

# ON THE EXISTENCE OF A POSITIVE SOLUTION FOR A SEMILINEAR ELLIPTIC EQUATION IN THE UPPER HALF SPACE

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We show the existence of a nontrivial solution to the semilinear elliptic equation  $-\Delta u + u = b(x)|u|^{p-2}u$ ,  $u > 0$ ,  $u \in H_0^1(\mathbb{R}_+^N)$  under some suitable conditions.

## 1. Introduction

In this paper, we study the following problem:

$$-\Delta u + u = b(x)|u|^{p-2}u, \quad u > 0, \quad u \in H_0^1(\mathbb{R}_+^N), \quad (1.1)$$

where  $2 < p < 2^*$ ,  $2^* = 2N/(N-2)$  when  $N > 2$  and  $2^* = +\infty$  when  $N = 2$ ,  $\mathbb{R}_+^N = \{(x_1, x_2, \dots, x_N) \mid x_N > 0\}$  is the upper half space of  $\mathbb{R}^N$ . We put the following conditions on  $b$ :

(B<sub>1</sub>)  $b \in C(\mathbb{R}_+^N)$ , and

$$1 = \inf_{x \in \mathbb{R}_+^N} b(x) = \lim_{|x| \rightarrow +\infty, x \in \mathbb{R}_+^N} b(x); \quad (1.2)$$

(B<sub>2</sub>) there exists  $\delta \in (0, 1)$ , such that

$$\lim_{|x| \rightarrow \infty} (b(x) - 1)e^{(1-\delta)|x|} = +\infty. \quad (1.3)$$

Our main result is the following result.

**THEOREM 1.1.** *If  $b$  satisfies (B<sub>1</sub>) and (B<sub>2</sub>), then (1.1) has a nontrivial solution.*

Similar problems have been studied extensively. In [3], Esteban and Lions showed that  $-\Delta u + u = |u|^{p-2}u$ ,  $u \in H_0^1(\mathbb{R}_+^N)$  does not have nontrivial solution.

However, with the help of the potential  $b(x)$  and the comparison technique derived in [2], we proved the existence of a nontrivial solution to problem (1.1).

## 2. Proof of Theorem 1.1

Let

$$\varphi(u) = \int_{\mathbb{R}_+^N} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u|^2 \right) dx - \frac{1}{p} \int_{\mathbb{R}_+^N} b(x) |u|^p dx. \quad (2.1)$$

LEMMA 2.1. Denote  $S_p = \inf_{u \in H_0^1(\mathbb{R}_+^N), |u|_p=1} \|u\|^2$ . Then

$$\inf_{u \in H_0^1(\mathbb{R}_+^N), |u|_p=1} \|u\|^2 = S_p. \quad (2.2)$$

LEMMA 2.2. If  $0 < c < c^* := (1/2 - 1/p) S_p^{p/(p-2)}$ , then  $\varphi$  satisfies  $(PS)_c$  condition.

One can see [2] or [5] for the proof of Lemmas 2.1 and 2.2.

LEMMA 2.3. The equation  $-\Delta u + u = |u|^{p-2}u$  has a unique positive solution  $w$  in  $\mathbb{R}^N$  and satisfies the following conditions:

- (i)  $w > 0$  in  $\mathbb{R}^N$ ,  $w \in C^\infty(\mathbb{R}^N)$ .
- (ii)  $w(x) = w(|x|)$ .
- (iii)  $w(r)r^{(N-1)/2} \exp(r) \rightarrow C > 0$  and  $w'(r)r^{(N-1)/2} \exp(r) \rightarrow -C$ , where  $r = |x| \rightarrow +\infty$ .

*Proof.* See [1, 4]. □

*Proof of Theorem 1.1.* By Lemma 2.2, we need only to prove that the Mountain Pass value of  $\varphi$ ,

$$c = \inf \left\{ \left( \frac{1}{2} - \frac{1}{p} \right) \left( \frac{\|v\|^2}{\left( \int_{\mathbb{R}_+^N} b(x) |v|^p dx \right)^{2/p}} \right)^{p/(p-2)} ; v \in H_0^1(\mathbb{R}_+^N), v \neq \theta \right\}, \quad (2.3)$$

satisfies that  $0 < c < c^*$ .

Set

$$\psi_n(x) = \begin{cases} 1, & |x| \leq n-1, \\ n-|x|, & n-1 < |x| \leq n, \\ 0, & |x| > n. \end{cases} \quad (2.4)$$

Let  $x_n = (0, 0, \dots, 0, n)$ ,  $u_n = \psi_n(\cdot - x_n)w(\cdot - x_n)$ . Then  $u_n \in H_0^1(\mathbb{R}_+^N)$ ,  $n = 1, 2, \dots$

Notice that

$$\begin{aligned}
\|u_n\|^2 &= \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} u_n^2 dx \\
&= \int_{\mathbb{R}^N} |\nabla(\psi_n(\cdot - x_n)w(\cdot - x_n))|^2 dx + \int_{\mathbb{R}^N} |\psi_n(\cdot - x_n)w(\cdot - x_n)|^2 dx \\
&= \int_{B(\theta, n)} |\nabla(\psi_n \cdot w)|^2 dx + \int_{B(\theta, n)} |\psi_n \cdot w|^2 dx \\
&= \int_{B(\theta, n-1)} |\nabla \omega|^2 dx + \int_{B(\theta, n-1)} \omega^2 dx \\
&\quad + \int_{B_{n-1, n}} \psi_n^2 \cdot |\nabla \omega|^2 dx + \int_{B_{n-1, n}} |\nabla \psi_n|^2 \cdot \omega^2 dx \\
&\quad + 2 \int_{B_{n-1, n}} \omega \cdot \psi_n \cdot \nabla \omega \cdot \nabla \psi_n dx + \int_{B_{n-1, n}} \psi_n^2 \cdot \omega^2 dx \\
&= \int_{\mathbb{R}^N} |\nabla \omega|^2 dx + \int_{\mathbb{R}^N} \omega^2 dx \\
&\quad + \int_{\mathbb{R}^N \setminus B(\theta, n-1)} |\nabla \omega|^2 dx + \int_{\mathbb{R}^N \setminus B(\theta, n-1)} |\omega|^2 dx \\
&\quad + \int_{B_{n-1, n}} \psi_n^2 \cdot |\nabla \omega|^2 dx + 2 \int_{B_{n-1, n}} \omega \cdot \psi_n \cdot \nabla \omega \cdot \nabla \psi_n dx \\
&\quad + \int_{B_{n-1, n}} |\nabla \psi_n|^2 \cdot \omega^2 dx + \int_{B_{n-1, n}} \psi_n^2 \cdot \omega^2 dx,
\end{aligned} \tag{2.5}$$

where  $B_{n-1, n} = B(\theta, n) \setminus B(\theta, n-1)$ .

From [Lemma 2.3](#) we know that, when  $n$  is big enough,

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus B(\theta, n-1)} |\nabla \omega|^2 dx &= C_1 \int_{n-1}^{+\infty} |\omega'(r)|^2 \cdot r^{N-1} dr \\
&\leq C_1' \int_{n-1}^{+\infty} e^{-2r} dr = \frac{C_1'}{e^{2(n-1)}}.
\end{aligned} \tag{2.6}$$

Similarly, when  $n$  is big enough, we get

$$\int_{\mathbb{R}^N \setminus B(\theta, n-1)} \omega^2 dx \leq \frac{C_2'}{e^{2(n-1)}}. \tag{2.7}$$

So

$$\|u_n\|^2 = \|\omega\|^2 + O\left(\frac{1}{e^{2(n-1)}}\right). \tag{2.8}$$

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Let  $a(x) = b(x) - 1$ . Then

$$\begin{aligned}
 \int_{\mathbb{R}_+^N} b(x) |u_n|^p dx &= \int_{\mathbb{R}_+^N} |u_n|^p dx + \int_{\mathbb{R}_+^N} a(x) |u_n|^p dx \\
 &= \int_{B(\theta, n-1)} |\omega|^p dx + \int_{B(\theta, n-1)} a(x+x_n) |\omega|^p dx \\
 &\quad + \int_{B_{n-1, n}} a(x+x_n) |\psi_n(x) \cdot \omega(x)|^p dx \\
 &= \int_{\mathbb{R}^N} |\omega|^p dx + \int_{B(\theta, n-1)} a(x+x_n) |\omega|^p dx \\
 &\quad + \int_{B_{n-1, n}} a(x+x_n) |\psi_n(x) \cdot \omega(x)|^p dx + \int_{\mathbb{R}^N \setminus B(\theta, n-1)} |\omega|^p dx.
 \end{aligned} \tag{2.9}$$

Thus

$$\begin{aligned}
 \int_{\mathbb{R}^N \setminus B(\theta, n-1)} |\omega|^p dx &= o\left(\frac{1}{e^{2(n-1)}}\right), \\
 \int_{B_{n-1, n}} a(x+x_n) |\psi_n(x) \cdot \omega(x)|^p dx &= o\left(\frac{1}{e^{2(n-1)}}\right).
 \end{aligned} \tag{2.10}$$

From (B<sub>2</sub>), we see that there exists a positive constant  $\lambda$ , such that

$$a(x) \geq \frac{\lambda}{e^{(1-\delta)|x|}}. \tag{2.11}$$

So there exists a positive constant  $C_3$ , such that

$$\int_{\mathbb{R}_+^N} b(x) |u_n|^p dx \geq \int_{\mathbb{R}^N} |\omega|^p dx + \frac{C_3}{e^{2(1-\delta)(n-1)}}. \tag{2.12}$$

Then

$$\begin{aligned}
 &\left( \int_{\mathbb{R}_+^N} b(x) |u_n|^p dx \right)^{2/p} \\
 &\geq \left( \int_{\mathbb{R}^N} |\omega|^p dx + \frac{C_3}{e^{2(1-\delta)(n-1)}} \right)^{2/p} \\
 &= \left( \int_{\mathbb{R}^N} |\omega|^p dx \right)^{2/p} \left( 1 + \frac{C_4}{e^{2(1-\delta)(n-1)}} \right)^{2/p} \\
 &= \left( \int_{\mathbb{R}^N} |\omega|^p dx \right)^{2/p} \left( 1 + \frac{C_5}{e^{2(1-\delta)(n-1)}} + o\left(\frac{1}{e^{2(1-\delta)(n-1)}}\right) \right).
 \end{aligned} \tag{2.13}$$

Thus

$$\frac{\|u_n\|^2}{(\int_{\mathbb{R}_+^N} b(x)|u_n|^p dx)^{2/p}} = \frac{\|\omega\|^2}{(\int_{\mathbb{R}^N} \omega^p dx)^{2/p}} - \frac{C_6}{e^{2(1-\delta)(n-1)}} + o\left(\frac{1}{e^{2(1-\delta)(n-1)}}\right). \quad (2.14)$$

So

$$\frac{\|u_n\|^2}{(\int_{\mathbb{R}_+^N} b(x)|u_n|^p dx)^{2/p}} < \frac{\|\omega\|^2}{(\int_{\mathbb{R}^N} \omega^p dx)^{2/p}}, \quad (2.15)$$

when  $n$  is big enough.

Thus, we come to the conclusion that

$$c < \left(\frac{1}{2} - \frac{1}{p}\right) S_p^{p/(p-2)}. \quad (2.16)$$

□

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