THE INTEGRAL WAVELET TRANSFORM IN WEIGHTED SOBOLEV SPACES

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The integral wavelet transform is defined in weighted Sobolev spaces, in which some properties of the transform as well as its asymptotical behaviour for small dilation parameter are studied.

1. Preliminaries and notations

It is well known that the integral wavelet transform is a very powerful tool to study sciences and technology. In [5] wavelet theory has been investigated in very much functional spaces, even in BMO, VMO (for further details of those spaces, please refer to [5] and the references therein) even for pseudodifferential operators, however, the integral wavelet transform in weighted Sobolev spaces has not been studied yet neither in [5] nor in any other work.

The aim of this paper is to study this unsolved problem.

Let $\omega_\mu(x) \in L^\infty(\mathbb{R}^n)$, $\omega_\mu(x) > 0$, for almost all $x \in \mathbb{R}^n$ and for each x,

$$\omega_\mu(x+y) \le C_{1,\mu}\omega_\mu(x), \tag{1.1}$$

for almost all $y \in \mathbb{R}^n$, where μ is a multi-index.

We use the Sobolev space with weighted norm defined as follows:

$$W_\omega^{m,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) \mid \partial^k f \in L^p(\mathbb{R}^n),\ |k| \le m \right\} \tag{1.2}$$

equipped with the norm

$$\|f\|_{m,p,\omega} = \sum_{|\mu| \le m} \left(\int_{\mathbb{R}^n} \omega_\mu(x) |\partial^\mu f(x)|^p \, dx \right)^{1/p} < \infty, \tag{1.3}$$

where $\mu = (\mu_1, \dots, \mu_n)$, $|\mu| = \mu_1 + \cdots + \mu_n$, $\mu_i \ge 0$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all differentiable functions φ on $\mathbb{R}^n$ such that for all multi-indices α and β

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha (D^\beta \varphi(x)) \right| < \infty. \tag{1.4}$$

The Fourier transform $\mathcal{F} : f \mapsto \hat{f}$ is given by

$$\hat{f}(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i(x,y)} \, dx, \tag{1.5}$$

where $(x, y) = x_1 y_1 + \cdots + x_n y_n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ (see [1, 2, 4]).

As traditionally, it is not difficult to prove that $\mathcal{S}(\mathbb{R}^n)$ is dense in $W_\omega^{m,p}(\mathbb{R}^n)$. Now we recall that a basic wavelet is a nontrivial function $\psi \in L^1(\mathbb{R}^n)$ such that its integral on $\mathbb{R}^n$ is 0 and its Fourier transform $\hat{\psi}(\xi)$ satisfies the condition

$$(2\pi)^n \int_0^\infty \frac{\left| \hat{\psi}(a\xi) \right|^2}{a} \, da, \tag{1.6}$$

denoted by C_ψ which is a constant for every $\xi \neq 0$ and $C_\psi \neq 0$.

With a basic wavelet ψ and a function $f \in \mathcal{S}(\mathbb{R}^n)$, we define the following integral:

$$(L_\psi f)(b, a) = \frac{1}{\sqrt[2n]{C_\psi}} \frac{1}{\sqrt{|a|^n}} \int_{\mathbb{R}^n} \bar{\psi} \left(\frac{t-b}{a} \right) f(t) \, dt, \tag{1.7}$$

where $b \in \mathbb{R}^n$ and $a \in \mathbb{R} \setminus \{0\}$ (see [3, 5, 6]).

2. Some properties

PROPOSITION 2.1. *If $f \in \mathcal{S}(\mathbb{R}^n)$ then*

$$\left\| (L_\psi f)(\cdot, a) \right\|_{m,p,\omega} \leq C \|f\|_{m,p,\omega}, \tag{2.1}$$

where $a \in \mathbb{R}$, $a \neq 0$ and fixed, C is a constant independent of f.

Proof. Obviously we have

$$(L_\psi f)(\cdot, a) = \frac{1}{\sqrt[2n]{C_\psi}} (f * D^{-a} \bar{\psi})(\cdot), \tag{2.2}$$

where $D^a : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $(D^a \psi)(a) = |a|^{-n/2} \psi(x/a)$; $a \neq 0$.

Since $f \in \mathcal{S}(\mathbb{R}^n)$ the differentiation and integration can be interchanged

$$\partial^\mu (L_\psi f)(\cdot, a) = \frac{1}{\sqrt[2n]{C_\psi}} (D^{-a} \bar{\psi} * \partial^\mu f)(\cdot). \tag{2.3}$$

It is not difficult to see that

$$
\|(L_\psi f)(\cdot, a)\|_{m,p,\omega}
$$

$$
= \sum_{|\mu| \le m} \left(\int_{\mathbb{R}^n} \omega_\mu(\cdot) |\partial^\mu (L_\psi f)(\cdot, a)|^p \, d(\cdot) \right)^{1/p}
$$

$$
= \sum_{|\mu| \le m} \frac{1}{\sqrt[2^n]{C_\psi}} \left(\int_{\mathbb{R}^n} \omega_\mu(x) \left| \int_{\mathbb{R}^n} (\partial^\mu f)(x-y)(D^{-a}\bar{\psi})(y) \, dy \right|^p dx \right)^{1/p}
$$

$$
\le \frac{1}{\sqrt[2^n]{C_\psi}} \sum_{|\mu| \le m} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |(\partial^\mu f)(x-y)|^p |(D^{-a}\bar{\psi})(y)|^p \omega_\mu(x) \, dx \right)^{1/p} dy
$$

$$
\le \frac{1}{\sqrt[2^n]{C_\psi}} \sum_{|\mu| \le m} \int_{\mathbb{R}^n} |D^{-a}\bar{\psi}(y)| \left(\int_{\mathbb{R}^n} \omega_\mu(x) |(\partial^\mu f)(x-y)|^p \, dx \right)^{1/p} dy.
$$

$$(2.4)$$

However,

$$
\int_{\mathbb{R}^n} \omega_\mu(x) |(\partial^\mu f)(x-y)|^p \, dx = \int_{\mathbb{R}^n} \omega_\mu(u+y) |(\partial^\mu f)(u)|^p \, du
$$

$$
\le C_{1,\mu} \int_{\mathbb{R}^n} \omega_\mu(u) |(\partial^\mu f)(u)|^p \, du.
$$

$$(2.5)$$

Consequently,

$$
\left(\int_{\mathbb{R}^n} \omega_\mu(\cdot) |\partial^\mu (L_\psi f)(\cdot, a)|^p \, d(\cdot) \right)^{1/p}
$$

$$
\le \frac{(C_{1,\mu})^{1/p}}{\sqrt[2^n]{C_\psi}} \int_{\mathbb{R}^n} |D^{-a}\bar{\psi}(y)| \, dy \left(\int_{\mathbb{R}^n} \omega_\mu(x) |(\partial^\mu f)(x)|^p \, dx \right)^{1/p} \qquad (2.6)
$$

$$
\le C_\mu \left(\int_{\mathbb{R}^n} \omega_\mu(x) |(\partial^\mu f)(x)|^p \, dx \right)^{1/p},
$$

where

$$
C_\mu = \frac{(C_{1,\mu})^{1/p}}{\sqrt[2^n]{C_\psi}} |a|^{n/2} \|\psi\|_1. \qquad (2.7)
$$

Therefore,

$$
\|(L_\psi f)(\cdot, a)\|_{m,p,\omega} \le C \sum_{|\mu| \le m} \left(\int_{\mathbb{R}^n} \omega_\mu(x) |(\partial^\mu f)(x)|^p \, dx \right)^{1/p}, \qquad (2.8)
$$

where

$$
C = \max_{|\mu| \le m} C_\mu, \qquad (2.9)
$$

that is,

$$\left\| (L_\psi f)(\cdot, a) \right\|_{m,p,\omega} \le C \|f\|_{m,p,\omega}. \tag{2.10}$$

$\square$

By Proposition 2.1, we extend $(L_\psi f(\cdot, a))$ for fixed a to a continuous mapping from $W_\omega^{m,p}(\mathbb{R}^n)$ to itself. It is called the *integral wavelet transform in weighted Sobolev space.*

THEOREM 2.2. *If ψ and φ are basic wavelets and f, g belong to $W_\omega^{m,p}(\mathbb{R}^n)$, then the following estimate holds true:*

$$\left\| (L_\psi f)(\cdot, a) - (L_\varphi g)(\cdot, a) \right\|_{m,p,\omega}$$

$$\le C^1 |a|^{n/2} \left(\left\| \frac{\psi}{\sqrt[2n]{C_\psi}} - \frac{\varphi}{\sqrt[2n]{C_\varphi}} \right\|_1 \|f\|_{m,p,\omega} + \frac{\|\varphi\|_1}{\sqrt[2n]{C_\varphi}} \|f - g\|_{m,p,\omega} \right), \tag{2.11}$$

where C^1 is a constant independent of f and g.

Proof. It is sufficient to prove the case $f, g \in \mathscr{S}(\mathbb{R}^n)$.
    Obviously

$$\left(\int_{\mathbb{R}^n} \omega_\mu(\cdot) \left| \partial^\mu \left[L_\psi f - L_\varphi g \right](\cdot, a) \right|^p d(\cdot) \right)^{1/p}$$

$$= \left\{ \int_{\mathbb{R}^n} \omega_\mu(\cdot) \left| \partial^\mu f * \left(\frac{D^{-a}\bar{\psi}}{\sqrt[2n]{C_\psi}} - \frac{D^{-a}\bar{\varphi}}{\sqrt[2n]{C_\varphi}} \right)(\cdot) \right|^p d(\cdot) \right\}^{1/p}$$

$$= \left\{ \int_{\mathbb{R}^n} \omega_\mu(x) \left| \int_{\mathbb{R}^n} (\partial^\mu f)(x - y) \left(\frac{D^{-a}\bar{\psi}}{\sqrt[2n]{C_\psi}} - \frac{D^{-a}\bar{\varphi}}{\sqrt[2n]{C_\varphi}} \right)(y) \, dy \right|^p dx \right\}^{1/p}$$

$$\le \int_{\mathbb{R}^n} \left| \left(\frac{D^{-a}\bar{\psi}}{\sqrt[2n]{C_\psi}} - \frac{D^{-a}\bar{\varphi}}{\sqrt[2n]{C_\varphi}} \right)(y) \right| \left(\int_{\mathbb{R}^n} \omega_\mu(x) \left| (\partial^\mu f)(x - y) \right|^p dx \right)^{1/p} dy$$

$$\le C_{2,\mu} \left(\int_{\mathbb{R}^n} \omega_\mu(x) \left| (\partial^\mu f)(x) \right|^p dx \right)^{1/p}, \tag{2.12}$$

where

$$C_{2,\mu} = |a|^{n/2} \left\| \frac{\psi}{\sqrt[2n]{C_\psi}} - \frac{\varphi}{\sqrt[2n]{C_\varphi}} \right\|_1 (C_{1,\mu})^{1/p}. \tag{2.13}$$

So

$$\left\| (L_\psi f)(\cdot, a) - (L_\varphi f)(\cdot, a) \right\|_{m,p,\omega} \le C^1 |a|^{n/2} \left\| \frac{\psi}{\sqrt[2n]{C_\psi}} - \frac{\varphi}{\sqrt[2n]{C_\varphi}} \right\|_1 \|f\|_{m,p,\omega}, \quad (2.14)$$

where

$$C^1 = \max_{|\mu| \le m} (C_{1,\mu})^{1/p}. \tag{2.15}$$

Similarly we obtain

$$\left\| (L_\varphi f)(\cdot, a) - (L_\varphi g)(\cdot, a) \right\|_{m,p,\omega} \le C^1 |a|^{n/2} \frac{\|\varphi\|_1}{\sqrt[2n]{C_\varphi}} \|f - g\|_{m,p,\omega}. \tag{2.16}$$

By the triangle inequality we get

$$\left\| (L_\psi f)(\cdot, a) - (L_\varphi g)(\cdot, a) \right\|_{m,p,\omega}$$

$$\le C^1 |a|^{n/2} \left(\left\| \frac{\psi}{\sqrt[2n]{C_\psi}} - \frac{\varphi}{\sqrt[2n]{C_\varphi}} \right\|_1 \|f\|_{m,p,\omega} + \frac{\|\varphi\|_1}{\sqrt[2n]{C_\varphi}} \|f - g\|_{m,p,\omega} \right). \tag{2.17}$$

$$\square$$

3. Symptotical behaviour for small dilation parameter

From Theorem 2.2 the following proposition follows immediately.

PROPOSITION 3.1. *If ψ is a basic wavelet and $f \in W_\omega^{m,p}(\mathbb{R}^n)$, then*

$$\left\| (L_\psi f)(\cdot, a) \right\|_{m,p,\omega} = O(|a|^{n/2}). \tag{3.1}$$

Now consider the operator

$$(\Lambda_\psi f)(b, a) = (\psi_a * f)(b) = \frac{1}{a^n} \int_{\mathbb{R}^n} f(t) \psi\left(\frac{b-t}{a}\right) dt, \tag{3.2}$$

where $\psi \in L^1(\mathbb{R}^n)$, $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, and

$$\psi_a(x) = \frac{1}{a^n} \psi\left(\frac{x}{a}\right), \quad a \ne 0. \tag{3.3}$$

In the sequel, the following lemma is needed.

LEMMA 3.2. *Let $f \in W_\omega^{m,p}(\mathbb{R}^n)$, $1 \le p < \infty$, $\psi \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, with*

$$\int_{\mathbb{R}^n} \psi(t) \, dt = 1, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{3.4}$$

Then

(i) $(\Lambda_\psi f)(\cdot, a) \to f(\cdot)$ in $W_\omega^{m,p}(\mathbb{R}^n)$ as $a \to 0^+$;

(ii) $[\partial^\mu (\Lambda_\psi f)](\cdot, a) = [\Lambda_\psi (\partial^\mu f)](\cdot, a) = a^{-|\mu|}(\Lambda_{\partial^\mu \psi} f)(\cdot, a)$ if $\partial^\mu \psi \in L^1(\mathbb{R}^n)$
$\cap L^q(\mathbb{R}^n)$, $|\mu| < m$, $a > 0$.

Proof. Since $(\Lambda_\psi f)(\cdot, a) = (\psi_a * f)(\cdot)$, and for each multi-index α, $\sum_{i=1}^n \alpha_i \le m$ such that $\partial^\mu f \in L^p(\mathbb{R}^n)$, $\psi \in L^q(\mathbb{R}^n)$, we obtain

$$[\partial^\mu (\psi_a * f)](\cdot) = (\psi_a * \partial^\mu f)(\cdot),$$

$$\|(\Lambda_\psi f)(\cdot, a) - f(\cdot)\|_{m,p,\omega}$$

$$= \sum_{|\mu| \le m} \left(\int_{\mathbb{R}^n} \omega_\mu(\cdot) |\partial^\mu [(\Lambda_\psi f)(\cdot, a) - f(\cdot)]|^p \, d(\cdot) \right)^{1/p} \qquad (3.5)$$

$$= \sum_{|\mu| \le m} \left(\int_{\mathbb{R}^n} \omega_\mu(\cdot) |\Lambda_\psi (\partial^\mu f)(\cdot, a) - (\partial^\mu f)(\cdot)|^p \, d(\cdot) \right)^{1/p}.$$

Taking into account that $\partial^\mu f \in L^p(\mathbb{R}^n)$, it is easy to see that

$$\int_{\mathbb{R}^n} \omega_\mu(\cdot) |\Lambda_\psi (\partial^\mu f)(\cdot, a) - (\partial^\mu f)(\cdot)|^p \, d(\cdot)$$
$$\le \|\omega_\mu\|_\infty \int_{\mathbb{R}^n} |\Lambda(\partial^\mu f)(\cdot, a) - (\partial^\mu f)(\cdot)|^p \, d(\cdot) \longrightarrow 0 \quad \text{as } a \longrightarrow 0^+. \qquad (3.6)$$

Consequently, $(\Lambda_\psi f)(\cdot, a) \to f(\cdot)$ in $W_\omega^{m,p}(\mathbb{R}^n)$ as $a \to 0^+$, that is, (i) is proved.

To check (ii) take $f_r \in \mathscr{S}(\mathbb{R}^n)$, $f_r \to f$ in $W_\omega^{m,p}(\mathbb{R}^n)$. It is obvious that in $W_\omega^{m-|\mu|,p}(\mathbb{R}^n)$

$$[\partial^\mu (\Lambda_\psi f_r)](\cdot, a) = [\Lambda_\psi (\partial^\mu f_r)](\cdot, a) = a^{-|\mu|}(\Lambda_{\partial^\mu \psi} f_r)(\cdot, a). \qquad (3.7)$$

By the continuity of the operators

$$\Lambda_\psi : W_\omega^{m,p}(\mathbb{R}^n) \longrightarrow W_\omega^{m,p}(\mathbb{R}^n),$$
$$\partial^\mu : W_\omega^{m,p}(\mathbb{R}^n) \longrightarrow W_\omega^{m-|\mu|,p}(\mathbb{R}^n), \qquad (3.8)$$

for $\psi \in L^q(\mathbb{R}^n)$, $|\mu| < m$, letting $r \to \infty$ we get (ii). $\qquad \square$

THEOREM 3.3. *Let* $f \in W_\omega^{m,p}(\mathbb{R}^n)$, $1 \le p < \infty$, $\psi \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ *such that*

$$\int_{\mathbb{R}^n} \psi(t) \, dt = 1. \qquad (3.9)$$

Moreover, assume that $\partial^\mu \psi$ *is a basic wavelet for each multi-index* $\mu, |\mu| \in \{0, 1, 2, \ldots, m\}$, *and* $\partial^\mu \psi \in L^q(\mathbb{R}^n)$, *with* $1/p + 1/q = 1$. *Suppose additionally that* f *and* ψ *are real-valued functions and* $a > 0$. *Then*

$$\lim_{a \to 0^+} \left\| \frac{1}{a^{|\mu|+n/2}} (L_{\partial^\mu \psi} f)(\cdot, -a) - \frac{1}{\sqrt[2n]{C_{\partial^\mu \psi}}} \partial^\mu f(\cdot) \right\|_{m-|\mu|, p, \omega} = 0. \qquad (3.10)$$

Proof. Obviously

$$
\begin{aligned}
(L_{\partial^\mu \psi} f)(\cdot, -a) &= \frac{a^{n/2}}{\sqrt[2^n]{C_{\partial^\mu \psi}}} \left[(\partial^\mu \psi)_a * f \right](\cdot) \\
&= \frac{a^{n/2}}{\sqrt[2^n]{C_{\partial^\mu \psi}}} (\psi_a * \partial^\mu f)(\cdot) = \frac{a^{|\mu|+n/2}}{\sqrt[2^n]{C_{\partial^\mu \psi}}} \left[\partial^\mu (\psi_a * f) \right](\cdot).
\end{aligned}
\tag{3.11}
$$

Under the assumptions of the theorem, the differentiation and integration can be interchanged, and furthermore by the continuity of the operator

$$
\partial^\mu : W_\omega^{m,p}(\mathbb{R}^n) \longrightarrow W_\omega^{m-|\mu|,p}(\mathbb{R}^n),
\tag{3.12}
$$

for $|\mu| < m$, we get

$$
\begin{aligned}
&\left\| \frac{1}{a^{|\mu|+n/2}} (L_{\partial^\mu \psi} f)(\cdot, -a) - \frac{1}{\sqrt[2^n]{C_{\partial^\mu \psi}}} (\partial^\mu f)(\cdot) \right\|_{m-|\mu|,p,\omega} \\
&\qquad \le \frac{1}{\sqrt[2^n]{C_{\partial^\mu \psi}}} \left\| \left[\partial^\mu (\psi_a * f) \right](\cdot) - (\partial^\mu f)(\cdot) \right\|_{m-|\mu|,p,\omega} \\
&\qquad \le \frac{C}{\sqrt[2^n]{C_{\partial^\mu \psi}}} \left\| (\psi_a * f)(\cdot) - f(\cdot) \right\|_{m,p,\omega}.
\end{aligned}
\tag{3.13}
$$

Lemma 3.2 implies now that the last term in (3.13) tends to 0 as $a \to 0^+$. $\square$

Acknowledgment

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