

# AN EXISTENCE RESULT FOR A SEMIPOSITONE PROBLEM WITH A SIGN CHANGING WEIGHT

JAFFAR ALI AND R. SHIVAJI

Received 5 March 2005; Accepted 5 September 2005

We establish an existence result on positive solution for a class of reaction-diffusion equation with semipositone structure. In particular, our results apply to the diffusive logistic equation with a class of sign changing weight and constant yield harvesting. We establish the result via the method of subsuper solutions.

Copyright © 2006 J. Ali and R. Shivaji. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

## 1. Introduction

In this paper we discuss the existence of positive classical solutions ( $u \in C^{2,\alpha}(\bar{\Omega})$ ) of the boundary value problem

$$\begin{aligned} -\Delta u &= \lambda(g(x)[u(1-u^p)] - ch(x)), & x \in \Omega, \\ u &= 0, & x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where  $p > 0$ ,  $c > 0$ , and  $\lambda > 0$  are parameters and  $\Omega$  is an open bounded region with boundary  $\partial\Omega$  in class  $C^2$  in  $\mathbb{R}^n$  for  $n \geq 1$ . Here  $g : \bar{\Omega} \rightarrow \mathbb{R}$  is a  $C^\alpha$  function while  $h : \Omega \rightarrow \mathbb{R}$  is a nonnegative  $C^\alpha$  function with  $\|h\|_\infty = 1$ . When  $p = 1$ , (1.1) arises in population dynamics where  $1/\lambda$  is the diffusion coefficient and  $ch(x)$  represents the constant yield harvesting. In this case ( $p = 1$ ), when  $g(x)$  is a positive constant, various results have been established in [4]. Here we focus on sign changing weight functions  $g$ .

To precisely define our classes of weight functions, we first let  $\lambda_1 > 0$  be the principal eigenvalue and  $\phi > 0$  with  $\|\phi\|_\infty = 1$  the corresponding eigenfunction of  $-\Delta$  with the Dirichlet boundary conditions. It is well known that  $\partial\phi/\partial\eta < 0$  on  $\partial\Omega$  where  $\eta$  is the unit outward normal. Hence there exists  $\delta > 0$ ,  $\sigma > 0$ , and  $m > 0$  such that

$$|\nabla\phi|^2 - \lambda_1\phi^2 \geq m \quad \text{on } \bar{\Omega}_\delta, \tag{1.2}$$

$$\phi \geq \sigma \quad \text{on } \Omega - \bar{\Omega}_\delta, \tag{1.3}$$

where  $\Omega_\delta := \{x \in \Omega \mid d(x, \partial\Omega) < \delta\}$ .

## 2 A semipositone problem with a sign changing weight

In this paper we assume that the weight  $g$  takes negative values in  $\Omega_\delta$  but requires  $g$  to be strictly positive in  $\Omega - \Omega_\delta$ . Define  $\gamma := \min_{\Omega - \Omega_\delta} g(x)$ ,  $\mu := \min_{\overline{\Omega}_\delta} g(x)$ , and we assume that

$$|\mu| < \frac{m\gamma}{\lambda_1} \left( \frac{1}{p+1} \right)^{1/p}. \quad (1.4)$$

Further let  $0 < x_1 < x_2 < \gamma/2\lambda_1$  be the positive roots of  $q(x) = -\mu$  (see Figure 1.1), where

$$q(x) := x \left[ 1 - \frac{2\lambda_1}{\gamma} x \right]^{1/p} \left( \frac{p+1}{p} \right) 2m. \quad (1.5)$$

Then we establish the following.

**THEOREM 1.1.** *Suppose (1.4) holds,  $1/x_2 < \lambda < 1/x_1$  and  $c \leq c_0(\lambda)$ , where*

$$c_0(\lambda) := \min \left\{ \left( \frac{1}{p+1} \right)^{1/p} \left[ \frac{2m}{\lambda} \left( 1 - \frac{2\lambda_1}{\lambda\gamma} \right)^{1/p} + \frac{\mu p}{(p+1)} \right], \frac{p\gamma\sigma^2}{(p+1)^{(p+1)/p}} \left[ 1 - \frac{2\lambda_1}{\lambda\gamma} \right]^{(p+1)/p} \right\}. \quad (1.6)$$

*Then (1.1) has at least one positive solution  $u$  such that  $\|u\|_\infty < 1$ .*

Note that when  $c > 0$ , (1.1) is a semipositone problem and it is well known in the literature that the study of positive solutions is mathematically challenging (see [2–4]). Here we also include the additional challenge of dealing with a sign changing weight function  $g$ .

Finally, we also deduce a result for the case when  $g(x) \geq 0$  on  $\overline{\Omega}_\delta$ . In particular we prove the following.

**COROLLARY 1.2.** *If  $g(x) \geq 0$  on  $\overline{\Omega}_\delta$  and  $c = 0$ , then for any  $\lambda \geq 2\lambda_1/\gamma$  (1.1) has a positive solution.*

We establish our results by the method of subsuper solutions. By a subsolution we mean a function  $w \in C^2(\overline{\Omega})$  such that

$$\begin{aligned} -\Delta w &\leq \lambda(g(x)[w(1-w^p)] - \chi_h(x)), & x \in \Omega, \\ w &\leq 0, & x \in \partial\Omega, \end{aligned} \quad (1.7)$$

and by a supersolution a function  $v \in C^2(\overline{\Omega})$  such that

$$\begin{aligned} -\Delta v &\geq \lambda(g(x)[v(1-v^p)] - \chi_h(x)), & x \in \Omega, \\ v &\geq 0, & x \in \partial\Omega. \end{aligned} \quad (1.8)$$

Then it is well known (see [1, 5]) that if there exists a subsolution  $w$  and a supersolution  $v$  such that  $w < v$ , then there exists a solution  $u \in C^2(\overline{\Omega})$  such that  $w \leq u \leq v$ .

We will prove Theorem 1.1 in Section 2 and Corollary 1.2 in Section 3.

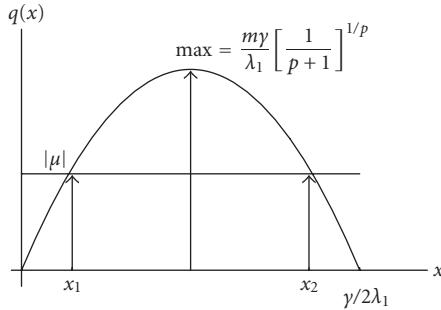


Figure 1.1

## 2. Proof of Theorem 1.1

*Proof.* Let  $w = k_0\phi^2$ , where

$$k_0 = \left( \frac{1}{p+1} \right)^{1/p} \left[ 1 - \frac{2\lambda_1}{\lambda\gamma} \right]^{1/p}. \quad (2.1)$$

We will prove that  $w$  is a subsolution. Now

$$-\Delta w = -\nabla \cdot \nabla (k_0\phi^2) = -\nabla \cdot (2k_0\phi\nabla\phi) = -2k_0(\nabla\phi \cdot \nabla\phi + \phi\Delta\phi) = 2k_0(\lambda_1\phi^2 - |\nabla\phi|^2). \quad (2.2)$$

First we consider the case when  $x \in \overline{\Omega}_\delta$ . Since the maximum of  $s(1-s^p)$  is  $p/(p+1)^{(p+1)/p}$ , we have

$$\lambda(g(x)[w(1-w^p)] - ch(x)) \geq \lambda \left( \mu \left[ \frac{p}{(p+1)^{(p+1)/p}} \right] - c \right). \quad (2.3)$$

Since

$$c < c_0 \leq \left( \frac{1}{p+1} \right)^{1/p} \left[ \frac{2m}{\lambda} \left( 1 - \frac{2\lambda_1}{\lambda\gamma} \right)^{1/p} + \frac{\mu p}{(p+1)} \right] = \frac{2k_0 m}{\lambda} + \frac{\mu p}{(p+1)^{(p+1)/p}}, \quad (2.4)$$

combining (2.3)-(2.4) and using (1.2)-(2.2), we have

$$\lambda \left( \mu \left[ \frac{p}{(p+1)^{(p+1)/p}} \right] - c \right) \geq -\Delta w. \quad (2.5)$$

Hence

$$-\Delta w \leq (g(x)[w(1-w^p)] - ch(x)) \quad \text{on } \overline{\Omega}_\delta. \quad (2.6)$$

#### 4 A semipositone problem with a sign changing weight

Next consider the case when  $x \in \Omega - \overline{\Omega}_\delta$ . By the definition of  $\gamma$ , we have

$$\begin{aligned}
& \lambda(g(x)[w(1 - w^p)] - \text{ch}(x)) \\
& \geq \lambda(\gamma[k_0\phi^2(1 - k_0^p\phi^{2p})] - c) \geq \lambda(\gamma[k_0\phi^2(1 - k_0^p)] - c) \\
& \geq \lambda\left(\gamma[k_0\phi^2(1 - k_0^p)] - \frac{p\gamma}{(p+1)^{(p+1)/p}}\left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]^{(p+1)/p}\sigma^2\right) \quad \text{since } c \leq c_0 \\
& \geq \lambda\left(\gamma[k_0\phi^2(1 - k_0^p)] - \frac{p\gamma}{(p+1)}\left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]k_0\phi^2\right) \quad \text{using (1.3), (2.1)} \\
& = \lambda\gamma k_0\phi^2\left\{1 - k_0^p - \frac{p}{(p+1)}\left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]\right\} \\
& = \lambda\gamma k_0\phi^2\{1 - k_0^p - pk_0^p\} \quad \text{by (2.1)} \\
& = \lambda\gamma k_0\phi^2\{1 - [p+1]k_0^p\} \\
& = \lambda\gamma k_0\phi^2\left\{1 - \left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]\right\} \quad \text{by (2.1)} \\
& = 2k_0\lambda_1\phi^2 \geq 2k_0[\lambda_1\phi^2 - |\nabla\phi|^2] \\
& = -\Delta w \quad \text{using (2.2).}
\end{aligned} \tag{2.7}$$

Hence

$$-\Delta w \leq (g(x)[w(1 - w^p)] - \text{ch}(x)) \quad \text{on } \Omega - \overline{\Omega}_\delta. \tag{2.8}$$

From (2.6) and (2.8) we have

$$-\Delta w \leq (g(x)[w(1 - w^p)] - \text{ch}(x)) \quad \text{on } \Omega. \tag{2.9}$$

Thus  $w = k_0\phi^2$  is a subsolution of (1.1).

Next it is easy to see that  $v \equiv 1$  is a supersolution of (1.1) and  $v > w$  on  $\overline{\Omega}$ . Thus we have a positive solution  $u$  such that  $\|u\|_\infty < 1$ .  $\square$

#### 3. Proof of Corollary 1.2

*Proof.* Since  $g(x) \geq 0$  and  $c = 0$ , on  $\overline{\Omega}_\delta$ ,  $\lambda(g(x)[w(1 - w^p)]) \geq 0$ . But  $-\Delta w \leq -2k_0m$  and is negative; hence, on  $\overline{\Omega}_\delta$ , we have

$$-\Delta w \leq g(x)[w(1 - w^p)] \quad \text{on } \overline{\Omega}_\delta, \tag{3.1}$$

and on  $\Omega - \overline{\Omega}_\delta$ , we have

$$\begin{aligned}
& \lambda g(x)[w(1 - w^p)] \\
& \geq \lambda \gamma [k_0 \phi^2 (1 - k_0^p \phi^{2p})] \geq \lambda \gamma [k_0 \phi^2 (1 - k_0^p)] \\
& \geq \lambda \gamma k_0 \phi^2 \left[ 1 - \frac{1}{p+1} \left[ 1 - \frac{2\lambda_1}{\lambda \gamma} \right] \right] \quad \text{by (2.1)} \\
& = \frac{k_0 \phi^2}{p+1} [p \lambda \gamma + 2\lambda_1] \\
& \geq \frac{k_0 \phi^2}{p+1} [2\lambda_1(p+1)] \quad \text{since } \lambda \geq \frac{2\lambda_1}{\gamma} \\
& = 2\lambda_1 k_0 \phi^2 \\
& \geq 2k_0 [\lambda_1 \phi^2 - |\nabla \phi|^2] = -\Delta w.
\end{aligned} \tag{3.2}$$

Hence we have

$$-\Delta w \leq g(x)[w(1 - w^p)] \quad \text{on } \Omega - \overline{\Omega}_\delta. \tag{3.3}$$

Using (3.1)–(3.3) we have that  $w = k_0 \phi^2$  is a subsolution. Again we note that  $v \equiv 1$  is a supersolution. Hence the result holds.  $\square$

## References

- [1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review **18** (1976), no. 4, 620–709.
- [2] H. Berestycki, L. A. Caffarelli, and L. Nirenberg, *Inequalities for second-order elliptic equations with applications to unbounded domains. I*, Duke Mathematical Journal **81** (1996), no. 2, 467–494, A celebration of John F. Nash Jr.
- [3] P. L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Review. A Publication of the Society for Industrial and Applied Mathematics **24** (1982), no. 4, 441–467.
- [4] S. Oruganti, J. Shi, and R. Shivaji, *Diffusive logistic equation with constant yield harvesting. I. Steady states*, Transactions of the American Mathematical Society **354** (2002), no. 9, 3601–3619.
- [5] D. H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana University Mathematics Journal **21** (1971/1972), 979–1000.

Jaffar Ali: Department of Mathematics, Mississippi State University, Mississippi State, MS 39762, USA

*E-mail address:* js415@ra.msstate.edu

R. Shivaji: Department of Mathematics, Mississippi State University, Mississippi State, MS 39762, USA

*E-mail address:* shivaji@ra.msstate.edu

## Special Issue on Boundary Value Problems on Time Scales

### Call for Papers

The study of dynamic equations on a time scale goes back to its founder Stefan Hilger (1988), and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics; moreover, it often reveals the reasons for the discrepancies between two theories.

In recent years, the study of dynamic equations has led to several important applications, for example, in the study of insect population models, neural network, heat transfer, and epidemic models. This special issue will contain new researches and survey articles on Boundary Value Problems on Time Scales. In particular, it will focus on the following topics:

- Existence, uniqueness, and multiplicity of solutions
- Comparison principles
- Variational methods
- Mathematical models
- Biological and medical applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/ade/guidelines.html>. Authors should follow the Advances in Difference Equations manuscript format described at the journal site <http://www.hindawi.com/journals/ade/>. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	April 1, 2009
First Round of Reviews	July 1, 2009
Publication Date	October 1, 2009

### Lead Guest Editor

**Alberto Cabada**, Departamento de Análise Matemática, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain; [alberto.cabada@usc.es](mailto:alberto.cabada@usc.es)

### Guest Editor

**Victoria Otero-Espinar**, Departamento de Análise Matemática, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain; [mvictoria.oter@usc.es](mailto:mvictoria.oter@usc.es)