

# AN EXISTENCE RESULT FOR A SEMIPOSITONE PROBLEM WITH A SIGN CHANGING WEIGHT

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We establish an existence result on positive solution for a class of reaction-diffusion equation with semipositone structure. In particular, our results apply to the diffusive logistic equation with a class of sign changing weight and constant yield harvesting. We establish the result via the method of subsuper solutions.

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## 1. Introduction

In this paper we discuss the existence of positive classical solutions ( $u \in C^{2,\alpha}(\overline{\Omega})$ ) of the boundary value problem

$$\begin{aligned} -\Delta u &= \lambda(g(x)[u(1 - u^p)] - ch(x)), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where  $p > 0$ ,  $c > 0$ , and  $\lambda > 0$  are parameters and  $\Omega$  is an open bounded region with boundary  $\partial\Omega$  in class  $C^2$  in  $\mathbb{R}^n$  for  $n \geq 1$ . Here  $g: \overline{\Omega} \rightarrow \mathbb{R}$  is a  $C^\alpha$  function while  $h: \Omega \rightarrow \mathbb{R}$  is a nonnegative  $C^\alpha$  function with  $\|h\|_\infty = 1$ . When  $p = 1$ , (1.1) arises in population dynamics where  $1/\lambda$  is the diffusion coefficient and  $ch(x)$  represents the constant yield harvesting. In this case ( $p = 1$ ), when  $g(x)$  is a positive constant, various results have been established in [4]. Here we focus on sign changing weight functions  $g$ .

To precisely define our classes of weight functions, we first let  $\lambda_1 > 0$  be the principal eigenvalue and  $\phi > 0$  with  $\|\phi\|_\infty = 1$  the corresponding eigenfunction of  $-\Delta$  with the Dirichlet boundary conditions. It is well known that  $\partial\phi/\partial\eta < 0$  on  $\partial\Omega$  where  $\eta$  is the unit outward normal. Hence there exists  $\delta > 0$ ,  $\sigma > 0$ , and  $m > 0$  such that

$$|\nabla\phi|^2 - \lambda_1\phi^2 \geq m \quad \text{on } \overline{\Omega}_\delta, \tag{1.2}$$

$$\phi \geq \sigma \quad \text{on } \Omega - \overline{\Omega}_\delta, \tag{1.3}$$

where  $\Omega_\delta := \{x \in \Omega \mid d(x, \partial\Omega) < \delta\}$ .

## 2 A semipositone problem with a sign changing weight

In this paper we assume that the weight  $g$  takes negative values in  $\Omega_\delta$  but requires  $g$  to be strictly positive in  $\Omega - \Omega_\delta$ . Define  $\gamma := \min_{\Omega - \Omega_\delta} g(x)$ ,  $\mu := \min_{\overline{\Omega}_\delta} g(x)$ , and we assume that

$$|\mu| < \frac{m\gamma}{\lambda_1} \left( \frac{1}{p+1} \right)^{1/p}. \quad (1.4)$$

Further let  $0 < x_1 < x_2 < \gamma/2\lambda_1$  be the positive roots of  $q(x) = -\mu$  (see Figure 1.1), where

$$q(x) := x \left[ 1 - \frac{2\lambda_1}{\gamma} x \right]^{1/p} \left( \frac{p+1}{p} \right) 2m. \quad (1.5)$$

Then we establish the following.

**THEOREM 1.1.** *Suppose (1.4) holds,  $1/x_2 < \lambda < 1/x_1$  and  $c \leq c_0(\lambda)$ , where*

$$c_0(\lambda) := \min \left\{ \left( \frac{1}{p+1} \right)^{1/p} \left[ \frac{2m}{\lambda} \left( 1 - \frac{2\lambda_1}{\lambda\gamma} \right)^{1/p} + \frac{\mu p}{(p+1)} \right], \frac{p\gamma\sigma^2}{(p+1)^{(p+1)/p}} \left[ 1 - \frac{2\lambda_1}{\lambda\gamma} \right]^{(p+1)/p} \right\}. \quad (1.6)$$

*Then (1.1) has at least one positive solution  $u$  such that  $\|u\|_\infty < 1$ .*

Note that when  $c > 0$ , (1.1) is a semipositone problem and it is well known in the literature that the study of positive solutions is mathematically challenging (see [2–4]). Here we also include the additional challenge of dealing with a sign changing weight function  $g$ .

Finally, we also deduce a result for the case when  $g(x) \geq 0$  on  $\overline{\Omega}_\delta$ . In particular we prove the following.

**COROLLARY 1.2.** *If  $g(x) \geq 0$  on  $\overline{\Omega}_\delta$  and  $c = 0$ , then for any  $\lambda \geq 2\lambda_1/\gamma$  (1.1) has a positive solution.*

We establish our results by the method of subsuper solutions. By a subsolution we mean a function  $w \in C^2(\overline{\Omega})$  such that

$$\begin{aligned} -\Delta w &\leq \lambda(g(x)[w(1-w^p)] - ch(x)), \quad x \in \Omega, \\ w &\leq 0, \quad x \in \partial\Omega, \end{aligned} \quad (1.7)$$

and by a supersolution a function  $v \in C^2(\overline{\Omega})$  such that

$$\begin{aligned} -\Delta v &\geq \lambda(g(x)[v(1-v^p)] - ch(x)), \quad x \in \Omega, \\ v &\geq 0, \quad x \in \partial\Omega. \end{aligned} \quad (1.8)$$

Then it is well known (see [1, 5]) that if there exists a subsolution  $w$  and a supersolution  $v$  such that  $w < v$ , then there exists a solution  $u \in C^2(\overline{\Omega})$  such that  $w \leq u \leq v$ .

We will prove Theorem 1.1 in Section 2 and Corollary 1.2 in Section 3.

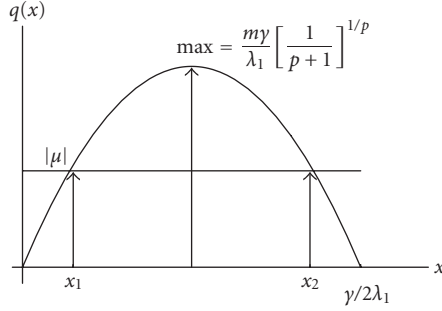


Figure 1.1

## 2. Proof of Theorem 1.1

*Proof.* Let  $w = k_0\phi^2$ , where

$$k_0 = \left( \frac{1}{p+1} \right)^{1/p} \left[ 1 - \frac{2\lambda_1}{\lambda\gamma} \right]^{1/p}. \quad (2.1)$$

We will prove that  $w$  is a subsolution. Now

$$-\Delta w = -\nabla \cdot \nabla (k_0\phi^2) = -\nabla \cdot (2k_0\phi \nabla \phi) = -2k_0(\nabla \phi \cdot \nabla \phi + \phi \Delta \phi) = 2k_0(\lambda_1\phi^2 - |\nabla \phi|^2). \quad (2.2)$$

First we consider the case when  $x \in \overline{\Omega}_\delta$ . Since the maximum of  $s(1-s^p)$  is  $p/(p+1)^{(p+1)/p}$ , we have

$$\lambda(g(x)[w(1-w^p)] - ch(x)) \geq \lambda \left( \mu \left[ \frac{p}{(p+1)^{(p+1)/p}} \right] - c \right). \quad (2.3)$$

Since

$$c < c_0 \leq \left( \frac{1}{p+1} \right)^{1/p} \left[ \frac{2m}{\lambda} \left( 1 - \frac{2\lambda_1}{\lambda\gamma} \right)^{1/p} + \frac{\mu p}{(p+1)} \right] = \frac{2k_0 m}{\lambda} + \frac{\mu p}{(p+1)^{(p+1)/p}}, \quad (2.4)$$

combining (2.3)-(2.4) and using (1.2)-(2.2), we have

$$\lambda \left( \mu \left[ \frac{p}{(p+1)^{(p+1)/p}} \right] - c \right) \geq -\Delta w. \quad (2.5)$$

Hence

$$-\Delta w \leq (g(x)[w(1-w^p)] - ch(x)) \quad \text{on } \overline{\Omega}_\delta. \quad (2.6)$$

#### 4 A semipositone problem with a sign changing weight

Next consider the case when  $x \in \Omega - \overline{\Omega}_\delta$ . By the definition of  $\gamma$ , we have

$$\begin{aligned}
& \lambda(g(x)[w(1 - w^p)] - \text{ch}(x)) \\
& \geq \lambda(\gamma[k_0\phi^2(1 - k_0^p\phi^{2p})] - c) \geq \lambda(\gamma[k_0\phi^2(1 - k_0^p)] - c) \\
& \geq \lambda\left(\gamma[k_0\phi^2(1 - k_0^p)] - \frac{p\gamma}{(p+1)^{(p+1)/p}}\left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]^{(p+1)/p}\sigma^2\right) \quad \text{since } c \leq c_0 \\
& \geq \lambda\left(\gamma[k_0\phi^2(1 - k_0^p)] - \frac{p\gamma}{(p+1)}\left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]k_0\phi^2\right) \quad \text{using (1.3), (2.1)} \\
& = \lambda\gamma k_0\phi^2\left\{1 - k_0^p - \frac{p}{(p+1)}\left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]\right\} \\
& = \lambda\gamma k_0\phi^2\{1 - k_0^p - pk_0^p\} \quad \text{by (2.1)} \\
& = \lambda\gamma k_0\phi^2\{1 - [p+1]k_0^p\} \\
& = \lambda\gamma k_0\phi^2\left\{1 - \left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]\right\} \quad \text{by (2.1)} \\
& = 2k_0\lambda_1\phi^2 \geq 2k_0[\lambda_1\phi^2 - |\nabla\phi|^2] \\
& = -\Delta w \quad \text{using (2.2).}
\end{aligned} \tag{2.7}$$

Hence

$$-\Delta w \leq (g(x)[w(1 - w^p)] - \text{ch}(x)) \quad \text{on } \Omega - \overline{\Omega}_\delta. \tag{2.8}$$

From (2.6) and (2.8) we have

$$-\Delta w \leq (g(x)[w(1 - w^p)] - \text{ch}(x)) \quad \text{on } \Omega. \tag{2.9}$$

Thus  $w = k_0\phi^2$  is a subsolution of (1.1).

Next it is easy to see that  $v \equiv 1$  is a supersolution of (1.1) and  $v > w$  on  $\overline{\Omega}$ . Thus we have a positive solution  $u$  such that  $\|u\|_\infty < 1$ .  $\square$

### 3. Proof of Corollary 1.2

*Proof.* Since  $g(x) \geq 0$  and  $c = 0$ , on  $\overline{\Omega}_\delta$ ,  $\lambda(g(x)[w(1 - w^p)]) \geq 0$ . But  $-\Delta w \leq -2k_0m$  and is negative; hence, on  $\overline{\Omega}_\delta$ , we have

$$-\Delta w \leq g(x)[w(1 - w^p)] \quad \text{on } \overline{\Omega}_\delta, \tag{3.1}$$

and on  $\Omega - \overline{\Omega}_\delta$ , we have

$$\begin{aligned}
 & \lambda g(x)[w(1 - w^p)] \\
 & \geq \lambda \gamma [k_0 \phi^2 (1 - k_0^p \phi^{2p})] \geq \lambda \gamma [k_0 \phi^2 (1 - k_0^p)] \\
 & \geq \lambda \gamma k_0 \phi^2 \left[ 1 - \frac{1}{p+1} \left[ 1 - \frac{2\lambda_1}{\lambda \gamma} \right] \right] \quad \text{by (2.1)} \\
 & = \frac{k_0 \phi^2}{p+1} [p\lambda \gamma + 2\lambda_1] \\
 & \geq \frac{k_0 \phi^2}{p+1} [2\lambda_1 (p+1)] \quad \text{since } \lambda \geq \frac{2\lambda_1}{\gamma} \\
 & = 2\lambda_1 k_0 \phi^2 \\
 & \geq 2k_0 [\lambda_1 \phi^2 - |\nabla \phi|^2] = -\Delta w.
 \end{aligned} \tag{3.2}$$

Hence we have

$$-\Delta w \leq g(x)[w(1 - w^p)] \quad \text{on } \Omega - \overline{\Omega}_\delta. \tag{3.3}$$

Using (3.1)–(3.3) we have that  $w = k_0 \phi^2$  is a subsolution. Again we note that  $v \equiv 1$  is a supersolution. Hence the result holds.  $\square$

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