

# ON DYNAMICS OF VISCOELASTIC MULTIDIMENSIONAL MEDIUM WITH VARIABLE BOUNDARY

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We prove the existence and uniqueness theorems for solutions of an initial-boundary value problem to the system of equations, which describes dynamics of viscoelastic continuous medium with a variable boundary and a memory along the trajectories of particles in classes of summable functions.

## 1. Introduction and main results

Let  $\Omega_t \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a family of the bounded domains with boundary  $\Gamma_t$ . We consider the following initial-boundary value problem:

$$\begin{aligned} \rho(t, x) \left( V_t + \sum_{k=1}^n V_k \frac{\partial V}{\partial x_k} \right) - \mu \hat{\Delta} V - \operatorname{Div} G(B(V)) + \operatorname{grad} \Phi(\rho(t, x)) &= \rho(t, x) F(t, x); \\ \rho_t(t, x) + \operatorname{div}(\rho(t, x) V(t, x)) &= 0, \quad (t, x) \in Q = \{(t, x) : 0 \leq t \leq t_0, x \in \Omega_t\}; \\ V(0, x) &= V^0(x), \quad x \in \Omega_0; \\ V(t, x) &= Y(t, x), \quad (t, x) \in \Gamma = \{(t, x) : 0 \leq t \leq t_0, x \in \Gamma_t\}. \end{aligned} \tag{1.1}$$

Here  $V(t, x) = (V_1, \dots, V_n)$ ,  $\rho(t, x)$  are a vector function and a scalar function which denote the velocity and the density of the medium,  $G(Z)$  is an  $n \times n$  matrix function with the coefficients  $g_{ij} = g_{ij}(z_{11}, \dots, z_{nn})$  whose arguments are the coefficients of an  $n \times n$  matrix  $Z$ ,  $\Phi$  is a scalar function,  $G$  and  $\Phi$  are supposed to be smooth,  $\mu > 0$ . Next,  $B(V) = z_x(0, t, x)$ , where  $z(\tau, t, x)$  is a solution to the Cauchy problem (in the integral form)

$$z(\tau, t, x) = x + \int_t^\tau V(s, z(s; t, x)) ds, \quad (t, x) \in Q; \tag{1.2}$$

$$\hat{\Delta} V = \frac{1}{2} \operatorname{Div} (V_x + V_x^*). \tag{1.3}$$

Here  $V_x$  is the Jacobi matrix of a vector function  $V(t, x)$ ,  $\text{Div} Z$  means the divergence of a matrix function  $Z$ , that is, the vector whose coefficients are the divergences of the rows of a matrix function  $Z$ ,  $Z^*$  is the adjoint matrix to  $Z$ .

Problem (1.1) describes the dynamics of a viscoelastic continuous medium which occupies at the moment  $t$  the position  $\Omega_t$ . Particles  $y \in \Omega_0$  of the medium move in the field of velocities  $V(t, x)$  along trajectories determined by the function

$$x = z(t, 0, y) \equiv u(t, y), \quad y \in \Omega_0, \quad (1.4)$$

with the velocity  $v(t, y) = u_t(t, y)$ . Thus, the field of velocities  $V(t, x)$  is associated with velocities of particles  $v(t, y)$  by the relations

$$v(t, y) = V(t, u(t, y)), \quad V(t, x) = v(t, U(t, x)), \quad U(t, x) = z(0, t, x). \quad (1.5)$$

Recall that the variables  $x$  and  $y$  are called Euler and Lagrange coordinates, respectively. It is supposed that the stress tensor  $T_s$  of the medium has the form

$$T_s = \mu T_v + \nu T_e + I\Phi(\rho). \quad (1.6)$$

Here  $T_v = (1/2)(V_x + V_x^*)$  is the rate of strain tensor; the strain tensor has the form (see [3, Chapter 1, page 78])

$$T_e = \omega_1 I + \omega_2 U_x U_x^* + \omega_3 (U_x U_x^*)^2, \quad (1.7)$$

where the functions  $\omega_i = \omega_i(I_1, I_2, I_3)$  are smooth functions of the principal invariants  $I_i$  of the matrix  $U_x U_x^*$ ,  $U(t, x) = z(0, t, x)$ . We consider a more general situation assuming that  $T_e = G(U_x U_x^*)$ .

The study of nonlinear viscous continuous mediums with a memory along trajectories of particles was initiated in [6] for the stationary case. The dynamics case was considered in [2, 15]. The viscoelastic problem (1.1) and its generalization in the case of cylindrical domain was studied in [8, 9, 10, 11]. The existence and uniqueness theorems and the stability of solutions in classes of summable functions under necessary conditions on the data were established. Dynamics problems in the case of a variable boundary for viscous medium (the case of a noncylindrical domain) was studied in [7]. An important moment in these works was proceeding from the Euler to Lagrange coordinates. This approach allows to investigate viscoelastic mediums in the case of variable boundary as well as free boundary problems (cf. [14]).

Now we give the main results. It is convenient to exclude  $\rho(t, x)$  setting

$$\rho(t, x) = R(V), \quad R(V) = |U_x(t, x)| = |z_x(0, t, x)|, \quad (1.8)$$

and consider the density as an operator  $R(V)$ . Hereinafter  $|Z|$  denotes the determinant of a matrix  $Z$ . Then the original problem takes the form

$$R(V) \left( V_t + \sum_{k=1}^n V_k \frac{\partial V}{\partial x_k} \right) - \mu \hat{\Delta} V - \text{Div} G(B(V)) \quad (1.9)$$

$$+ \text{grad} \Phi(R(V)) = R(V)F(t, x), \quad (t, x) \in Q;$$

$$V(0, x) = V^0(x), \quad x \in \Omega_0; \quad V(t, x) = Y(t, x), \quad (t, x) \in \Gamma. \quad (1.10)$$

Let the function  $v(t, x) \in W_q^{1,2}(Q_0)$ ,  $Q_0 = \{(t, x) : 0 \leq t \leq t_0, x \in \Omega_0\}$ ,  $\Gamma_0 \in C^2$ . Then, the trace of the function  $v(t, x)$  on  $S_0 = [0, t_0] \times \Gamma_0$  belongs to  $W_q^{1-1/q, 2-2/q}(S_0) \equiv W$ . We assume that the domains  $\Omega_t$ ,  $0 \leq t \leq t_0$ , are defined as

$$\Omega_t = \{x : x = u(t, y), y \in \Omega_0\}, \quad (1.11)$$

where  $u(t, y) = y + \int_0^t v(s, y) ds$ ,  $y \in \Omega_0$ ,  $\Gamma_t = \{x : x = u(t, y), y \in \Gamma_0\}$ , respectively.

The value  $Y$  of the function  $V$  on  $\Gamma$  is defined as

$$Y(t, x) = v(t, U(t, x)), \quad (1.12)$$

where  $U(t, x)$  for each  $t$  is the inverse to  $u(t, y)$  map. The solution to problem (1.9), (1.10) is called summable in  $Q$  with the power  $q$  function  $V(t, x)$ , having generalized derivatives  $V_t(t, x)$  and  $V_{x_i, x_j}(t, x)$  whose  $q$ th power are summable, satisfying (1.9) and conditions (1.10). It follows necessarily from (1.10) that

$$V^0(y) = v(0, y), \quad (t, y) \in \Gamma_0. \quad (1.13)$$

The basic result reads as follows.

**THEOREM 1.1.** *Let  $q \in (n, \infty)$ ,  $F \in L_q(Q_0)$ ,  $V^0 \in W_q^{2-2/q}(\Omega_0)$ ,  $v \in W_q^{1,2}(Q_0)$ . Let (1.12) and (1.13) hold. Then problem (1.9), (1.10) has a unique solution  $V(t, x)$ , if  $t_0$  is small enough.*

It will be convenient to study problem (1.9), (1.10) by means of proceeding to Lagrange coordinates. For this purpose we make the change of variable (1.4)

in (1.9). Then problem (1.9), (1.10) takes the form

$$\begin{aligned}
 v_t(t, y) - \frac{1}{2}\mu \operatorname{Div} & \left( \left| I + \int_0^t v_y(s, y) ds \right| \right. \\
 & \times \left( v_y(s, y) \left( I + \int_0^t v_y(s, y) ds \right)^{-1} \right. \\
 & \quad \left. \left. + \left( I + \int_0^t v_y^*(s, y) ds \right)^{-1} v_y^*(t, y) \right) \left( I + \int_0^t v_y^*(s, y) ds \right)^{-1} \right) \\
 & - \operatorname{Div} \left( \left| I + \int_0^t v_y(s, y) ds \right| G \left( I + \int_0^t v_y(s, y) ds \right) \right. \\
 & \quad \left. \times \left( I + \int_0^t v_y^*(s, y) ds \right)^{-1} \right) \\
 & = f(t, y), \quad (t, x) \in Q_0;
 \end{aligned} \tag{1.14}$$

$$\begin{aligned}
 & - \left( I + \int_0^t v_y^*(s, y) ds \right)^{-1} \operatorname{grad} \Phi \left( \left| I + \int_0^t v_y(s, y) ds \right|^{-1} \right) \\
 & = f(t, y), \quad (t, x) \in Q_0;
 \end{aligned} \tag{1.15}$$

$$v(0, y) = V^0(y), \quad y \in \Omega_0; \quad v(t, y) = v(t, y), \quad (t, y) \in S_0.$$

Here

$$f(t, y) = F \left( t, y + \int_0^t v(s, y) ds \right). \tag{1.16}$$

A function  $v(t, y) \in W_q^{1,2}(Q_0)$  is called a solution to problem (1.14), (1.15), and (1.16) if it satisfies a.e. (1.14) and conditions (1.15).

**THEOREM 1.2.** *Let  $F \in L_q(Q_0)$ ,  $V^0 \in W_q^{2-2/q}(\Omega_0)$ ,  $v \in W_q^{1,2}(Q_0)$ ,  $q \in (n, \infty)$ . Let (1.13) hold. Then problem (1.14), (1.15), and (1.16) has a unique solution  $V(t, x)$ , if  $t_0$  is small enough.*

For the proof of Theorem 1.2, we consider the case when  $f \in L_q(Q_0)$  in (1.14) is an arbitrary function but not the function defined by (1.16).

**THEOREM 1.3.** *Let the conditions of Theorem 1.2 hold. Then for the sufficiently small  $t_0$ , problem (1.14), (1.15) has a unique solution.*

Thus, having Theorems 1.2 and 1.3 proved, we get Theorem 1.1. The paper is organized as follows: the proofs of Theorems 1.3, 1.2, and 1.1 are given in Sections 2, 3, and 4, respectively. In Section 5, the necessity of conditions on the data of the problems is discussed.

Below,  $\|\cdot\|_{k,m}$ ,  $\|\cdot\|_0$ ,  $|\cdot|_m$ ,  $|\cdot|_0$ , and  $]\cdot[$  stand for the norms in  $W_q^{k,m}(Q_0)$ ,  $L_q(Q_0)$ ,  $W_q^m(\Omega_0)$ ,  $L_q(\Omega_0)$ , and  $W$ , respectively,  $\|\cdot\|$  denotes the norm in  $\mathbb{R}^n$ . The function in this paper are scalar, vector, or matrix-valued and we do not

distinguish them in our notations; the situation is clear from the context. For definitions and properties of Sobolev spaces  $W$  see [1, Chapter 3, page 123].

## 2. Proof of Theorem 1.3

We give a number of the known facts for linear problems. The linear problem

$$\begin{aligned}\mathfrak{L}(v) &\equiv v_t - \mu \hat{\Delta} = \phi(t, y), \quad (t, y) \in Q_0; \\ v(0, y) &= 0, \quad y \in \Omega_0; \\ v(t, y) &= 0, \quad (t, y) \in S_0,\end{aligned}\tag{2.1}$$

has a unique solution for every  $\phi \in L_q(Q_0)$  (see [12, 13]) and the estimate

$$\|v\|_{1,2} + \sup_t |v(t, y)|_{2-2/q} \leq M \|\phi\|_0 \tag{2.2}$$

holds. If  $\phi = \text{Div } \psi$ , where  $\psi$  is a matrix function, and  $\psi \in W_q^{0,1}(Q_0)$ , then (2.2) implies  $\|v(t, y)\|_{0,1} \leq M \|\psi\|_0$ . Let  $v = L\phi$  be a solution to problem (2.1). Thus,  $L$  is the inverse to  $\mathfrak{L}$  operator. Consider the problem

$$\begin{aligned}\tilde{v}_t - \mu \hat{\Delta} \tilde{v} &= 0, \quad (t, y) \in Q_0; \\ \tilde{v}(0, y) &= V^0(y), \quad y \in \Omega_0; \\ \tilde{v}(t, y) &= v(t, y), \quad (t, y) \in S_0.\end{aligned}\tag{2.3}$$

Problem (2.3) has a unique solution (see [4, Chapter 4, page 388] and [12]), and for it, the estimate

$$\|\tilde{v}\|_{1,2} + \sup_t |\tilde{v}(t, y)|_{2-2/q} \leq M(|v_0|_{2-2/q} + \|v\|) \tag{2.4}$$

holds. Set  $v = \tilde{v} + w$ . Then  $w$  is a solution to the problem

$$\begin{aligned}\mathfrak{L}(w) &\equiv w_t - \mu \hat{\Delta} w = \tilde{\phi} + K(w); \\ w(0, y) &= 0, \quad y \in \Omega_0; \\ w(t, y) &= 0, \quad (t, y) \in S_0.\end{aligned}\tag{2.5}$$

Here

$$\tilde{\phi}(t, y) = f(t, y) - \mu \hat{\Delta} \tilde{v}, \quad K(w) = \mu K_1(w) + K_2(w) + K_3(w), \tag{2.6}$$

where by the notations

$$\begin{aligned}\hat{w}(t, y) &= \int_0^t \tilde{v}_y(s, y) ds + \int_0^t w_y(s, y) ds, \\ \tilde{w} &= I + \hat{w}, \quad \tilde{w}_* = (\tilde{w})^*, \quad \tilde{w}_*^{-1} = (\tilde{w}^{-1})^*,\end{aligned}\tag{2.7}$$

we set

$$\begin{aligned}
K_1(w) &= \operatorname{Div}(|\tilde{w}|((\tilde{v}_y + w_y)\tilde{w}^{-1} + \tilde{w}_*^{-1}(\tilde{v}_y^* + w_y^*))\tilde{w}_*^{-1}) \\
&\quad - \operatorname{Div}(w_y + w_y^* + \tilde{v}_y + \tilde{v}_y^*); \\
K_2(w) &= \operatorname{Div}(|w|G(\tilde{w})\tilde{w}_*^{-1}); \\
K_3(w) &= \tilde{w}_*^{-1} \operatorname{grad} \Phi(|I + \tilde{w}|^{-1}).
\end{aligned} \tag{2.8}$$

Applying the operator  $L$  to both parts of (2.5), we have

$$w = \tilde{K}(w), \quad \tilde{K}(w) = L\tilde{\phi} + LK(w). \tag{2.9}$$

Let  $S(R) = \{w : \|w\|_{1,2} \leq R\}$ . Consider (2.9) as an operator equation and show that on the ball  $S(R)$  it has a unique solution. For this purpose, we apply the fixed point theorem for contraction. Note that it cannot be applied to  $S(R)$  endowed with the  $\|\cdot\|_{1,2}$ -metric. However, considering  $S(R)$  as the metric space  $\mathfrak{M}$  endowed with the  $\|\cdot\|_{0,1}$ -metrics, it is possible to apply it.

LEMMA 2.1. *The metric space  $\mathfrak{M}$  is complete.*

*Proof.* Let a sequence  $w_n$ ,  $n = 1, 2, \dots$ , be a Cauchy sequence in  $\mathfrak{M}$ . Since  $\|w_n\|_{1,2} \leq R$ , then  $w_n$  (or its subsequence) weakly converges in  $W_q^{1,2}(Q_0)$  to  $\tilde{w} \in W_q^{1,2}(Q_0)$  and  $\|\tilde{w}\|_{1,2} \leq \liminf_{n \rightarrow \infty} \|w_n\|_{1,2} \leq R$ . Hence,  $\tilde{w} \in \mathfrak{M}$ . From the compact embedding  $W_q^{1,2}(Q_0) \subset W_q^{0,1}(Q_0)$ , it follows that  $w_n$  converges strongly to  $\tilde{w}$  in  $W_q^{0,1}(Q_0)$ . Hence,  $w_n$  converges to  $\tilde{w} \in \mathfrak{M}$  in  $\mathfrak{M}$ .  $\square$

LEMMA 2.2. *Let  $R$  be large enough and let  $t_0$  be small enough. Then the operator  $\tilde{K}$  transforms  $\mathfrak{M}$  into itself.*

*Proof.* Let  $w \in \mathfrak{M}$ . It follows from (2.2) that

$$\begin{aligned}
\|\tilde{K}(w)\|_{0,2} &\leq \|L\tilde{\phi}\|_{1,2} + \|LK(w)\|_{1,2} \\
&\leq M(\|\tilde{\phi}\|_0 + \|K(w)\|_0) \\
&\leq M\left(\|\tilde{\phi}\|_0 + \sum_{i=1}^3 \|K_i(w)\|_0\right).
\end{aligned} \tag{2.10}$$

We denote by  $M$  constants, which values are not important

$$\|K_i(w)\|_0 \leq M(R)t_0^{1-1/q}, \quad i = 1, 2, 3. \tag{2.11}$$

First, establish (2.11) for  $i = 1$ . Letting

$$\begin{aligned}
A_0 &= \tilde{w}, & A_1 &= |\tilde{w}|I, & A_2 &= w_y + w_y^* + \tilde{v}_y + \tilde{v}_y^*, & A_3 &= \tilde{w}^{-1}, \\
A_4 &= \tilde{w}_*^{-1}, & A_5 &= (\tilde{v}_y^* + w_y^*)\hat{w} + \hat{w}^*(\tilde{v}_y + w_y),
\end{aligned} \tag{2.12}$$

rewrite  $K_1(w)$  in the form

$$\begin{aligned}
 K_1(w) &= \operatorname{Div} (A_1 A_4 A_2 A_3 (A_4 - I)) + \operatorname{Div} (A_1 A_4 A_2 (A_3 - I)) \\
 &\quad + \operatorname{Div} ((A_1 - I) A_4 A_2) + \operatorname{Div} ((A_4 - I) A_2) \\
 &\quad + \operatorname{Div} (A_1 A_4 A_5 A_3 A_4) \\
 &= \sum_1^5 S_i.
 \end{aligned} \tag{2.13}$$

Here every term involves a small multiplier at small  $t_0$ . We need the following fact to estimate  $S_i$ . Let  $a_i$  be an arbitrary coefficient of a matrix  $A_i$ .

**PROPOSITION 2.3.** *If  $t_0 = t_0(R)$  is small enough, then the following inequalities hold:*

$$\max_{t,y} |a_i(t, y)| \leq M(R), \quad i = 0, 1, 2, 3, 4, 5; \tag{2.14}$$

$$\max_{t,y} |a_i(t, y) - 1| \leq M(R) t_0^{1-1/q}, \quad i = 0, 1, 3, 4; \tag{2.15}$$

$$\max_t \left| \frac{\partial}{\partial y_k} a_i(t, y) \right|_0 \leq M(R) t_0^{1-1/q}, \quad i \neq 2, 1 \leq k \leq n. \tag{2.16}$$

*Proof.* First, establish (2.14) for  $i = 0$ . Let  $\|\tilde{v}\|_{1,2} + \|\phi\|_0 \leq \delta$ . Using the continuous embedding

$$W_q^1(\Omega) \subset C(\bar{\Omega}), \quad q > n, \tag{2.17}$$

and Hölder's inequality, we have

$$\begin{aligned}
 \|\hat{w}\|_{C(\Omega)} &\leq \int_0^t \|\tilde{v}_y(s, y)\|_{C(\Omega)} ds + \int_0^t \|w_y(s, y)\|_{C(\Omega)} ds \\
 &\leq M \left( \int_0^{t_0} \|\tilde{v}_y(s, y)\|_2 ds + \int_0^{t_0} \|w_y(s, y)\|_2 ds \right) \\
 &\leq M t_0^{1-1/q} (\|\tilde{v}\|_{1,2} + \|w\|_{0,2}) \\
 &\leq M t_0^{1-1/q} (R + \delta).
 \end{aligned} \tag{2.18}$$

Since  $\tilde{w} = I + \hat{w}$ , estimate (2.14) for  $i = 0$  follows from (2.18). It follows from (2.7) that the matrix  $\tilde{w}$  at small  $t_0 = t_0(R)$  has the inverse matrix  $\tilde{w}^{-1} = A_3$  and for every coefficient  $a_3$  inequality (2.14) holds. Since the matrix  $A_4$  is adjoint to  $A_3$ , then estimate (2.14) for  $i = 4$  follows from (2.14) for  $i = 3$ . Since nonzero coefficient  $a_1$  of the matrix  $A_1$  is the determinant of the matrix  $\tilde{w}$ , estimate (2.14) for  $i = 1$  follows from (2.18) and (2.7). Estimates (2.14) are proved.

Since  $A_0 = I + \hat{w}$ , estimate (2.15) for  $i = 0$  follows from (2.18). It follows from (2.12) that  $A_3 - I = -\hat{w}(t, y)\tilde{w}^{-1}(t, y)$ . Estimate (2.15) for  $i = 3$  follows from (2.14) and (2.15) for  $i = 0$ . Estimates (2.15) for  $i = 1, 4$  follows from estimate (2.14) for  $i = 3$ . Estimates (2.15) are proved.

Taking an  $y_k$  derivative, we have

$$\frac{\partial}{\partial y_k}(\tilde{w}^{-1}) = -\tilde{w}^{-1}\left(\frac{\partial}{\partial y_k}\tilde{w}\right)\tilde{w}^{-1} = -\tilde{w}^{-1}\left(\frac{\partial}{\partial y_k}\hat{w}\right)\tilde{w}^{-1}. \quad (2.19)$$

It is easy to see that

$$\begin{aligned} \left|\frac{\partial}{\partial y_k}\tilde{w}\right|_0 &= \left|\frac{\partial}{\partial y_k}\hat{w}\right|_0 \\ &\leq M\left(\int_0^t \|\tilde{v}(s, y)\|_2 ds + \int_0^t \|w(s, y)\|_2 ds\right) \\ &\leq Mt_0^{1-1/q}(\|\tilde{v}\|_{0,2} + \|w\|_{0,2}) \\ &\leq Mt_0^{1-1/q}(R + \delta). \end{aligned} \quad (2.20)$$

Estimate (2.16) for  $i = 0$  follows from the above relation. It follows from (2.20) and inequality (2.14) for  $i = 3$  that estimate (2.16) for a coefficient  $a_3$  of the matrix  $A_3 = \tilde{w}^{-1}$  holds. Estimate (2.16) for  $i = 4$  is proved in a similar way.

Since nonzero coefficients of the matrix  $A_1$  are determinants of the matrix  $A_0$ , then in force of (2.7) and (2.20), we get

$$\left|\frac{\partial}{\partial y_k}a_1(t, y)\right|_0 \leq M \max_k \left|\frac{\partial}{\partial y_k}\hat{w}\right|_0 \leq Mt_0^{1-1/q}. \quad (2.21)$$

Estimate (2.16) for  $i = 1$  is established. This completes the proof of Proposition 2.3.  $\square$

Now estimate the terms  $S_i$  from the right-hand side of (2.13). First, consider  $S_1$ . After differentiation,  $S_1$  has a form of a sum of vectors whose coefficients look like

$$\begin{aligned} S_{11} &= \left(\frac{\partial}{\partial y_k}a_1\right)a_4a_2a_3(a_4 - 1), & S_{12} &= a_1\left(\frac{\partial}{\partial y_k}a_4\right)a_2a_3(a_4 - 1), \\ S_{13} &= a_1a_4\left(\frac{\partial}{\partial y_k}a_2\right)a_3(a_4 - 1), & S_{14} &= a_1a_4a_2\left(\frac{\partial}{\partial y_k}a_3\right)(a_4 - 1), \\ S_{15} &= a_1a_4a_2a_3\left(\frac{\partial}{\partial y_k}a_4\right). \end{aligned} \quad (2.22)$$



Using a continuity on  $y$  of  $a_i$  and the imbedding (2.17), we have

$$\begin{aligned} |S_{13}(t, y)|_0 &\leq \|a_1\|_{C(\Omega)} \|a_4\|_{C(\Omega)} \left| \frac{\partial}{\partial y_k} a_2 \right|_0 \|a_3\|_{C(\Omega)} \|a_4 - 1\|_{C(\Omega)} \\ &\leq \|a_1\|_{C(\Omega)} \|a_4\|_{C(\Omega)} \|a_2\|_1 \|a_3\|_{C(\Omega)} \|a_4 - 1\|_{C(\Omega)}. \end{aligned} \quad (2.23)$$

It follows from estimates (2.14), (2.15), (2.16), and the form of  $a_2$  that

$$|S_{13}(t, y)|_0 \leq M t_0^{1-1/q} (|\tilde{v}|_2 + |w|_2). \quad (2.24)$$

Integration over  $[0, t_0]$  yields

$$\|S_{13}\|_0 \leq M t_0^{1-1/q} (\|\tilde{v}\|_{0,2} + \|w\|_{0,2}) \leq M t_0^{1-1/q} (R + \delta). \quad (2.25)$$

In a similar way, we get

$$\begin{aligned} |S_{15}(t, y)|_0 &\leq \|a_1\|_{C(\Omega)} \|a_4\|_{C(\Omega)} \|a_2\|_{C(\Omega)} \|a_3\|_{C(\Omega)} \left| \frac{\partial}{\partial y_k} a_4 \right|_0 \\ &\leq M t_0^{1-1/q} (R + \delta). \end{aligned} \quad (2.26)$$

It follows from here that  $\|S_{15}\|_0 \leq M t_0^{1-1/q} (R + \delta)$ . The similar estimates hold for other  $S_{1i}$ . It follows from the estimates of  $S_{1i}$  that

$$\|S_1\|_0 \leq M t_0^{1-1/q} (R + \delta). \quad (2.27)$$

The analogous estimates for  $S_2, S_3, S_4$ , and  $S_5$  are established as in (2.27).

Inequality (2.11) for  $i = 1$  follows from the estimates of  $S_i$ .

Establish estimate (2.11) for  $i = 2$ . Let  $B_1 = |\tilde{w}|I$ ,  $B_2 = G(\tilde{w})$ , and  $B_3 = \tilde{w}^{-1}$ . Let  $b_i$  denote an arbitrary coefficient of the matrix  $B_i$ . Differentiation of  $K_2(w)$  gives an expressions of the form  $Z_i = b_i(\partial b_j / \partial y_k) b_l$ . First, establish the estimates

$$|B_i(t, y)| \leq M(R), \quad (2.28)$$

$$\left| \frac{\partial}{\partial y_k} b_i(t, y) \right|_0 \leq M(R) t_0^{1-1/q}. \quad (2.29)$$

Since  $A_i = B_i$  for  $i = 2$ , estimates (2.28), (2.29) for  $i = 2$  follow from estimates (2.14), (2.15), and (2.16) for  $i = 2$ . Estimate (2.28) for  $i = 2$  follows from the boundedness of coefficients of the matrix  $\tilde{w}$  and the continuity of  $G$ . Since  $(\partial / \partial y_k) \tilde{w} = (\partial / \partial y_k) \tilde{w}$ , estimate (2.29) for  $i = 2$  follows from a continuity of the derivatives of  $G$ , boundedness of  $\tilde{w}$  and estimate (2.18). Inequalities (2.28) and (2.29) are established.

This implies that  $\|z_i\|_0 \leq M t_0^{1-1/q} (R + \delta)$  whence (2.11) for  $i = 2$  follows. Estimate (2.11) for  $i = 3$  is handled similarly.

It follows from estimates (2.11) and relation (2.10) that if  $R$  is large enough and  $t_0$  is small enough, then the operator  $\tilde{K}$  transforms  $\mathfrak{M}$  into itself. This completes the proof of Lemma 2.2.  $\square$

LEMMA 2.4. *If  $t_0$  is small enough, then there exists  $\varkappa \in (0, 1)$  such that for every  $w^1, w^2 \in \mathfrak{M}$  the following inequality holds:*

$$\|\tilde{K}(w^1) - \tilde{K}(w^2)\|_{0,1} \leq \varkappa \|w^1 - w^2\|_{0,1}. \quad (2.30)$$

*Proof.* It follows from (2.2) that

$$\begin{aligned} \|\tilde{K}(w^1) - \tilde{K}(w^2)\|_{0,1} &= \|L(K(w^1) - K(w^2))\|_{0,1} \\ &\leq M \sum_{i=1}^3 \|L(K_i(w^1) - K_i(w^2))\|_{0,1} \\ &\equiv M \sum_{i=1}^3 \|L\text{Div}(Z_i(w^1, w^2))\|_{0,1} \\ &\leq M \sum_{i=1}^3 \|Z_i(w^1, w^2)\|_{0,1}. \end{aligned} \quad (2.31)$$

We show that

$$\|Z_i(w^1, w^2)\|_0 \leq M(R)t_0^{1-1/q} \|w^1 - w^2\|_{0,1}, \quad i = 1, 2, 3. \quad (2.32)$$

Denote by  $A_i^1$  the matrices (2.12) for  $w = w^1$ , the similar matrices for  $w = w^2$  are denoted by  $A_i^2$ . Elementary calculations yield

$$\begin{aligned} Z_1(w^1, w^2) &= A_1^1 A_4^1 A_2^1 A_3^1 - A_1^2 A_4^2 A_2^2 A_3^2 - (A_2^1 - A_2^2) \\ &= A_1^1 A_4^1 A_2^1 (A_3^1 - A_3^2) + A_1^1 (A_4^1 - A_4^2) A_2^1 A_3^2 + (A_1^1 - A_1^2) A_4^2 A_2^1 A_3^2 \\ &\quad + A_1^2 (A_4^2 - I) (A_2^1 - A_2^2) A_3^2 + (A_1^2 - I) (A_2^1 - A_2^2) A_3^2 \\ &\quad + (A_2^1 - A_2^2) A_3^2 + A_1^1 A_4^1 A_5^1 (A_3^1 - A_3^2) + A_1^1 (A_4^1 - A_4^2) A_5^1 A_3^2 \\ &\quad + (A_1^1 - A_1^2) A_4^2 A_5^1 A_3^2 + A_1^2 A_4^2 (A_5^1 - A_5^2) A_3^2. \end{aligned} \quad (2.33)$$

We need the following relations to estimate the norms of  $Z_1(w^1, w^2)$ :

$$\|A_i^1 - A_i^2\|_0 \leq M(R)t_0^{1-1/q} \|w^1 - w^2\|_{0,1}, \quad i \neq 2, \quad (2.34)$$

$$\|A_i^k(t, y) - I\| \leq M(R)t_0^{1-1/q}, \quad i \neq 2. \quad (2.35)$$

In fact,

$$\begin{aligned} \|\hat{w}^1(t, y) - \hat{w}^2(t, y)\|_0 &\leq \int_0^t \|w_y^1(s, y) - w_y^2(s, y)\|_0 ds \\ &\leq M t_0^{1-1/q} \|w^1 - w^2\|_{0,1}. \end{aligned} \quad (2.36)$$

It is easy to see that

$$\begin{aligned} A_3^1 - A_3^2 &= (\tilde{w}^1)^{-1} - (\tilde{w}^2)^{-1} \\ &= (\tilde{w}^1)^{-1}(\tilde{w}^2 - \tilde{w}^1)(\tilde{w}^1)^{-1} \\ &= (\tilde{w}^1)^{-1}(\hat{w}^2 - \hat{w}^1)(\tilde{w}^1)^{-1}. \end{aligned} \quad (2.37)$$

It follows from (2.14) and (2.36) that

$$\begin{aligned} |A_3^1 - A_3^2|_0 &\leq \max\left(\|\tilde{w}^1(t, y)\|_{C(\Omega)}, \|\tilde{w}^2(t, y)\|_{C(\Omega)}\right) \|\hat{w}^2(t, y) - \hat{w}^1(t, y)\|_0 \\ &\leq M(R + \delta)t_0^{1-1/q} \|w^1 - w^2\|_{0,1}. \end{aligned} \quad (2.38)$$

Integration over  $[0, t_0]$  yields

$$\|A_3^1(t, y) - A_3^2(t, y)\|_0 \leq M(R + \delta)t_0^{1-1/q} \|w^1 - w^2\|_{0,1}. \quad (2.39)$$

Estimate (2.34) for  $i = 3$  is established.

Estimate (2.34) for  $i = 1, 4, 5$  follows from estimate (2.34) for  $i = 3$  in view of the facts that  $A_1^1$  and  $A_1^2$  are products of the determinants of matrices  $(\partial/\partial y)\tilde{w}^1$  and  $(\partial/\partial y)\tilde{w}^2$ ,  $A_4^1$  and  $A_4^2$  are adjoint to  $A_3^1$  and  $A_3^2$ , respectively,  $A_5^1, A_5^2$  are integrals of  $w_y^1, w_y^2$ . Inequality (2.34) is established.

Inequality (2.35) follows from inequality (2.15) and the definition of matrices  $A_i^1$  and  $A_i^2$ .

Applying inequalities (2.34) and (2.35) to the right-hand side of (2.33), we get estimate (2.33) for  $i = 1$ . By this, the multipliers  $A_i^1 - A_i^2$  are estimated in  $L_q(Q)$ -norm while others in  $C(\Omega)$ -norm.

For proving inequality (2.32) for  $Z_2$ , note that it can be rewritten in the form (2.33), where  $A_i^1, A_i^2, i \neq 2$ , are from (2.33) while  $A_2^1 = G(w^1), A_2^2 = G(w^2)$ . It follows from the Newton-Leibnitz formula that

$$A_2^1 - A_2^2 = \int_0^1 G_z(\hat{w}^1(t, y) + \lambda(\hat{w}^2(t, y) - \hat{w}^1(t, y)))(\hat{w}^2(t, y) - \hat{w}^1(t, y)) d\lambda. \quad (2.40)$$

Then, the boundedness of  $\|\hat{w}^i(t, y)\|$  and continuity of  $G_z(Z)$  imply that

$$\|A_2^1 - A_2^2\| \leq M(R) \|w^1 - w^2\|. \quad (2.41)$$

Bearing this in mind, inequality (2.32) for  $i = 2$  as well as for  $i = 3$  is proved in a similar way to the case where  $i = 1$ .

It follows from inequality (2.32) that

$$\|\tilde{K}(w^1) - \tilde{K}(w^2)\|_{0,1} \leq M(R)t_0^{1-1/q} \|w^1 - w^2\|_{0,1}, \quad (2.42)$$

whence at small  $t_0$  Lemma 2.4 follows.

It follows from Lemmas 2.1, 2.2, and 2.4 that at small  $t_0$ , the operator  $\tilde{K}$  satisfies on  $\mathfrak{M}$  the conditions of the fixed point theorem for contractions. This completes the proof of Theorem 1.3.  $\square$

*Remark 2.5.* It follows from the proof of Theorem 1.3 that if

$$f(t, y) \in S_0(R_0) = \{\phi : \phi \in L_q(Q), \|\phi\|_0 \leq R_0\}, \quad (2.43)$$

then at sufficiently small  $t_0$ , depending only on  $R_0$ , problem (1.14), (1.15) has a unique solution  $v$ . Moreover, we may assume that  $v \in \mathfrak{M}$  and  $\|v\|_{1,2} \leq R$ , if  $R$  is large enough and  $t_0$  is small enough.

If  $t_0$  is those, then for each  $f(t, y) \in S_0(R_0)$ , problem (1.14), (1.15) has a unique solution. Denote by  $\mathfrak{N}$  the operator assigning in accordance to function  $F \in S_0(R_0)$  the solution  $v$  to problem (1.14), (1.15) so that

$$v = \mathfrak{N}(f). \quad (2.44)$$

### 3. Proof of Theorem 1.2

**3.1. Proof of existence.** Denoting  $w = F(t, y + \int_0^t v(s, y) ds)$  and supposing  $w$  is known, we obtain (by the assumption  $w \in S_0(R_0)$ ) that  $v = \mathfrak{N}(w)$ . Substituting  $v = \mathfrak{N}(w)$  by  $w$ , we get

$$w = F\left(t, y + \int_0^t \mathfrak{N}(w) ds\right) \equiv \mathfrak{S}(w). \quad (3.1)$$

Thus, problem (1.14), (1.15) is reduced to the operator (3.7) in  $L_q(Q_0)$ .

**LEMMA 3.1.** *Let  $R_0$  be large enough and let  $t_0$  be small enough. Then the operator  $\mathfrak{S}$  transforms  $S_0(R_0)$  into itself and is compact on  $S_0(R_0)$ .*

*Proof.* First, we show that operator  $\mathfrak{S}$  transforms  $S_0(R_0)$  into itself. Let  $w \in S_0(R_0)$  and let  $v = \mathfrak{N}(w)$  be a solution to problem (1.14), (1.15),  $v \in \mathfrak{M}$ . Consider the map of  $\Omega_0$  onto  $\Omega_t$

$$x = u(t, y) = x + \int_0^t v(s, y) ds. \quad (3.2)$$

It is easy to see that

$$\|I + u_y(t, y)\|_{C(\Omega_0)} \leq \int_0^t \|v_y(s, y)\|_{C(\Omega_0)} ds \leq M_1 t_0^{1-1/q} \|v\|_{1,2} \leq M_1 R t_0^{1-1/q}. \quad (3.3)$$

Assuming that  $t_0$  is small, we have the inequalities

$$\frac{1}{2} \leq \|u_y(t, y)\|_{C(\Omega_0)}, \quad \|U_x(t, x)\|_{C(\Omega_t)} \leq 2. \quad (3.4)$$

Making the change of variable (3.2), we get

$$\begin{aligned}
 \|\mathfrak{S}(w)\|_0^q &= \int_0^{t_0} \int_{\Omega_0} \left\| F\left(t, y + \int_0^t v(s, y) ds\right) \right\|^q dy dt \\
 &= \int_0^{t_0} \int_{\Omega_t} \|F(t, x)\|^q |U_x(t, x)| dx dt \\
 &\leq 2\|F\|_0^q.
 \end{aligned} \tag{3.5}$$

Supposing that  $R_0 \geq 2\|F\|_0$ , we obtain that  $\|\mathfrak{S}(w)\|_0 \leq R_0$ .

Establish now a compactness of  $\mathfrak{S}(w)$ . Specify a sequence of smooth finite functions  $F_n(t, x)$  in  $Q$  such that  $\|F - F_n\|_0 \rightarrow 0$  at  $n \rightarrow \infty$ . Without loss of generality, we may assume that  $\|F_n\|_0 \leq 2\|F\|_0$  for all  $n$ . Therefore, operators  $\mathfrak{S}_n$ , which for every  $F_n$  put in correspondence the solutions to problem (1.14), (1.15), transform  $S_0(R_0)$  into itself. Note that the existence of such sequence  $F_n(t, x)$  follows from the fact that the map  $(t, x) \rightarrow (t, U(t, x))$  is a homeomorphism of  $Q$  onto  $Q_0$  with the lateral surface  $S_0 = [0, t_0] \times \Gamma_0$ ,  $\Gamma_0 \in C^2$ . Arranging a sequence of smooth finite functions  $F_n(t, y)$ , convergent in  $L_q(Q_0)$  to  $f_n(t, y) = F_n(t, \zeta(t, 0y))$ , we obtain the desired sequence  $F_n(t, x) = f_n(t, \zeta(0, t, y))$ .

We show that the operator  $\mathfrak{S}_n$  is compact on  $S_0(R_0)$ . Let  $w \in S_0(R_0)$  and  $v = \mathfrak{N}(w)$ . Then the function  $u(t, y) = y + \int_0^t v(s, y) ds$  is continuously differentiable with respect to every variable and uniformly bounded together with its derivatives since  $\|v\|_{1,2} \leq R$ . From this and the smoothness of  $F_n$ , there follow uniform boundedness and equicontinuity of the set of functions  $F_n(t, y + \int_0^t v(s, y) ds)$  (or  $\mathfrak{S}_n(S_0(R_0))$ ). This implies the compactness of  $\mathfrak{S}_n$  in  $L_q(Q_0)$ .

Now we show that  $\mathfrak{S}_n$  uniformly on  $S_0(R_0)$  converges to  $\mathfrak{S}$ . In fact, making use of the change (3.2) by  $v = \mathfrak{N}(w)$ ,  $w \in S_0(R_0)$  and estimate (3.12), we have

$$\begin{aligned}
 \|\mathfrak{S}_n(w) - \mathfrak{S}(w)\|_0 &= \left\| F_n\left(t, y + \int_0^t v(s, y) ds\right) - F\left(t, y + \int_0^t v(s, y) ds\right) \right\|_0 \\
 &\leq 2^{1/q} \|F_n(t, x) - F(t, x)\|_0.
 \end{aligned} \tag{3.6}$$

The uniform convergence of  $\mathfrak{S}_n$  follows from the convergence of  $F_n$  to  $F$ . The compactness of  $\mathfrak{S}$  on  $S_0(R_0)$  follows from the compactness of  $\mathfrak{S}_n$  and uniform convergence of  $\mathfrak{S}_n$  to  $\mathfrak{S}$  on  $S_0(R_0)$ . This completes the proof of Lemma 3.1.  $\square$

Finally, the existence of a fixed point for  $\mathfrak{S}$  follows from the Schauder fixed point theorem.

**3.2. Proof of uniqueness.** The proof of the uniqueness is accomplished by a series of auxiliary propositions for solutions to the Cauchy problem in noncylindrical domain.

PROPOSITION 3.2. Let  $V, V^1, V^2 \in W_q^{1,2}(Q)$ ,  $\|V^1\|_{W_q^{1,2}(Q)}, \|V^2\|_{W_q^{1,2}(Q)}$ , and  $\|V\|_{W_q^{1,2}(Q)} \leq R$ . Let  $z^i(\tau, t, x)$ ,  $i = 1, 2$  and let  $z(\tau, t, x)$  be solutions to the Cauchy problem

$$z^i(\tau, t, x) = x + \int_t^\tau V^i(s, z(s; t, x)) ds \quad (3.7)$$

and (1.2), respectively. Then at  $\tau, t \in [0, t_0]$  the following inequalities hold:

$$\begin{aligned} \|z_x(\tau, t, x)\| &\leq M; \\ \|z_x(\tau, t, x) - I\|_{C(\Omega_t)} &\leq \left| \int_t^\tau \|V_x(s, x)\|_{C(\Omega_s)} ds \right|; \\ \|z_{xx}(\tau, t, x)\|_{L_q(\Omega_t)} &\leq M \left| \int_t^\tau \|V(s, x)\|_{W_q^2(\Omega_s)} ds \right|; \\ 0 \leq |z_x(\tau, t, x)| &\leq \left| \int_t^\tau \|\text{Spur } V_x(s, x)\|_{C(\Omega_s)} ds \right|; \\ \|z_x^1(\tau, t, x) - z_x^2(\tau, t, x)\|_{L_r(\Omega_t)} &\leq M \left| \int_t^\tau \|V^1(s, x) - V^2(s, x)\|_{L_r(\Omega_s)} ds \right|, \\ &1 \leq r \leq q; \\ \|z_x^1(\tau, t, x) - z_x^2(\tau, t, x)\|_{W_r^1(\Omega_t)} &\leq M \left| \int_t^\tau \|V^1(s, x) - V^2(s, x)\|_{W_r^1(\Omega_s)} ds \right|, \\ &1 \leq r \leq q. \end{aligned} \quad (3.8)$$

For the case of cylindrical domain  $Q = Q_0$  and  $V(t, x) = 0$  on  $\Gamma_0$ , Proposition 3.2 is proved in [10]. The proof in the general case is reduced to the case of cylindrical domain  $Q = Q_0$  in the following way. In what follows  $u(t, y) = x + \int_0^t v(s, y) ds$ , where  $v(t, x)$  satisfies the conditions of Theorem 1.1. The function  $u(t, y)$  defines, at fixed  $t \in [0, t_0]$  and small  $t_0$ , the homeomorphism  $x = u(t, y)$  of domain  $\Omega_0$  onto  $\Omega_t$  and we may assume that for  $u(t, y)$  and for inverse to it map  $y = U(t, x)$ , estimates (3.4) hold. Let  $z(\tau, t, x)$  be a solution to Cauchy problem (1.2). Then the function

$$\bar{z}(\tau, t, y) = U(\tau, z(\tau, t, u(t, y))) \quad (3.9)$$

is the solution to Cauchy problem

$$\bar{z}(\tau, t, y) = y + \int_t^\tau \bar{V}(s, \bar{z}(s, t, y)) ds, \quad (3.10)$$

where

$$\bar{V}(t, y) = -U_x(t, u(t, y))v(t, y) + U_x(t, u(t, y))V(t, u(t, y)). \quad (3.11)$$

Thus, the study of  $z(\tau, t, x)$  in the case of noncylindrical domain  $Q$  is reduced to the study of a solution  $\bar{z}(\tau, t, y)$  to Cauchy problem in the cylindrical domain  $Q = Q_0$ . Note that  $\bar{V}(t, y) = 0$  on  $S_0$ . The statement of Proposition 3.2

for  $z(\tau, t, y)$  follows from the statement of [Proposition 3.2](#) for  $\bar{z}(\tau, t, x)$  due to inequalities (3.2) and the relation  $z(\tau, t, x) = u(\tau, \bar{z}(\tau, t, U(t, x)))$ .

PROPOSITION 3.3. For each  $\Theta(x) \in W_q^1(\Omega_t)$ ,

$$\|\Theta(x)\|_{C(\Omega_t)} \leq M \|\Theta(x)\|_{W_q^1(\Omega_t)}, \quad M \neq M(t). \quad (3.12)$$

*Proof.* Let  $\theta(y) = \Theta(u(t, y))$ . Due to the change of variable  $x = u(t, y)$  and the continuity of the imbedding (2.17) and (3.2) we have

$$\|\Theta(x)\|_{C(\Omega_t)} = \|\theta(y)\|_{C(\Omega_0)} \leq M \|\theta(y)\|_{W_q^1(\Omega_0)} \leq M \|\Theta(x)\|_{W_q^1(\Omega_t)}, \quad M \neq M(t). \quad (3.13)$$

□

In what follows, we denote by  $|\cdot|_1$  the norm in  $W_2^1(\Omega_t)$ . This norm depends on  $t$ ; however, the situation is clear from the context. The norm in the dual to  $W_2^1(\Omega_t)$  space  $W_2^{-1}(\Omega_t)$  will be denoted by  $|\cdot|_{-1}$ .

PROPOSITION 3.4. Let  $a \in W_q^1(\Omega_t)$ ,  $u \in W_q^1(\Omega_t)$ . Then, at every fixed  $t$ ,

$$|au|_{-1} \leq K|a|_0|u|_1, \quad K \neq K(t); \quad (3.14)$$

$$|au|_{-1} \leq K|a|_1|u|_{-1}, \quad K \neq K(t). \quad (3.15)$$

If  $a \in W_q^1(\Omega_t)$ ,  $u \in W_q^2(\Omega_t)$ , then

$$|au_{xx}|_{-1} \leq K|a|_0|u|_2, \quad K \neq K(t). \quad (3.16)$$

*Proof.* Given  $v \in W_2^1(\Omega_t)$ , using the definition of  $|\cdot|_{-1}$ -norm and (2.17) we get

$$|au|_{-1} = \sup_v \frac{(au, v)}{|v|_1} \leq \sup_v \frac{(u, av)}{|av|_0} \sup_v \frac{|av|_0}{|v|_1} \leq M|a|_0|u|_1. \quad (3.17)$$

Here we have used the inequality  $|av|_0 \leq M|a|_0|v|_1$ . Inequality (3.14) is proved. Making use of the inequality  $|av|_1 \leq M|a|_1|v|_1$ , we have

$$|au|_{-1} = \sup_v \frac{(au, v)}{|v|_1} \leq \sup_v \frac{(u, av)}{|av|_1} \sup_v \frac{|av|_1}{|v|_1} \leq M|a|_1|u|_{-1}. \quad (3.18)$$

Inequality (3.15) is proved. Using (3.12), (3.14), (3.15), and the inclusion  $u \in W_2^2(\Omega_t)$ , we get

$$\begin{aligned} |au_{xx}|_{-1} &\leq |(au_x)_x|_{-1} + |a_x u_x|_{-1} \\ &\leq |au_x|_{-1} + |a_x|_{-1} |u_x|_1 \leq |a|_0 |u_x|_2. \end{aligned} \quad (3.19)$$

Inequality (3.16) is proved. □

PROPOSITION 3.5. *The following inequalities hold:*

$$\|R(V^i)\|, \|R^{-1}(V^i)\| \leq M, \quad (t, x) \in Q, \quad i = 1, 2; \quad (3.20)$$

$$|R(V^i)|_1 \leq Mt_0^{1/2}, \quad i = 1, 2; \quad (3.21)$$

$$\|R(V^i) - I\|_{C(\Omega_t)} \leq Mt_0^{1-1/q}; \quad (3.22)$$

$$|R(V^1) - R(V^2)|_0 \leq Mt_0^{1/2} \|V^1 - V^2\|_{0,1}; \quad (3.23)$$

$$|B(V^1) - B(V^2)|_0 \leq Mt_0^{1/2} \|V^1 - V^2\|_{0,1}. \quad (3.24)$$

The above proposition follows from [Proposition 3.2](#) and (1.8).

Consider now the linear problem

$$\begin{aligned} V_t - \hat{\Delta} V &= \Xi(t, y), & (t, y) &\in Q; \\ V(0, y) &= 0, & y &\in \Omega_0; \\ V(t, y) &= 0, & (t, y) &\in \Gamma. \end{aligned} \quad (3.25)$$

In what follows,  $\|\cdot\|_{0,-1}$  stands for the norm in  $L_2(0, t_0 : W_2^{-1}(\Omega_t))$ .

LEMMA 3.6. *Let  $V \in W_q^{1,2}(Q)$ ,  $q \in (0, +\infty)$ , and  $V$  satisfy (3.25). Then*

$$\|V\|_{0,1} \leq M \|\Xi\|_{0,-1}, \quad (3.26)$$

$$\sup_t |V(t, y)|_0 \leq M \|\Xi\|_0. \quad (3.27)$$

*Proof.* Multiplying (3.25) in  $L_2(\Omega_t)$  by  $V(t, x)$ , we get

$$(V_t, V) + (\hat{\Delta} V, V) = (\Xi, V). \quad (3.28)$$

Hereinafter,  $(\cdot, \cdot)$  denotes the scalar product in  $L_2(\Omega_t)$  while  $(\cdot)$  denotes the scalar product in  $\mathbb{R}^n$ . It is easy to see that

$$(V_t, V) = \frac{1}{2} \int_{\Omega_t} \frac{d}{dt} (V \cdot V) dx = \frac{1}{2} \int_{\Omega_t} \Psi_t(t, x) dx, \quad \Psi(t, x) = (V \cdot V). \quad (3.29)$$

Let  $V^1 \in W_q^{1,2}(Q)$  be a solution to problem (1.9), (1.10). Making use of the change of variable

$$x = U(t, y), \quad u(t, y) = y + \int_0^t V^1(s, us, y) ds, \quad (3.30)$$



and differentiating the integral by parameter, we have

$$\begin{aligned}
 \int_{\Omega_t} \Psi_t(t, x) dx &= \int_{\Omega_0} \Psi_t(t, u(t, y)) I(t, y) dy \\
 &= \frac{d}{dt} \int_{\Omega_0} \Psi(t, u(t, y)) I(t, y) dy \\
 &\quad - \int_{\Omega_0} (\nabla_x \Psi(t, u(t, y))) v^1(t, y) I(t, y) dy \\
 &\quad - \int_{\Omega_0} \Psi(t, u(t, y)) \frac{d}{dt} I(t, y) dy.
 \end{aligned} \tag{3.31}$$

Here  $v^1(t, y) = V^1(t, u(t, y))$ ,  $I(t, y) = |u_y(t, y)|$ . It follows from the Ostrogradskii-Liouville formula  $I(t, y) = \exp(\int_0^t \operatorname{div} V^1(s, u(s, y)) ds)$  that

$$\frac{d}{dt} I(t, y) = I(t, y) \frac{d}{dt} \ln I(t, y) = \operatorname{div} V^1(t, u(t, y)) I(t, y). \tag{3.32}$$

Making use of the inverse to (3.30) change of variable, using (3.32) and integrating in the second integral in parts, we obtain

$$\begin{aligned}
 \int_{\Omega_t} \Psi_t(t, x) dx &= \frac{d}{dt} \int_{\Omega_t} \Psi(t, x) dx - \int_{\Omega_t} (\nabla_x \Psi(t, x)) V^1(t, x) dx \\
 &\quad - \int_{\Omega_t} \Psi(t, x) \operatorname{div} V^1(t, x) dx \\
 &= \frac{d}{dt} \int_{\Omega_t} \Psi(t, x) dx \\
 &= \frac{d}{dt} \|V(t, x)\|_0^2.
 \end{aligned} \tag{3.33}$$

Consider the second term in (3.28). Integrating by parts, we have

$$\begin{aligned}
 -(\hat{\Delta} V, V) &= \frac{1}{2} (\nabla V + (\nabla V)^*, \nabla V) \\
 &= \frac{1}{4} (\nabla V + (\nabla V)^*, \nabla V + (\nabla V)^*) \\
 &= \frac{1}{4} \|\nabla V + (\nabla V)^*\|_0^2.
 \end{aligned} \tag{3.34}$$

It follows from the Corn inequality [5, Theorem 2.2, page 30] that

$$m_1 \|V\|_1 \leq \|\nabla V + (\nabla V)^*\|_0 \leq m_2 \|V\|_1. \tag{3.35}$$

Thus,

$$M_1 \|V\|_1^2 \leq -(\hat{\Delta} V, V) \leq M_2 \|V\|_1^2. \tag{3.36}$$

Next, using, for small  $\epsilon > 0$ , the inequality

$$(\Xi, V) \leq |\Xi|_{-1} \|V\|_1 \leq C(\epsilon) |\Xi|_{-1}^2 + \epsilon \|V\|_1^2 \tag{3.37}$$

and relations (3.28), (3.33), (3.36), and (3.37), we have

$$\frac{d}{dt} |V(t, x)|_0^2 + |V|_1^2 \leq M |\Xi|_{-1}^2. \quad (3.38)$$

Integrating the last inequality over  $[0, t_0]$ , we obtain inequality (3.26). Making use of the inequality  $(\Xi, V) \leq |\Xi|_0 |V|_0$  and relations (3.28), (3.37), and (3.33), we get  $(d/dt)|V(t, x)|_0 \leq |\Xi|_0$ . Integrating the last inequality on  $t$  on interval  $[0, t_0]$ , we obtain inequality (3.27). Thus, Lemma 3.6 is proved.  $\square$

Now we proceed directly to the proof of the uniqueness. Arguing by contradiction, suppose that there are two solutions  $V^1$  and  $V^2$  of (1.9), (1.10). Then, for  $V = V^1 - V^2$ , we have

$$\begin{aligned} V_t - \hat{\Delta} V &= - \sum_i^n V_i \frac{\partial V^1}{\partial x_i} - \sum_i^n V_i^2 \frac{\partial V}{\partial x_i} + (1 - R^{-1}(V^1)) \hat{\Delta} V \\ &\quad + R^{-1}(V^1) R^{-1}(V^2) (R^{-1}(V^1) - R^{-1}(V^2)) \hat{\Delta} V^2 \\ &\quad + R^{-1}(V^1) R^{-1}(V^2) (R^{-1}(V^1) - R^{-1}(V^2)) \operatorname{Div} B(V^1) \\ &\quad + R^{-1}(V^2) \operatorname{Div} B(V^1) - \operatorname{Div} B(V^2) \\ &\equiv \sum_{i=1}^6 Z_i. \end{aligned} \quad (3.39)$$

Without loss of generality, we set above that  $G(Z) \equiv Z$ . Using Lemma 3.6, we obtain

$$\|V\|_{0,1} \leq M \sum_{i=1}^6 \|Z_i\|_{0,-1}. \quad (3.40)$$

Estimate  $\|Z_i\|_{0,-1}$ . Using (3.14), we have

$$|Z_1|_{-1} \leq M |V|_0 |V_x^1|_1 \leq |V|_0 |V^1|_2. \quad (3.41)$$

Since it follows from the Newton-Leibnitz formula that

$$|V(t, x)|_0 \leq \int_0^{t_0} |V_t(s, x)|_0 ds \leq M t_0^{1/2} \|V\|_{1,2}, \quad (3.42)$$

then we have

$$\begin{aligned} \|Z_1\|_{0,-1} &\leq M \left( \int_0^{t_0} |V|_0^2 |V_x^1|_1^2 ds \right)^{1/2} \\ &\leq \sup_t |V(t, x)|_0^{2/q} \left( \int_0^{t_0} |V(s, x)|_0^{2(q-2)/q} |V^1(s, x)|_2^2 ds \right)^{1/2}. \end{aligned} \quad (3.43)$$

Making use of the Hölder inequality, we get

$$\begin{aligned} \|Z_1\|_{0,-1} &\leq M t_0^{1/2q} \left( \int_0^{t_0} |V_t(s, x)|_0^2 ds \right)^{(q-2)/2q} \left( \int_0^{t_0} |V^1(s, x)|_0^2 ds \right)^{1/2q} \\ &\leq M t_0^{1/2q} \|V\|_{0,1}^{(q-2)/q} \|V^1\|_{W_q^{1,2}(Q)}^{q/2}. \end{aligned} \quad (3.44)$$

Similarly the term  $Z_2$  is handled. For other  $Z_i$ , the following inequalities hold:

$$\|Z_i\|_{0,-1} \leq M t_0^{1/2} \|V\|_{0,1}, \quad i = 3, \dots, 6. \quad (3.45)$$

Prove (3.45) for  $I = 3$ . Using (3.16) and (3.20), we get

$$\begin{aligned} |Z_3|_{-1} &\leq M |R^{-1}(V^1)R^{-1}(V^2)(R^{-1}(V^1) - R^{-1}(V^2))|_0 |V^2|_2 \\ &\leq M |R^{-1}(V^1) - R^{-1}(V^2)|_0 |V^2|_2. \end{aligned} \quad (3.46)$$

Due to (3.23), it follows from here that

$$\|Z_3\|_{0,-1} \leq M \sup_t |R^{-1}(V^1) - R^{-1}(V^2)|_0 \|V^2\|_{0,2} \leq M t_0^{1/2} \|V\|_{0,1}. \quad (3.47)$$

Making use of (3.15), (3.20), (3.21), and (3.24), we obtain

$$\begin{aligned} \|Z_6\|_{0,-1} &\leq M \left( \int_0^{t_0} |R^{-1}(V^2)|_1^2 |\operatorname{Div}(B(V^2) - B(V^2))|_{-1}^2 ds \right)^{1/2} \\ &\leq M \left( \int_0^{t_0} |R(V^2)|_1^2 |\operatorname{Div}(B(V^2) - B(V^2))|_0^2 ds \right)^{1/2} \\ &\leq M t_0^{1/2} \|V\|_{0,1}. \end{aligned} \quad (3.48)$$

It follows from (3.40), (3.44), and (3.45) that

$$\|V\|_{0,1} \leq M t_0^{1/2q} \|V\|_{0,1}^{(q-2)/q} + M t_0^{1/2} \|V\|_{0,1}. \quad (3.49)$$

Choosing  $t_0$  sufficiently small, we have that  $\|V\|_{0,1} \leq 0$ . Hence,  $V = 0$ . The uniqueness of the solution is established. This completes the proof of Theorem 1.2.

#### 4. Proof of Theorem 1.3

The proof of Theorem 1.3 follows from the proof of Theorem 1.2. We should make the change of variable  $x = z(t, 0, y)$ — $z(\tau, t, x)$  is a solution to Cauchy problem (1.2), where  $V$  is a solutions to problem (1.14), (1.15), and (1.16)—and then make use of uniform boundedness from above and from below of the Jacobian of  $z(\tau, t, x)$  which implies an isomorphism of Sobolev spaces on  $Q$  onto Sobolev spaces on  $Q_10$ .

## 5. Remarks

It follows from the definition of the solution and the form of the left-hand side of (1.14) that necessarily  $f \in L_q(Q_0)$ . Rewrite problem (1.14), (1.15), and (1.16) in the form

$$\begin{aligned} v_t - \mu \hat{\Delta} v &= w; \\ v(0, y) &= V^0(y), \quad y \in \bar{\Omega}, \\ v(t, y) &= Y_1(t, y), \quad (t, y) \in S, \end{aligned} \quad (5.1)$$

where  $w$  is defined as in the proof of Theorem 1.3. It follows from (1.14) and the definition of the solution that necessarily  $w \in L_q(Q_0)$ . From the properties of the solutions of the linear problem (5.1) and from [4, Chapter 4, page 388], we get that necessarily  $V^0 \in W_{q,0}^{2-2/q}(\Omega_0)$ ,  $Y \in W$ . Note also that the proofs of Theorems 1.2 and 1.3 are carried out without an assumption about any smoothness of  $F$  (cf. [14]).

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