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*Research Article*

# Stabilization for a Periodic Predator-Prey System

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A reaction-diffusion system modelling a predator-prey system in a periodic environment is considered. We are concerned in stabilization to zero of one of the components of the solution, via an internal control acting on a small subdomain, and in the preservation of the nonnegativity of both components.



## 1. Introduction

This paper concerns the internal zero stabilization of the predator population of a predator-prey system in a periodic environment. Our starting point is the system describing the evolution of a predator population and a prey population distributed over the habitat $\Omega$:

$$\begin{aligned} h_t - d_1\Delta h &= r(t)h - k(t)h^2 - f_1(t,h,p)hp, \quad x\in\Omega,\ t>0,\\ p_t - d_2\Delta p &= -a(t)p + f_2(t,h,p)hp, \quad x\in\Omega,\ t>0,\\ \frac{\partial h}{\partial \nu} &= \frac{\partial p}{\partial \nu} = 0, \quad x\in\partial\Omega,\ t>0,\\ h(x,0) &= h_0(x), \quad p(x,0) = p_0(x), \quad x\in\Omega, \end{aligned} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ is a bounded domain with a smooth enough boundary $\partial\Omega$. Here $h(x,t)$ is the density of preys at position $x\in\overline{\Omega}$ and time $t\geq 0$ and $p(x,t)$ is the density of predators at position $x\in\overline{\Omega}$ and time $t\geq 0$; $h$ and $p$ are both nonnegative functions. $d_1$, $d_2>0$ are the diffusivity constants of the two populations. $r(t)$ is the intrinsic growth rate of preys in the absence of predators, at the moment $t\geq 0$ (which can be positive, zero, or negative) and is $T$-periodic ($T > 0$). Usually, the period $T$ is of one year. $a(t)$ is the decay rate of predators in the absence of preys, at the moment $t$, and is also $T$-periodic. $k$ is a $T$-periodic and positive function. $k(t)h(x,t)$ represents an additional mortality rate of the preys due to the overpopulation.

Homogeneous Neumann boundary conditions mean that there is no flux of species through the boundary $\partial\Omega$ (this corresponds to isolated populations). $h_0$ and $p_0$ are the initial densities of the two populations.

The following cases are well known in the literature.

When $f_1(t,h,p) = \theta_1$ and $f_2(t,h,p) = \theta_2$, where $\theta_1$, $\theta_2$ are positive constants, the standard Lotka-Volterra system is obtained.

For $f_1(t,h,p) = \theta_1/(1+qh)$ and $f_2(t,h,p) = \theta_2/(1+qh)$, for every $h,p \geq 0$, where $\theta_1$, $\theta_2$, $q$ are positive constants, we obtain a Holling II functional response to predation.

Finally, in the case $f_1(t,h,p) = \theta_1/(1+qh+\tilde{q}p)$ and $f_2(t,h,p) = \theta_2/(1+qh+\tilde{q}p)$, for every $h,p \geq 0$, and $\theta_1$, $\theta_2$, $q$, $\tilde{q}$ positive constants, a Beddington-De Angelis functional response for predation is obtained. For a complete study of the solutions to this model we refer to [1]. For a description of the predator-prey systems and some basic results we refer to [2, 3].

Throughout this paper, the following assumptions will be considered:

(H1) $h_0, p_0 \in L^\infty(\Omega)$, $h_0(x) \geq 0$, $p_0(x) \geq 0$, a.e. $x \in \Omega$,

$$||h_0(x)||_{L^\infty(\Omega)}, \qquad ||p_0(x)||_{L^\infty(\Omega)} > 0; \tag{1.2}$$

(H2) $r,k,a \in C([0,+\infty))$ satisfy

$$\begin{gathered} r(t) = r(t+T), \quad k(t) = k(t+T), \quad a(t) = a(t+T), \quad \forall t \geq 0, \\ k(t) \geq k_0 > 0, \quad \forall t \geq 0 \text{ (where } k_0 \text{ is a constant)}, \\ \int_0^T r(t)\,dt > 0, \\ a(t) \geq a_0 > 0, \quad \forall t \geq 0 \text{ (where } a_0 \text{ is a constant)}; \end{gathered} \tag{1.3}$$

(H3) $f_1, f_2 : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and locally Lipschitz continuous with respect to $(h,p)$ and satisfy

$$\begin{gathered} f_1(t,h,p) = f_1(t+T,h,p), \quad f_2(t,h,p) = f_2(t+T,h,p), \quad \forall t \geq 0,\ h \geq 0,\ p \geq 0, \\ \exists C > 0 \text{ such that } 0 \leq f_1(t,h,p), \quad f_2(t,h,p) \leq C, \quad \forall t \geq 0,\ h \geq 0,\ p \geq 0; \end{gathered} \tag{1.4}$$

(H4) the application $h \mapsto hf_2(t,h,p)$ is nondecreasing on $[0,+\infty)$, $\forall t \geq 0$, $\forall p \geq 0$;

(H5) the application $p \mapsto f_2(t,h,p)$ is nonincreasing on $[0,+\infty)$, $\forall t \geq 0$, $\forall h \geq 0$.

Condition $\int_0^T r(t)\,dt > 0$ is a persistence condition for the preys in the absence of predators. So, if $p_0 \equiv 0$ and $h_0(x) > 0$ a.e. in $\Omega$, then the necessary and sufficient condition for the persistence of preys is the above-mentioned one.

For basic results concerning the solutions of periodic predator-prey systems (without diffusion) we refer to [4].

Let $\omega \subset \mathbb{R}^N$ be a nonempty domain with a smooth-enough boundary $\partial\omega$ and satisfying $\omega \subset\subset \Omega$. We denote by $m$ the characteristic function of $\omega$.

The questions we want to investigate are the following.

(1) Is there any control $u \in L^\infty_{\text{loc}}(\overline{\omega} \times [0,\infty))$ such that the solution to the initial-boundary value problem

$$\begin{gathered} h_t - d_1\Delta h = r(t)h - k(t)h^2 - f_1(t,h,p)hp, \quad x \in \Omega,\ t > 0, \\ p_t - d_2\Delta p = -a(t)p + f_2(t,h,p)hp + m(x)u(x,t), \quad x \in \Omega,\ t > 0, \\ \frac{\partial h}{\partial \nu} = \frac{\partial p}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ h(x,0) = h_0(x), \quad p(x,0) = p_0(x), \quad x \in \Omega, \end{gathered} \tag{1.5}$$

satisfies

$$\begin{gathered} h(x,t) \geq 0, \quad p(x,t) \geq 0 \quad \text{a.e. } x \in \Omega,\ \forall t \geq 0, \\ \lim_{t\to\infty} p(t) = 0 \quad \text{in } L^\infty(\Omega)? \end{gathered} \tag{1.6}$$

(2) Is there any control $v \in L^\infty_{\text{loc}}(\overline{\omega} \times [0,\infty))$ such that the solution to the initial-boundary value problem

$$\begin{gathered} h_t - d_1\Delta h = r(t)h - k(t)h^2 - f_1(t,h,p)hp + m(x)v(x,t), \quad x \in \Omega,\ t > 0, \\ p_t - d_2\Delta p = -a(t)p + f_2(t,h,p)hp, \quad x \in \Omega,\ t > 0, \\ \frac{\partial h}{\partial \nu} = \frac{\partial p}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ h(x,0) = h_0(x), \quad p(x,0) = p_0(x), \quad x \in \Omega, \end{gathered} \tag{1.7}$$

satisfies (1.6)?

*Definition 1.1.* Say that the predator population is *p-zero stabilizable* if for any $h_0$, $p_0$ satisfying (H1), the answer to the first question is affirmative. *p-zero stabilizable* means that the zero stabilizability holds for controls acting only on the predator population.

*Definition 1.2.* Say that the predator population is *h-zero stabilizable* if for any $h_0$, $p_0$ satisfying (H1), the answer to the second question is affirmative. *h-zero stabilizable* means that the zero stabilizability holds for controls acting only on the prey population.

We are dealing here with some results of zero stabilizability with state constraints.

First notice that, due to assumption (H3) and to the comparison principle for parabolic equations, the solution $(h,p)$ to (1.1) satisfies

$$0 \le h(x,t) \le \overline{h}(x,t) \quad \text{a.e. } x \in \Omega,\ \forall t \ge 0, \tag{1.8}$$

where $\overline{h}$ is the solution to

$$\begin{gathered}\overline{h}_t - d_1\Delta\overline{h} = r(t)\overline{h} - k(t)\overline{h}^2, \quad x \in \Omega,\ t > 0,\\ \frac{\partial\overline{h}}{\partial\nu} = 0, \quad x \in \Omega,\ t > 0,\\ \overline{h}(x,0) = h_0(x), \quad x \in \Omega.\end{gathered} \tag{1.9}$$

Lemma 1.3. *The solution $\overline{h}$ to (1.9) satisfies*

$$\lim_{t\to\infty} \|\overline{h}(t) - \widetilde{h}(t)\|_{L^\infty(\Omega)} = 0, \tag{1.10}$$

*where $\widetilde{h}$ is the unique nontrivial nonnegative solution to the following problem:*

$$\begin{gathered}\widetilde{h}_t - d_1\Delta\widetilde{h} = r(t)\widetilde{h} - k(t)\widetilde{h}^2, \quad x \in \Omega,\ t > 0,\\ \frac{\partial\widetilde{h}}{\partial\nu} = 0, \quad x \in \Omega,\ t > 0,\\ \widetilde{h}(x,t) = \widetilde{h}(x,t+T), \quad x \in \Omega,\ t > 0.\end{gathered} \tag{1.11}$$

*Remark 1.4.* In fact, we will show that (1.11) has exactly two nonnegative solutions, the trivial one and the unique nontrivial and nonnegative solution to

$$\begin{gathered}g_t = r(t)g - k(t)g^2, \quad t > 0,\\ g(t) = g(t+T), \quad t > 0.\end{gathered} \tag{1.12}$$

If $\int_0^T r(t)dt \le 0$, then (1.12) has a unique nonnegative solution (the trivial one). This follows by a simple calculation and taking into account that the first equation in (1.12) is a Bernoulli equation.

*Proof of Lemma 1.3.* Since $\|h_0\|_{L^\infty(\Omega)} > 0$, it follows that there exists a positive constant $\rho_1 > 0$ such that

$$\overline{h}(x,T) \ge \rho_1 > 0 \quad \text{a.e. } x \in \Omega \tag{1.13}$$

(this is a consequence of a result in [5]). Therefore, we can assert that

$$\overline{h}(x,t) \ge h^{\rho_1}(t), \quad \text{a.e. } x \in \Omega,\ \forall t \ge T, \tag{1.14}$$

where $h^{\rho_1}(t)$ is the solution to

$$\begin{gathered}(h^\rho)_t - d_1\Delta h^\rho = r(t)h^\rho - k(t)(h^\rho)^2, \quad x\in\Omega,\ t>T,\\ \frac{\partial h^\rho}{\partial\nu} = 0, \quad x\in\Omega,\ t>T,\\ h^\rho(x,T) = \rho, \quad x\in\Omega,\end{gathered} \tag{1.15}$$

corresponding to $\rho := \rho_1$ ($h^{\rho_1}$ does not depend explicitly on $x$).

If we choose $\rho_1 > 0$ sufficiently small and taking into account that $\int_0^T r(t)dt > 0$, it follows that

$$h^{\rho_1}(T) < h^{\rho_1}(2T). \tag{1.16}$$

By mathematical induction, we get that

$$h^{\rho_1}(t+T+nT) \le h^{\rho_1}(t+T+(n+1)T), \quad \forall t\in[0,T],\ \forall n\in\mathbb{N} \tag{1.17}$$

and consequently

$$h_n^{\rho_1}(t) \le h_{n+1}^{\rho_1}(t), \quad \text{a.e. } x\in\Omega,\ \forall t\in[0,T], \tag{1.18}$$

for any $n\in\mathbb{N}$, where $h_n^{\rho_1}(t) = h^{\rho_1}(t+T+nT)$, $\forall t\in[0,T]$. Obviously, $h_n^{\rho_1}$ is the solution of

$$\begin{gathered}(h_n^{\rho_1})_t - d_1\Delta h_n^{\rho_1} = r(t)h_n^{\rho_1} - k(t)(h_n^{\rho_1})^2, \quad x\in\Omega,\ t\in(0,T),\\ \frac{\partial h_n^{\rho_1}}{\partial\nu} = 0, \quad x\in\Omega,\ t\in(0,T),\\ h_n^{\rho_1}(x,0) = h_{n-1}^{\rho_1}(x,T) = h^{\rho_1}(x,T+nT), \quad x\in\Omega,\end{gathered} \tag{1.19}$$

for any $n\in\mathbb{N}^*$.

In the same manner, taking $\rho_2 > 0$ sufficiently large, we can obtain a nonincreasing bounded sequence $h_n^{\rho_2}$, where $h_n^{\rho_2}(t) = h^{\rho_2}(t+T+nT)$, for all $t\in[0,T]$, for all $n\in\mathbb{N}$ and $h^{\rho_2}$ is the solution to (1.15) corresponding to $\rho := \rho_2$.

Using the comparison result for parabolic equations, we have that

$$h_n^{\rho_1}(t) \le \overline{h}(x,t+(n+1)T) \le h_n^{\rho_2}(t), \quad \text{a.e. } x\in\Omega,\ \forall t\in[0,T],\ \forall n\in\mathbb{N}. \tag{1.20}$$

Taking into account (1.20), we may pass to the limit in (1.19) and get that

$$h_n^{\rho_1} \longrightarrow \widetilde{h}_1, \tag{1.21}$$

in $C([0,T])$, as $n\to+\infty$, where $\widetilde{h}_1$ is a positive solution (has only positive values) of

$$\begin{gathered}\widetilde{h}_t - d_1\Delta\widetilde{h} = r(t)\widetilde{h} - k(t)\widetilde{h}^2, \quad x\in\Omega,\ t\in(0,T),\\ \frac{\partial\widetilde{h}}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t\in(0,T),\\ \widetilde{h}(x,0) = \widetilde{h}(x,T), \quad x\in\Omega,\end{gathered} \tag{1.22}$$

where $\widetilde{h}_1$ does not depend explicitly on $x$ (because $h_n^{\rho_1}$ does not). We may extend $\widetilde{h}_1$ by $T$-periodicity to $[0,+\infty)$ and we deduce that $\widetilde{h}_1$ is a positive solution to (1.11) and to (1.12). Since (1.12) has a unique nontrivial nonnegative solution, we may infer that this one is $\widetilde{h}_1$. So,

$$\lim_{t\to+\infty}\left|h^{\rho_1}(t) - \widetilde{h}_1(t)\right| = 0. \tag{1.23}$$

In the same manner, it follows that

$$\lim_{t\to+\infty}\left|h^{\rho_2}(t) - \widetilde{h}_1(t)\right| = 0. \tag{1.24}$$

By (1.20) we conclude that

$$\lim_{t\to\infty}\|\overline{h}(t) - \widetilde{h}_1(t)\|_{L^\infty(\Omega)} = 0. \tag{1.25}$$

Let us prove that there is only one nontrivial and nonnegative solution to (1.11).

Let $\widetilde{h}_2$ be a nontrivial and nonnegative solution to (1.11). It follows immediately that there exists $\rho_0>0$ (see [5]) such that $\widetilde{h}_2(x,T)\geq\rho_0$ a.e. $x\in\Omega$. If we choose $\rho_1$ and $\rho_2$ such that $0<\rho_1<\rho_0\leq\widetilde{h}_2(x,0)=\widetilde{h}_2(x,T)\leq\rho_2$ a.e. $x\in\Omega$ with $\rho_1$ small enough and $\rho_2$ large enough, then it follows as before that $\widetilde{h}_2\equiv\widetilde{h}_1$ (because $h_n^{\rho_1}(t)\leq\widetilde{h}_2(x,t)\leq h_n^{\rho_2}(t)$ a.e. $x\in\Omega$, for all $t\in[0,T]$, for all $n\in\mathbb{N}$) and so we get the conclusion of the lemma. □

Let us consider now the corresponding equation in $p$ for $h:=\widetilde{h}$, that is,

$$\begin{gathered}p_t - d_2\Delta p = -a(t)p + f_2(t,\widetilde{h}(t),p)\widetilde{h}(t)p, \quad x\in\Omega,\ t>0,\\ \frac{\partial p}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t>0,\\ p(x,0) = p_0(x), \quad x\in\Omega.\end{gathered} \tag{1.26}$$

Having in mind (H5), we obtain that

$$f_2(t,h,p)\leq f_2(t,h,0), \quad \forall t,h,p\geq0, \tag{1.27}$$

therefore, the solution $p$ to (1.26) satisfies (using the comparison principle for parabolic equations)

$$0\leq p(x,t)\leq\overline{p}(x,t), \quad \text{a.e. } x\in\Omega,\ \forall t\geq0, \tag{1.28}$$

where $\overline{p}$ is a solution to

$$\begin{aligned} \overline{p}_t - d_2\Delta\overline{p} &= -a(t)\overline{p} + f_2\big(t,\widetilde{h}(t),0\big)\widetilde{h}(t)\overline{p}, \quad x \in \Omega,\ t > 0, \\ \frac{\partial\overline{p}}{\partial\nu} &= 0, \quad x \in \partial\Omega,\ t > 0, \\ \overline{p}(x,0) &= p_0(x), \quad x \in \Omega. \end{aligned} \tag{1.29}$$

This may be rewritten as

$$\begin{aligned} \overline{p}_t - d_2\Delta\overline{p} &= l(t)\overline{p}, \quad x \in \Omega,\ t > 0, \\ \frac{\partial\overline{p}}{\partial\nu} &= 0, \quad x \in \partial\Omega,\ t > 0, \\ \overline{p}(x,0) &= p_0(x), \quad x \in \Omega, \end{aligned} \tag{1.30}$$

where

$$l(t) = f_2\big(t,\widetilde{h}(t),0\big)\widetilde{h}(t) - a(t), \quad \forall t \geq 0. \tag{1.31}$$

Thus, the solution $\overline{p}$ can be written as

$$\overline{p}(x,t) = \exp\left\{\int_0^t l(\tau)\mathrm{d}\tau\right\} f(x,t), \quad x \in \Omega,\ t \geq 0 \tag{1.32}$$

with $f$ solution to

$$\begin{aligned} f_t - d_2\Delta f &= 0, \quad x \in \Omega,\ t > 0, \\ \frac{\partial f}{\partial\nu} &= 0, \quad x \in \partial\Omega,\ t > 0, \\ f(x,0) &= p_0(x), \quad x \in \Omega. \end{aligned} \tag{1.33}$$

Lemma 1.5. *There exist a real constant $\alpha^*$ and a $T$-periodic continuous function $w : [0,+\infty) \to \mathbb{R}$ such that*

$$\exp\left\{\int_0^t l(\tau)\mathrm{d}\tau\right\} = \exp\{\alpha^* t\} w(t), \quad \forall t \geq 0. \tag{1.34}$$

Indeed, one can check directly that, due to the periodicity assumptions made on $a$ and $f_2$, for $\alpha^* = (1/T)\int_0^T l(\tau)\mathrm{d}\tau$, the function

$$w(t) = \exp\left\{\int_0^t \big(l(s) - \alpha^*\big)\mathrm{d}s\right\}, \quad \forall t \geq 0, \tag{1.35}$$

is a $T$-periodic function.

Let us denote by $\lambda_1$ the principal eigenvalue of the following eigenvalue problem

$$-d_2\Delta\varphi = \lambda\varphi, \quad x \in \Omega,$$
$$\frac{\partial\varphi}{\partial\nu} = 0, \quad x \in \partial\Omega. \tag{1.36}$$

Remark that $\lambda_{-1} = 0$. Now, we notice that if $\lambda_1 > \alpha^*$, then (1.32) and (1.34) imply that the predator population goes to extinction without any control. Therefore, in the rest of this paper we will assume

(H6) $0 < \alpha^*$.

For basic results concerning the solutions to predator-prey systems we refer to [1, 6]. Stabilization of predator-prey systems with $r$, $k$, $a$ constants has been investigated in [7, 8]. If in (1.1) the predator is an alien population, then our main goal is to eliminate this population. This problem and its importance have been discussed in [9]. We will investigate next what happens in the cases when we act with a control with support in $\overline{\omega}$.

Section 2 is devoted to the study of $p$-zero stabilization, while Section 3 concerns the $h$-zero stabilization. Some remarks are given in Section 4.

## 2. The $p$-zero stabilization of the predator population

Denote by $\lambda_1^{\omega,p}$ the principal eigenvalue of the next problem

$$-d_2\Delta\varphi = \lambda\varphi \quad \text{in } \Omega \setminus \overline{\omega},$$
$$\varphi = 0 \quad \text{on } \partial\omega, \tag{2.1}$$
$$\frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega.$$

Then, according to Rayleigh's principle (see [10]), $\lambda_1^{\omega,p}$ satisfies

$$\lambda_1^{\omega,p} = \min\left\{d_2\int_{\Omega\setminus\overline{\omega}} |\nabla\varphi|^2 dx;\ \varphi \in H^1(\Omega \setminus \omega),\ \varphi = 0 \text{ on } \partial\omega,\ \|\varphi\|_{L^2(\Omega\setminus\overline{\omega})} = 1\right\}. \tag{2.2}$$

Here is one of the main results of our paper.

Theorem 2.1. *If the predator population is p-zero stabilizable, then* $\lambda_1^{\omega,p} \geq \alpha^*$, *where*

$$\alpha^* = \frac{1}{T}\int_0^T l(s)\, ds \tag{2.3}$$

*and l is defined by (1.31).*

*Conversely, if* $\lambda_1^{\omega,p} > \alpha^*$, *then the predator population is p-zero stabilizable and, for* $\gamma > 0$ *large enough, the feedback control* $u := -\gamma p$ *realizes (1.6), where* $(h, p)$ *is the nonnegative solution to (1.5) corresponding to* $u := -\gamma p$.

In order to prove Theorem 2.1, we need first to establish two auxiliary results. For any $\gamma \geq 0$ we consider the following problem:

$$-d_2\Delta\varphi + m(x)\gamma\varphi = \lambda\varphi \quad \text{in } \Omega,$$
$$\frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega \tag{2.4}$$

and denote by $\lambda_{1,\gamma}^{p}$ its principal eigenvalue.

Lemma 2.2.

$$\lim_{\gamma\to\infty}\lambda_{1,\gamma}^{p} = \lambda_1^{\omega,p}. \tag{2.5}$$

*Proof of Lemma 2.2.* By Rayleigh's principle, one gets

$$\lambda_{1,\gamma}^{p} = \min\left\{d_2\int_\Omega |\nabla\varphi|^2 \mathrm{d}x + \gamma\int_\omega |\varphi|^2 \mathrm{d}x;\ \varphi \in H^1(\Omega),\ \|\varphi\|_{L^2(\Omega)} = 1\right\}. \tag{2.6}$$

Hence, for every $0 \leq \gamma_1 \leq \gamma_2$, we have

$$\lambda_{1,\gamma_1}^{p} \leq \lambda_{1,\gamma_2}^{p}. \tag{2.7}$$

Now, denoting by $\varphi_1$ the corresponding eigenfunction to $\lambda_1^{\omega,p}$ satisfying $\|\varphi_1\|_{L^2(\Omega)} = 1$, $\varphi_1(x) \geq 0$ a.e. $x \in \Omega$, we get that $\varphi_1$ is the minimum point for the right-hand side of (2.2).

We extend $\varphi_1$ to $\Omega$ as follows:

$$\tilde{\varphi}(x) = \begin{cases} \varphi_1(x), & x \in \Omega\setminus\overline{\omega}, \\ 0, & x \in \omega. \end{cases} \tag{2.8}$$

Then

$$\lambda_1^{\omega,p} = d_2\int_\Omega |\nabla\tilde{\varphi}|^2 \mathrm{d}x + \gamma\int_\omega |\tilde{\varphi}|^2 \mathrm{d}x \geq \lambda_{1,\gamma}^{p}, \quad \forall\gamma \geq 0. \tag{2.9}$$

Thus one obtains

$$\lim_{\gamma\to\infty}\lambda_{1,\gamma}^{p} \leq \lambda_1^{\omega,p}. \tag{2.10}$$

To prove the equality, let us consider $\varphi_\gamma \in H^1(\Omega)$ such that $\|\varphi_\gamma\|_{L^2(\Omega)} = 1$ and

$$\lambda_{1,\gamma}^{p} = d_2\int_\Omega |\nabla\varphi_\gamma|^2 \mathrm{d}x + \gamma\int_\omega |\varphi_\gamma|^2 \mathrm{d}x \leq \lambda_1^{\omega,p}. \tag{2.11}$$

It follows that there exists a constant $M \geq 0$ such that

$$\int_\Omega |\nabla\varphi_\gamma|^2 \mathrm{d}x \leq M, \quad \gamma\int_\omega |\varphi_\gamma^2| \mathrm{d}x \leq M, \quad \forall\gamma \geq 0. \tag{2.12}$$

Therefore, there exists a subsequence (also denoted by $\{\varphi_\gamma\}$), such that

$$\begin{aligned} &\varphi_\gamma \longrightarrow \varphi^* \quad \text{weakly in } H^1(\Omega), \\ &\varphi_\gamma \longrightarrow \varphi^* \quad \text{in } L^2(\Omega), \\ &\varphi_\gamma \longrightarrow 0 \quad \text{in } L^2(\omega). \end{aligned} \tag{2.13}$$

Hence, $\varphi^* \in H^1(\Omega \setminus \overline{\omega})$, $\|\varphi^*\|_{L^2(\Omega\setminus\overline{\omega})} = 1$, $\varphi^* \equiv 0$ in $\omega$, and one may infer that $\varphi^* = 0$ on $\partial\omega$. Thus by (2.11) we get that

$$\lim_{\gamma\to\infty} \lambda_{1,\gamma}^{p} \geq \lambda_1^{\omega,p}. \tag{2.14}$$

By (2.10) and (2.14) we get the conclusion of Lemma 2.2. □

Lemma 2.3. *Let $(h,p)$ be a nonnegative solution to (1.5), corresponding to the control $u \in L^\infty_{\mathrm{loc}}(\overline{\omega}\times[0,\infty))$. If*

$$\lim_{t\to\infty} p(t) = 0 \quad \textit{in } L^\infty(\Omega), \tag{2.15}$$

*then*

$$\lim_{t\to\infty} (h(t) - \widetilde{h}(t)) = 0 \quad \textit{in } L^\infty(\Omega), \tag{2.16}$$

*where $\widetilde{h}$ is the unique nontrivial nonnegative solution to (1.11).*

*Proof.* Since

$$\lim_{t\to\infty} p(t) = 0 \quad \text{in } L^\infty(\Omega), \tag{2.17}$$

it follows that, for every small enough $\delta > 0$, there exists $t_\delta > 0$ such that

$$0 \leq p(t,x) \leq \delta \quad \text{a.e. } x \in \Omega,\ \forall t \geq t_\delta. \tag{2.18}$$

By (H3) we get that

$$0 \leq f_1(t, h(x,t), p(x,t))\, p \leq C\delta, \quad \text{a.e. } x \in \Omega,\ \forall t \geq t_\delta. \tag{2.19}$$

Let us denote now by $h_1$ and $h_2$ the solutions to the following problems, respectively:

$$\begin{aligned}
&(h_1)_t - d_1\Delta h_1 = r(t)h_1 - k(t)h_1^2 - C\delta h_1, \quad x \in \Omega,\ t > t_\delta, \\
&\frac{\partial h_1}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > t_\delta, \\
&h_1(x,t_\delta) = \rho_1, \quad x \in \Omega, \\
&(h_2)_t - d_1\Delta h_2 = r(t)h_2 - k(t)h_2^2, \quad x \in \Omega,\ t > t_\delta, \\
&\frac{\partial h_2}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > t_\delta, \\
&h_2(x,t_\delta) = \rho_2, \quad x \in \Omega,
\end{aligned} \tag{2.20}$$

where $\rho_1 > 0$ is a small enough constant and $\rho_2$ is a large enough constant, such that

$$0 < \rho_1 < h(x,t_\delta) < \rho_2 \quad \text{a.e. } x \in \Omega \tag{2.21}$$

(existence of such $\rho_1$ is a consequence of a result in [5]).

Then, by the comparison principle for the parabolic equations, we obtain

$$h_1(x,t) \le h(x,t) \le h_2(x,t), \quad \text{a.e. } x \in \Omega,\ \forall t \ge t_\delta. \tag{2.22}$$

As in the proof of Lemma 1.3 we can prove that $h_2$ satisfies

$$\begin{aligned}
&\lim_{t\to\infty} |h_2(t) - \widetilde{h}(t)| = 0, \\
&\lim_{t\to\infty} |h_1(t) - \widetilde{h}_\delta(t)| = 0,
\end{aligned} \tag{2.23}$$

where $\widetilde{h}_\delta$ is the unique nontrivial nonnegative solution to

$$\begin{aligned}
&\widetilde{h}_t - d_1\Delta\widetilde{h} = r(t)\widetilde{h} - k(t)\widetilde{h}^2 - C\delta\widetilde{h}, \quad x \in \Omega,\ t > 0, \\
&\frac{\partial\widetilde{h}}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\
&\widetilde{h}(x,t) = \widetilde{h}(x,t+T), \quad x \in \Omega,\ t \ge 0.
\end{aligned} \tag{2.24}$$

Since $\delta \mapsto \widetilde{h}_\delta$ is a decreasing function, then we may pass to the limit in (2.24) and get that

$$\lim_{t\to\infty} |\widetilde{h}_\delta(t) - \widetilde{h}(t)| = 0. \tag{2.25}$$

By (2.22)–(2.24) we get the conclusion. □

*Proof of Theorem 2.1.* Assume that $p_0(x) > 0$ a.e. $x \in \Omega$ and let $(h,p)$ be a nonnegative solution to (1.5) corresponding to the $p$-stabilizing control $u \in L^\infty_{\text{loc}}(\overline{\omega} \times [0,\infty))$. Since

$$\lim_{t\to\infty} ||p(t)||_{L^\infty(\Omega)} = 0, \tag{2.26}$$

it follows by Lemma 2.3 that

$$\lim_{t\to\infty}\|h(t)-\widetilde{h}(t)\|_{L^\infty(\Omega)}=0, \tag{2.27}$$

which implies, due to the continuity of the function $f_2$, that, for any $\varepsilon>0$, there exists $t_\varepsilon \geq 0$ such that

$$\|h(t)f_2(t,h(t),p(t))-\widetilde{h}(t)f_2(t,\widetilde{h}(t),0)\|_{L^\infty(\Omega)}<\varepsilon, \tag{2.28}$$

for any $t\geq t_\varepsilon$.

Let $\varepsilon>0$ be arbitrary but fixed. Denoting now by $p_1$ the solution to the following problem:

$$\begin{gathered}
(p_1)_t-d_2\Delta p_1=-a(t)p_1+f_2(t,\widetilde{h}(t),0)\widetilde{h}(t)p_1-\varepsilon p_1,\quad x\in\Omega\setminus\overline{\omega},\ t>t_\varepsilon,\\
p_1=0,\quad x\in\partial\omega,\ t>t_\varepsilon,\\
\frac{\partial p_1}{\partial\nu}=0,\quad x\in\partial\Omega,\ t>t_\varepsilon,\\
p_1(x,t_\varepsilon)=p(x,t_\varepsilon),\quad x\in\Omega\setminus\overline{\omega},
\end{gathered} \tag{2.29}$$

we obtain via the comparison principle for parabolic equations and using (2.28) that

$$0\leq p_1(x,t)\leq p(x,t),\quad \text{a.e. } x\in\Omega\setminus\overline{\omega},\ \forall t\geq t_\varepsilon. \tag{2.30}$$

Let $\varphi_1$ be an eigenfunction corresponding to $\lambda_1^{\omega,p}$ and satisfying $\|\varphi_1\|_{L^2(\Omega\setminus\overline{\omega})}=1$, $\varphi_1(x)\geq 0$ a.e. $x\in\Omega\setminus\overline{\omega}$ and denote by $\langle\cdot,\cdot\rangle$ the usual inner product in $L^2(\Omega\setminus\overline{\omega})$. Then

$$\langle p_1(t),\varphi_1\rangle'+(\lambda_1^{\omega,p}-l(t)+\varepsilon)\langle p_1(t),\varphi_1\rangle=0,\quad \forall t\geq t_\varepsilon. \tag{2.31}$$

We infer that

$$\langle p_1(t),\varphi_1\rangle=\exp\left\{-\lambda_1^{\omega,p}(t-t_\varepsilon)+\int_{t_\varepsilon}^t(l(s)-\varepsilon)ds\right\}\langle p(t_\varepsilon),\varphi_1\rangle,\quad \forall t\geq t_\varepsilon. \tag{2.32}$$

The $p$-zero stabilizability and (2.30) imply that

$$\lim_{t\to\infty}p_1(t)=0\quad \text{in } L^\infty(\Omega\setminus\overline{\omega}). \tag{2.33}$$

Since $p(x,t_\varepsilon)>0$ a.e. $x\in\Omega$ (see [5]), we conclude that

$$-\lambda_1^{\omega,p}T+\int_0^T l(t)dt-\varepsilon T<0. \tag{2.34}$$

Making $\varepsilon\to 0$ we get the conclusion. □

Conversely, assume that $\lambda_1^{\omega,p} > \alpha^*$. Then, by Lemma 2.2, we have that for $\varepsilon > 0$ small enough and for $\gamma \geq 0$ large enough

$$\lambda_{1,\gamma}^{p} - \varepsilon > \alpha^*. \tag{2.35}$$

Set now $u := -\gamma p$ and let $(h, p)$ be the corresponding solution to (1.5). Using (1.9) and Lemma 1.3, we get that for every $\varepsilon > 0$, there exists $T_\varepsilon \geq 0$, such that

$$h(t,x) f_2(t, h(t,x), p(t,x)) < \widetilde{h}(t) f_2(t, \widetilde{h}(t), 0) + \varepsilon, \quad \text{a.e. } x \in \Omega,\ \forall t \geq T_\varepsilon. \tag{2.36}$$

Denote by $p_2$ the solution to the following problem:

$$\begin{gathered}
(p_2)_t - d_2 \Delta p_2 = -a(t) p_2 + f_2(t, \widetilde{h}(t), 0) \widetilde{h}(t) p_2 + \varepsilon p_2 - m(x) \gamma p_2, \quad x \in \Omega,\ t > T_\varepsilon, \\
\frac{\partial p_2}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > T_\varepsilon, \\
p_2(x, T_\varepsilon) = \varphi_{1\gamma}(x), \quad x \in \Omega,
\end{gathered} \tag{2.37}$$

where $\varphi_{1\gamma}$ is an eigenfunction of (2.4) corresponding to $\lambda := \lambda_{1,\gamma}^{p}$ and satisfying $\varphi_{1\gamma}(x) \geq p(x, T_\varepsilon)$ a.e. $x \in \Omega$.

Applying the comparison result for parabolic equations, we conclude that

$$0 \leq p(x,t) \leq p_2(x,t), \quad \text{a.e. } x \in \Omega,\ \forall t \geq T_\varepsilon. \tag{2.38}$$

This yields

$$p_2(x,t) \leq \varphi_{1\gamma}(x) \exp\left\{ -\lambda_{1,\gamma}^{p}(t - T_\varepsilon) + \int_{T_\varepsilon}^{t} (l(s) + \varepsilon)\, \mathrm{d}s \right\}, \quad \text{a.e. } x \in \Omega,\ \forall t \geq T_\varepsilon. \tag{2.39}$$

Since $\lambda_{1,\gamma}^{p} > (1/T) \int_0^T l(s)\mathrm{d}s + \varepsilon$, it follows that

$$p_2(t) \longrightarrow 0 \quad \text{in } L^\infty(\Omega), \tag{2.40}$$

which implies that

$$p(t) \longrightarrow 0 \quad \text{in } L^\infty(\Omega), \tag{2.41}$$

as $t \to +\infty$, at the same rate as $\exp\{(-\lambda_{1,\gamma}^{p} + \alpha^* + \varepsilon)t\}$.

*Remark 2.4.* Since

$$\lim_{\gamma \to +\infty,\quad \varepsilon \to 0+} (\lambda_{1,\gamma}^{p} - \varepsilon) = \lambda_1^{\omega,p}, \tag{2.42}$$

we see how important it would be to maximize $\lambda_1^{\omega,p}$ with respect to the location and geometry of $\omega$ and $\Omega$.

## 3. The $h$-zero stabilization of the predator population

In this section, we are looking for a stabilizing control $v$ acting indirectly (acting on the prey population). Let us consider $(h,p)$ a solution to (1.7) corresponding to the feedback control $v := -\gamma h$. The system becomes

$$\begin{gathered}
h_t - d_1\Delta h = r(t)h - k(t)h^2 - f_1(t,h,p)hp - m(x)\gamma h, \quad x \in \Omega,\ t > 0, \\
p_t - d_2\Delta p = -a(t)p + f_2(t,h,p)hp, \quad x \in \Omega,\ t > 0, \\
\frac{\partial h}{\partial \nu} = \frac{\partial p}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\
h(x,0) = h_0(x), \quad p(x,0) = p_0(x), \quad x \in \Omega.
\end{gathered} \tag{3.1}$$

For any $\gamma \geq 0$ we consider the following eigenvalue problem:

$$\begin{gathered}
-d_1\Delta\Psi + m(x)\gamma\Psi = \lambda\Psi \quad \text{in } \Omega, \\
\frac{\partial\Psi}{\partial\nu} = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.2}$$

and denote by $\lambda_{1,\gamma}^h$ its principal eigenvalue. Next, we denote by $\lambda_1^{\omega,h}$ the principal eigenvalue to

$$\begin{gathered}
-d_1\Delta\Psi = \lambda\Psi, \quad x \in \Omega \setminus \overline{\omega}, \\
\Psi = 0, \quad x \in \partial\omega, \\
\frac{\partial\Psi}{\partial\nu} = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{3.3}$$

It is a consequence of Rayleigh's principle that the mapping $\gamma \mapsto \lambda_{1,\gamma}^h$ is increasing and continuous, and

$$\lambda_{1,\gamma}^h \longrightarrow \lambda_1^{\omega,h} \quad \text{as } \gamma \longrightarrow \infty. \tag{3.4}$$

Let

$$\tilde{\alpha}^* = \frac{1}{T}\int_0^T r(s)\mathrm{d}s. \tag{3.5}$$

In the same manner as in Section 2 it follows the next result.

Theorem 3.1. *If for a $\gamma \geq 0$ one has that $\lambda_{1,\gamma}^h > \tilde{\alpha}^*$, then the predator population is h-zero stabilizable and the feedback control $v := -\gamma h$ realizes (1.6), where $(h,p)$ is the solution to (1.7) corresponding to $v := -\gamma h$. Moreover,*

$$\lim_{t\to+\infty} h(t) = 0 \quad \text{in } L^\infty(\Omega). \tag{3.6}$$

*Remark 3.2.* Assume that the hypotheses in Theorem 3.1 hold. Since $h(t)\to 0$ in $L^\infty(\Omega)$, as $t\to+\infty$, then it follows (as in Section 2) that $p(t)\to 0$ in $L^\infty(\Omega)$, as $t\to+\infty$, at the rate of

$$\exp\left\{-\left(\frac{1}{T}\int_0^T a(s)ds+\varepsilon\right)t\right\} \tag{3.7}$$

(for $\varepsilon>0$ small enough).

If, in addition, $(1/T)\int_0^T a(s)ds>\lambda_1^{\omega,p}$, then the second strategy (when we act on prey) leads to a faster convergence to zero of $p$, so it is better.

*Remark 3.3.* If $\lambda_1^{\omega,h}>\tilde{\alpha}^*$, then there exists $\gamma\geq 0$ such that $\lambda_{1,\gamma}^h>\tilde{\alpha}^*$. The solution $(h,p)$ to (3.1) satisfies

$$h(t)\longrightarrow 0 \quad \text{in } L^\infty(\Omega), \tag{3.8}$$

as $t\to+\infty$. Therefore,

$$p(t)\longrightarrow 0 \quad \text{in } L^\infty(\Omega), \tag{3.9}$$

as $t\to+\infty$.

*Remark 3.4.* In general, the habitat of preys is larger than $\Omega$. The strategy to eradicate the predators via indirect control is the following one: we isolate the domain $\Omega$ (we do not permit migration through the boundary of it), then we eradicate firstly the preys in $\Omega$ and consequently the predators will extinct. Next, the preys are allowed to repopulate the domain $\Omega$.

## 4. Final comments

The results in Sections 2 (and 3) show how important is to find the position and the geometry of $\omega$ and $\Omega$ in order to get a great value for $\lambda_1^{\omega,p}$ (and $\lambda_1^{\omega,h}$).

This yields

$$\lambda_1^{\omega,p}=d_2\lambda_1(\omega,\Omega), \qquad \lambda_1^{\omega,h}=d_1\lambda_1(\omega,\Omega), \tag{4.1}$$

where $\lambda_1(\omega,\Omega)$ is the principal eigenvalue to

$$\begin{gathered} -\Delta\varphi(x)=\lambda\varphi(x), \quad x\in\Omega\setminus\overline{\omega}, \\ \varphi(x)=0, \quad x\in\partial\omega, \\ \frac{\partial\varphi}{\partial\nu}=0, \quad x\in\partial\Omega. \end{gathered} \tag{4.2}$$

The following result has been proved in [8] using rearrangement techniques and can be used to obtain upper and lower bounds for $\lambda_1(\omega,\Omega)$.

THEOREM 4.1. *Assume that $\varphi^*$ is an eigenfunction of (4.2), corresponding to $\lambda := \lambda_1(\omega,\Omega)$, that satisfies in addition*

$$\begin{aligned} 0 < \varphi^*(x) < M, &\quad \forall x \in \Omega \setminus \overline{\omega}, \\ \varphi^*(x) = M, &\quad \forall x \in \partial\Omega, \end{aligned} \tag{4.3}$$

*where $M > 0$ is a constant. Then*

$$\lambda_1(\omega,\Omega) > \lambda_1(\omega,\tilde{\Omega}), \tag{4.4}$$

*for any domain $\tilde{\Omega} \subset \mathbb{R}^N$ with smooth boundary and such that $\omega \subset\subset \tilde{\Omega}$,* $\text{meas}(\tilde{\Omega}) = \text{meas}(\Omega)$, *and $\tilde{\Omega} \not\equiv \Omega$.*

*Remark 4.2.* If $\omega$ and $\Omega$ are balls with the same center, there exists such $\varphi^*$.

*Remark 4.3.* If there exists $\varphi^*$ an eigenfunction of (4.2) corresponding to $\lambda := \lambda_1(\omega,\Omega)$ and satisfying (4.3), then

$$\begin{aligned} \lambda_1(\omega,\Omega) &= \max\{\lambda_1(\omega,\tilde{\Omega});\ \tilde{\Omega} \subset \mathbb{R}^N \text{ is a domain with smooth} \\ &\qquad \text{boundary and satisfying } \omega \subset\subset \tilde{\Omega},\ \text{meas}\,(\tilde{\Omega}) = \text{meas}(\Omega)\} \\ &= \max\{\lambda_1(\tilde{\omega},\Omega);\ \tilde{\omega} \subset\subset \Omega \text{ is an isometric transform of } \omega\}. \end{aligned} \tag{4.5}$$

*Remark 4.4.* If $\omega$ is a ball, $\omega \subset\subset \Omega$, then we may conclude by Theorem 4.1 that

$$\lambda_1(\omega,\Omega) \le \lambda_1(\omega,B), \tag{4.6}$$

where $B$ is a ball with the same measure as $\Omega$ and with the same center as $\omega$. Moreover, we have equality only for $\Omega \equiv B$ and we conclude that the maximal value for $\lambda_1(\omega,\Omega)$, subject to all domains $\Omega \subset \mathbb{R}^N$ with smooth boundary and satisfying $\omega \subset\subset \Omega$ and having a prescribed measure, is attained for the ball $B$ of the same measure and with the same center as $\omega$.

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