3^{n+j+1}}, \frac{x}{3^{n+j+1}}, -\frac{x}{3^{n+j+1}}\right) \right] \\ &\leq 2 \sum_{j=n}^{\infty} 3^j \left[ \phi\left(-\frac{x}{3^{j+1}}, \frac{x}{3^j}, -\frac{x}{3^{j+1}}\right) + \frac{3}{2}\phi\left(\frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}, -\frac{x}{3^{j+1}}\right) \right], \end{aligned} \quad (2.11)$$

which tends to zero as  $n \rightarrow \infty$ . So, we can conclude that  $A(x) = T(x)$  for all  $x \in G$ . This proves the uniqueness of  $A$ . Hence, the mapping  $A : G \rightarrow Y$  is a unique Cauchy-Jensen additive mapping satisfying (2.5).  $\square$

**THEOREM 2.3.** *Assume that a mapping  $f : G \rightarrow Y$  satisfies inequality (2.3) and that the map  $\phi : G \times G \times G \rightarrow [0, \infty)$  satisfies the condition*

$$\Phi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{3^j} \phi(3^j x, 3^j y, 3^j z) < \infty \quad (2.12)$$

for all  $x, y, z \in G$ .

Then, there exists a unique Cauchy-Jensen additive mapping  $A : G \rightarrow Y$  such that

$$\|A(x) - f(x)\| \leq \frac{1}{3} \Phi(-x, x, -x) + \frac{1}{2} \Phi(x, x, -x) + \frac{5}{4} \|f(0)\| \quad (2.13)$$

for all  $x \in G$ .

*Proof.* We get by (2.8)

$$\begin{aligned} & \left\| \frac{1}{3^l} f(3^l x) - \frac{1}{3^m} f(3^m x) \right\| \\ & \leq \sum_{j=l}^{m-1} \left\| \frac{1}{3^j} f(3^j x) - \frac{1}{3^{j+1}} f(3^{j+1} x) \right\| \\ & \leq \sum_{j=l}^{m-1} \left[ \left\| \frac{1}{3^j} f(3^j x) + \frac{1}{3^{j+1}} f(-3^{j+1} x) \right\| + \left\| \frac{1}{3^{j+1}} f(2^{j+1} x) + \frac{1}{3^{j+1}} f(-3^{j+1} x) \right\| \right] \\ & \leq \sum_{j=l}^{m-1} \frac{1}{3^{j+1}} \left[ \phi(-3^j x, 3^{j+1} x, -3^j x) + \frac{3}{2} \phi(3^j x, 3^j x, -3^j x) + \frac{5}{2} \|f(0)\| \right] \end{aligned} \quad (2.14)$$

for all nonnegative integers  $m$  and  $l$  with  $m > l$  and all  $x \in G$ . It means that a sequence  $\{(1/3^n)f(3^n x)\}$  is a Cauchy sequence for all  $x \in G$ . Since  $Y$  is complete, the sequence  $\{(1/3^n)f(3^n x)\}$  converges. So, one can define a mapping  $A : G \rightarrow Y$  by  $A(x) := \lim_{n \rightarrow \infty} (1/3^n)f(3^n x)$  for all  $x \in G$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (2.14), we get (2.13).

The remaining proof goes through by the similar argument to Theorem 2.2.  $\square$

**THEOREM 2.4.** *Assume that a mapping  $f : G \rightarrow Y$  satisfies inequality (2.3) and that the map  $\phi : G \times G \times G \rightarrow [0, \infty)$  satisfies the condition*

$$\lim_{n \rightarrow \infty} 3^n \phi\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) = 0 \quad (2.15)$$

for all  $x, y, z \in G$ . If there exists a number  $L$  with  $0 \leq L < 1$  such that the mapping  $x \mapsto \psi(x) := \phi(-x, 3x, -x) + (3/2)\phi(x, x, -x)$  satisfies

$$\psi(x) \leq \frac{L}{3}\psi(3x), \quad (2.16)$$

then there exists a unique Cauchy-Jensen additive mapping  $A : G \rightarrow Y$  such that

$$\|f(x) - A(x)\| \leq \frac{L \cdot \psi(x)}{3(1-L)} \quad (2.17)$$

for all  $x \in G$ .

*Proof.* We get by (2.8)

$$\|f(3x) - 3f(x)\| \leq \psi(x) = \phi(-x, 3x, -x) + \frac{3}{2}\phi(x, x, -x) \quad (2.18)$$

for all  $x \in G$ . Hence, we get

$$\begin{aligned} \left\| 3^l f\left(\frac{x}{3^l}\right) - 3^m f\left(\frac{x}{3^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 3^j f\left(\frac{x}{3^j}\right) - 3^{j+1} f\left(\frac{x}{3^{j+1}}\right) \right\| \\ &\leq \sum_{j=l}^{m-1} 3^j \psi\left(\frac{x}{3^{j+1}}\right) \leq \sum_{j=l}^{m-1} \frac{L^{j+1}}{3} \psi(x) \end{aligned} \quad (2.19)$$

for all nonnegative integers  $m$  and  $l$  with  $m > l$  and all  $x \in G$ . It means that a sequence  $\{3^n f(x/3^n)\}$  is a Cauchy sequence for all  $x \in G$ . Since  $Y$  is complete, the sequence  $\{3^n f(x/3^n)\}$  converges. So, one can define a mapping  $A : G \rightarrow Y$  by  $A(x) := \lim_{n \rightarrow \infty} 3^n f(x/3^n)$  for all  $x \in G$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (2.19), we get (2.17).

The remaining proof goes through by the similar argument to Theorem 2.2.  $\square$

**COROLLARY 2.5.** Assume that there exist nonnegative numbers  $\theta$  and a real  $p > 1$  such that a mapping  $f : G \rightarrow Y$  satisfies the inequality

$$\left\| f\left(\frac{x-y}{2} - z\right) + f(y) + 2f(z) \right\| \leq \left\| f\left(\frac{x+y}{2} + z\right) \right\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \quad (2.20)$$

for all  $x, y, z \in G$ .

Then, there exists a unique Cauchy-Jensen additive mapping  $A : G \rightarrow Y$  such that

$$\|f(x) - A(x)\| \leq \frac{\theta(13 + 2 \cdot 3^p)}{2(3^p - 3)} \|x\|^p \quad (2.21)$$

for all  $x \in G$ .

**THEOREM 2.6.** Assume that a mapping  $f : G \rightarrow Y$  satisfies inequality (2.3) and that the map  $\phi : G \times G \times G \rightarrow [0, \infty)$  satisfies the condition

$$\lim_{n \rightarrow \infty} \frac{1}{3^n} \phi(3^n x, 3^n y, 3^n z) = 0 \quad (2.22)$$

for all  $x, y, z \in G$ . If there exists a number  $L$  with  $0 \leq L < 1$  such that the mapping  $x \mapsto \psi(x) := \phi(-x, 3x, -x) + (3/2)\phi(x, x, -x)$  satisfies

$$\psi(3x) \leq 3L \cdot \psi(x), \quad (2.23)$$

then there exists a unique Cauchy-Jensen additive mapping  $A : G \rightarrow Y$  such that

$$\|f(x) - A(x)\| \leq \frac{\psi(x)}{3(1-L)} + \frac{5}{4}\|f(0)\| \quad (2.24)$$

for all  $x \in G$ .

*Proof.* We get by (2.8)

$$\begin{aligned} \left\| \frac{1}{3^l} f(3^l x) - \frac{1}{3^m} f(3^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{3^j} f(3^j x) - \frac{1}{3^{j+1}} f(3^{j+1} x) \right\| \\ &\leq \sum_{j=l}^{m-1} \frac{1}{3^{j+1}} \left[ \psi(3^j x) + \frac{5}{2} \|f(0)\| \right] \\ &\leq \sum_{j=l}^{m-1} \left[ \frac{L^j \psi(x)}{3} + \frac{5}{2 \cdot 3^{j+1}} \|f(0)\| \right] \end{aligned} \quad (2.25)$$

for all nonnegative integers  $m$  and  $l$  with  $m > l$  and all  $x \in G$ . It means that a sequence  $\{(1/3^n)f(3^n x)\}$  is a Cauchy sequence for all  $x \in G$ . Since  $Y$  is complete, the sequence  $\{(1/3^n)f(3^n x)\}$  converges. So, one can define a mapping  $A : G \rightarrow Y$  by  $A(x) := \lim_{n \rightarrow \infty} (1/3^n)f(3^n x)$  for all  $x \in G$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (2.25), we get (2.24).

The remaining proof goes through by the similar argument to Theorem 2.3.  $\square$

**COROLLARY 2.7.** Assume that there exist nonnegative numbers  $\theta$ ,  $\delta$ , and a real  $p < 1$  such that a mapping  $f : G \rightarrow Y$  satisfies the inequality

$$\left\| f\left(\frac{x-y}{2} - z\right) + f(y) + 2f(z) \right\| \leq \left\| f\left(\frac{x+y}{2} + z\right) \right\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p) + \delta \quad (2.26)$$

for all  $x, y, z \in G$ .

Then, there exists a unique Cauchy-Jensen additive mapping  $A : G \rightarrow Y$  such that

$$\|f(x) - A(x)\| \leq \frac{\theta(13 + 2 \cdot 3^p)\|x\|^p + 5\delta + 5\|f(0)\|}{2(3 - 3^p)} \quad (2.27)$$

for all  $x \in G$ .

### 3. Stability of functional inequality (1.8)

We prove the generalized Hyers-Ulam stability of a functional inequality (1.8) associated with a Jordan-von Neumann-type 3-variable Cauchy-Jensen additive functional equation.

THEOREM 3.1. Assume that a mapping  $f : G \rightarrow Y$  satisfies the inequality

$$\|f(x) + f(y) + 2f(z)\| \leq \left\| 2f\left(\frac{x+y}{2} + z\right) \right\| + \phi(x, y, z) \quad (3.1)$$

and that the map  $\phi : G \times G \times G \rightarrow [0, \infty)$  satisfies the conditions

$$(1) \rho(x) := \sum_{j=0}^{\infty} (1/2^{j+1}) [\phi(-2^{j+1}x, 0, 2^jx) + \phi_1(2^{j+1}x)] < \infty,$$

$$(2) \lim_{n \rightarrow \infty} (1/2^n) \phi(2^n x, 2^n y, 2^n z) = 0 \text{ for all } x, y, z \in G,$$

where

$$\phi_1(x) := \min \left\{ \phi(x, -x, 0) + 4\|f(0)\|, \frac{1}{2}\phi(x, x, -x) + \|f(0)\| \right\}. \quad (3.2)$$

Then, there exists a unique Cauchy-Jensen additive mapping  $A : G \rightarrow Y$  such that

$$\|A(x) - f(x)\| \leq \rho(x) \quad (3.3)$$

for all  $x \in G$ .

*Proof.* Letting  $x, y, z := 0$  in (3.1), we get  $\|f(0)\| \leq (1/2)\phi(0, 0, 0)$ .

And by setting  $x := 2x$ ,  $y := 0$ , and  $z := -x$  in (3.1), we get

$$\|f(2x) + 2f(-x)\| \leq 3\|f(0)\| + \phi(2x, 0, -x) \quad (3.4)$$

for all  $x \in G$ .

Also by letting  $y := -x$  and  $z := 0$  or by letting  $y := x$  and  $z := -x$  in (3.1), we get

$$\|f(x) + f(-x)\| \leq \phi_1(x) = \min \left\{ \phi(x, -x, 0) + 4\|f(0)\|, \frac{1}{2}\phi(x, x, -x) + \|f(0)\| \right\} \quad (3.5)$$

for all  $x \in G$ . Hence, we get by (3.4) and (3.5)

$$\begin{aligned} & \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| \\ & \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\| \\ & \leq \sum_{j=l}^{m-1} \left[ \left\| \frac{1}{2^j} f(2^j x) + \frac{1}{2^{j+1}} f(-2^{j+1} x) \right\| + \left\| \frac{1}{2^{j+1}} f(2^{j+1} x) + \frac{1}{2^{j+1}} f(-2^{j+1} x) \right\| \right] \\ & \leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} [\phi(-2^{j+1}x, 0, 2^jx) + \phi_1(2^{j+1}x)] \end{aligned} \quad (3.6)$$

for all nonnegative integers  $m$  and  $l$  with  $m > l$  and all  $x \in G$ . It means that a sequence  $\{(1/2^n)f(2^n x)\}$  is a Cauchy sequence for all  $x \in G$ . Since  $Y$  is complete, the sequence  $\{(1/2^n)f(2^n x)\}$  converges. So, one can define a mapping  $A : G \rightarrow Y$  by  $A(x) := \lim_{n \rightarrow \infty} (1/2^n)f(2^n x)$  for all  $x \in G$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (3.6), we get (3.3).

The remaining proof is similar to that of Theorem 2.3.  $\square$

**THEOREM 3.2.** *Assume that a mapping  $f : G \rightarrow Y$  satisfies inequality (3.1) and that the map  $\phi : G \times G \times G \rightarrow [0, \infty)$  satisfies the conditions*

$$(1) \rho(x) := \sum_{j=0}^{\infty} 2^j \phi(x/2^j, 0, -x/2^{j+1}) + 2^{j+1} \phi_2(x/2^{j+1}) < \infty,$$

$$(2) \lim_{n \rightarrow \infty} 2^n \phi(x/2^n, y/2^n, z/2^n) = 0 \text{ for all } x, y, z \in G,$$

where

$$\phi_2(x) := \min \left\{ \phi(x, -x, 0), \frac{1}{2} \phi(x, x, -x) \right\}. \quad (3.7)$$

Then, there exists a unique Cauchy-Jensen additive mapping  $A : G \rightarrow Y$  such that

$$\|A(x) - f(x)\| \leq \rho(x) \quad (3.8)$$

for all  $x \in G$ .

*Proof.* Letting  $x, y, z := 0$  in (3.1), we get  $\|f(0)\| \leq (1/2)\phi(0, 0, 0) = 0$ . So  $f(0) = 0$ .

Now, it follows from (3.4) and (3.5) that for all nonnegative integers  $m$  and  $l$  with  $m > l$ ,

$$\begin{aligned} & \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \\ & \leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ & \leq \sum_{j=l}^{m-1} \left[ \left\| 2^j f\left(\frac{x}{2^j}\right) + 2^{j+1} f\left(-\frac{x}{2^{j+1}}\right) \right\| + \left\| 2^{j+1} f\left(-\frac{x}{2^{j+1}}\right) + 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \right] \\ & \leq \sum_{j=l}^{m-1} \left[ 2^j \phi\left(\frac{x}{2^j}, 0, -\frac{x}{2^{j+1}}\right) + 2^{j+1} \phi_2\left(\frac{x}{2^{j+1}}\right) \right] \end{aligned} \quad (3.9)$$

for all  $x \in G$ . It means that a sequence  $\{2^n f(x/2^n)\}$  is a Cauchy sequence for all  $x \in G$ . Since  $Y$  is complete, the sequence  $\{2^n f(x/2^n)\}$  converges. So, one can define a mapping  $A : G \rightarrow Y$  by  $A(x) := \lim_{n \rightarrow \infty} 2^n f(x/2^n)$  for all  $x \in G$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (3.9), we get (3.8).

The rest of proof is similar to that of Theorem 2.2.  $\square$

*Remark 3.3.* Assume that a mapping  $f : G \rightarrow Y$  satisfies inequality (3.1) and that the map  $\phi : G \times G \times G \rightarrow [0, \infty)$  satisfies the conditions

$$(1) \rho(x) := \sum_{j=0}^{\infty} (1/2^{j+2}) [\phi(-2^{j+1}x, 0, 2^jx) + \phi(2^{j+1}x, 0, -2^jx)] < \infty,$$

$$(2) \lim_{n \rightarrow \infty} (1/2^n) \phi(2^n x, 2^n y, 2^n z) = 0 \text{ for all } x, y, z \in G.$$

Then, there exists a unique Cauchy-Jensen additive mapping  $L : G \rightarrow Y$  such that

$$\left\| L(x) - \frac{f(x) - f(-x)}{2} \right\| \leq \rho(x) + 3\|f(0)\| \quad (3.10)$$

for all  $x \in G$ .

*Proof.* Let  $g(x) := (f(x) - f(-x))/2$ . Then, we get by (3.4)

$$\begin{aligned} \|2g(x) - g(2x)\| &\leq \left\| f(x) + \frac{1}{2}f(-2x) \right\| + \left\| f(-x) + \frac{1}{2}f(2x) \right\| \\ &\leq \frac{1}{2} [\phi(-2x, 0, x) + \phi(2x, 0, -x)] + 3\|f(0)\| \end{aligned} \quad (3.11)$$

for all  $x \in G$ . Hence, we get by (3.11)

$$\begin{aligned} &\left\| \frac{1}{2^l}g(2^l x) - \frac{1}{2^m}g(2^m x) \right\| \\ &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j}g(2^j x) - \frac{1}{2^{j+1}}g(2^{j+1}x) \right\| = \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} \left[ \left\| 2g(2^j x) - g(2^{j+1}x) \right\| \right] \\ &\leq \sum_{j=l}^{m-1} \frac{1}{2^{j+2}} [\phi(-2^{j+1}x, 0, 2^jx) + \phi(2^{j+1}x, 0, -2^jx) + 6\|f(0)\|] \end{aligned} \quad (3.12)$$

for all nonnegative integers  $m$  and  $l$  with  $m > l$  and all  $x \in G$ . It means that a sequence  $\{(1/2^n)g(2^n x)\}$  is a Cauchy sequence for all  $x \in G$ . So, one can define a mapping  $L : G \rightarrow Y$  by  $L(x) := \lim_{n \rightarrow \infty} (1/2^n)g(2^n x) = \lim_{n \rightarrow \infty} (1/2^n)[(f(2^n x) - f(-2^n x))/2]$  for all  $x \in G$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (3.12), we get (3.10). Next, we claim that the mapping  $L : G \rightarrow Y$  is a Cauchy-Jensen additive mapping. Note that  $L(-x) = -L(x)$  because  $g(-x) = -g(x)$ . Then

$$\|L(x) + L(y) - L(x+y)\| = \lim_{n \rightarrow \infty} \frac{1}{2^n} \|g(2^n x) + g(2^n y) - g(2^n(x+y))\|, \quad (3.13)$$

and so we obtain by (3.1) and (3.4),

$$\begin{aligned}
& \frac{1}{2^n} \|g(2^n x) + g(2^n y) + g(2^n(-x - y))\| \\
& \leq \frac{1}{2^{n+1}} \|f(2^n x) + f(2^n y) + 2f(2^{n-1}(-x - y))\| \\
& \quad + \frac{1}{2^{n+1}} \|-f(-2^n x) - f(-2^n y) - 2f(2^{n-1}(x + y))\| \\
& \quad + \frac{1}{2^{n+1}} \|-2f(2^{n-1}(-x - y)) - f(2^n(x + y))\| \\
& \quad + \frac{1}{2^{n+1}} \|f(2^n(-x - y)) + 2f(2^{n-1}(x + y))\| \\
& \leq \frac{1}{2^{n+1}} [\phi(2^n x, 2^n y, 2^{n-1}(-x - y)) + \phi(-2^n x, -2^n y, 2^{n-1}(x + y)) + 4\|f(0)\|] \\
& \quad + \frac{1}{2^{n+1}} [\|6f(0)\| + \phi(-2^n(x + y), 0, 2^{n-1}(x + y)) + \phi(2^n(x + y), 0, -2^{n-1}(x + y))],
\end{aligned} \tag{3.14}$$

which tends to zero as  $n \rightarrow \infty$  for all  $x \in G$ . Hence, we see that  $L$  is additive.

The remaining proof is similar to the corresponding part of Theorem 2.3.  $\square$

*Remark 3.4.* Assume that a mapping  $f : G \rightarrow X$  satisfies inequality (3.1) and that the map  $\phi : G \times G \times G \rightarrow [0, \infty)$  satisfies the conditions

- (1)  $\rho(x) := \sum_{j=0}^{\infty} 2^{j-1} [\phi(-x/2^j, 0, x/2^{j+1}) + \phi(x/2^j, 0, -x/2^{j+1})] < \infty$ ,
- (2)  $\lim_{n \rightarrow \infty} 2^n \phi(x/2^n, y/2^n, z/2^n) = 0$  for all  $x, y, z \in G$ .

Then, there exists a unique Cauchy-Jensen additive mapping  $L : G \rightarrow Y$  such that

$$\left\| L(x) - \frac{f(x) - f(-x)}{2} \right\| \leq \rho(x) \tag{3.15}$$

for all  $x \in G$ .

*Proof.* Letting  $x, y, z := 0$  in (3.1), we get  $\|f(0)\| \leq (1/2)\phi(0, 0, 0) = 0$ . So  $f(0) = 0$ .

Let  $g(x) := (f(x) - f(-x))/2$ . Then, we get by (3.4)

$$\begin{aligned}
\|2g(x) - g(2x)\| & \leq \left\| f(x) + \frac{1}{2}f(-2x) \right\| + \left\| f(-x) + \frac{1}{2}f(2x) \right\| \\
& \leq \frac{1}{2} [\phi(-2x, 0, x) + \phi(2x, 0, -x)]
\end{aligned} \tag{3.16}$$

for all  $x \in G$ . Hence, we get by (3.16)

$$\begin{aligned}
\left\| 2^l g\left(\frac{x}{2^l}\right) - 2^m g\left(\frac{x}{2^m}\right) \right\| & \leq \sum_{j=l}^{m-1} \left\| 2^j g\left(\frac{x}{2^j}\right) - 2^{j+1} g\left(\frac{x}{2^{j+1}}\right) \right\| \\
& \leq \sum_{j=l}^{m-1} 2^{j-1} \left[ \phi\left(-\frac{x}{2^j}, 0, \frac{x}{2^{j+1}}\right) + \phi\left(\frac{x}{2^j}, 0, -\frac{x}{2^{j+1}}\right) \right]
\end{aligned} \tag{3.17}$$

for all nonnegative integers  $m$  and  $l$  with  $m > l$  and all  $x \in G$ . It means that the sequence  $\{2^n g(x/2^n)\}$  is a Cauchy sequence for all  $x \in G$ . So, one can define a mapping  $L : G \rightarrow Y$  by  $L(x) := \lim_{n \rightarrow \infty} 2^n g(x/2^n) = \lim_{n \rightarrow \infty} 2^n [(f(x/2^n) - f(-x/2^n))/2]$  for all  $x \in G$ . Moreover, letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (3.17), we get (3.15).

Next, we claim that the mapping  $L : G \rightarrow Y$  is a Cauchy-Jensen additive mapping. Note that  $L(-x) = -L(x)$  because  $g(-x) = -g(x)$ . So, we obtain by (3.1) and (3.4)

$$\begin{aligned}
& \|L(x) + L(y) - L(x+y)\| \\
&= \lim_{n \rightarrow \infty} 2^n \left\| g\left(\frac{x}{2^n}\right) + g\left(\frac{y}{2^n}\right) - g\left(\frac{x+y}{2^n}\right) \right\| \\
&= \lim_{n \rightarrow \infty} 2^n \left\| g\left(\frac{x}{2^n}\right) + g\left(\frac{y}{2^n}\right) + g\left(\frac{-x-y}{2^n}\right) \right\| \\
&\leq \lim_{n \rightarrow \infty} \frac{2^n}{2} \left[ \left\| f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) + 2f\left(\frac{-x-y}{2^{n+1}}\right) \right\| \right. \\
&\quad \left. + \left\| -f\left(\frac{-x}{2^n}\right) - f\left(\frac{-y}{2^n}\right) - 2f\left(\frac{x+y}{2^{n+1}}\right) \right\| \right] \\
&\quad + \lim_{n \rightarrow \infty} \frac{2^n}{2} \left[ \left\| -2f\left(\frac{-x-y}{2^{n+1}}\right) - f\left(\frac{x+y}{2^n}\right) \right\| + \left\| f\left(\frac{-x-y}{2^n}\right) + 2f\left(\frac{x+y}{2^{n+1}}\right) \right\| \right] \\
&\leq \lim_{n \rightarrow \infty} 2^{n-1} \left[ \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{-x-y}{2^{n+1}}\right) + \phi\left(\frac{-x}{2^n}, \frac{-y}{2^n}, \frac{x+y}{2^{n+1}}\right) \right] \\
&\quad + \lim_{n \rightarrow \infty} 2^{n-1} \left[ \phi\left(\frac{x+y}{2^n}, 0, \frac{-x-y}{2^{n+1}}\right) + \phi\left(\frac{-x-y}{2^n}, 0, \frac{x+y}{2^{n+1}}\right) \right] = 0
\end{aligned} \tag{3.18}$$

from the condition of  $\phi$ . So, we have  $L(x+y) = L(x) + L(y)$ .

The remaining proof is similar to that of Theorem 2.2. □

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