ORDINARY DIFFERENTIAL SYSTEMS DESCRIBING HYSTERESIS EFFECTS AND NUMERICAL SIMULATIONS

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We consider a general class of ordinary differential systems which describes input-output relations of hysteresis types, for instance, play or stop operators. The system consists of two first-order nonlinear ODEs and one of them includes a subdifferential operator depending on the unknowns. Our main objective of this paper is to give an existence-uniqueness result for the system as well as to give various numerical simulations of input-output relations which the system describes as typical cases.

1. Introduction

Consider a nonlinear system of ODEs of the following form:

$$a_1(u(t), w(t)) u'(t) + a_2(u(t), w(t)) w'(t)$$
$$= g(u(t), w(t)), \quad 0 < t < T, \tag{1.1}$$

$$b_1(u(t), w(t)) u'(t) + b_2(u(t), w(t)) w'(t) + \partial I_{u(t)}(w(t))$$
$$\ni h(u(t), w(t)), \quad 0 < t < T, \tag{1.2}$$

subject to the initial conditions

$$u(0) = u_0, \qquad w(0) = w_0, \tag{1.3}$$

where $0 < T < +\infty$ and $a_i(\cdot, \cdot)$, $b_i(\cdot, \cdot)$, $i = 1, 2$, $g(\cdot, \cdot)$, $h(\cdot, \cdot)$ are functions on $\mathbb{R}^2$ satisfying some conditions (see Section 2), and for each $u \in \mathbb{R}$, $\partial I_u(\cdot)$ is the subdifferential of the indicator function $I_u(\cdot)$ of the interval $[f_*(u), f^*(u)]$ in $\mathbb{R}$;

namely

$$\partial I_u(w) = \begin{cases} \varnothing & \text{for } w > f^*(u) \text{ or } w < f_*(u), \\ [0,+\infty) & \text{for } w = f^*(u) > f_*(u), \\ \{0\} & \text{for } f_*(u) < w < f^*(u), \\ (-\infty,0] & \text{for } w = f_*(u) < f^*(u), \\ (-\infty,+\infty) & \text{for } w = f_*(u) = f^*(u), \end{cases} \qquad (1.4)$$

$f_*(\cdot)$ and $f^*(\cdot)$ being nondecreasing functions such that $f_* \le f^*$ on $\mathbb{R}$ (see Section 2 for precise conditions).

Equation (1.2) describes a lot of input-output relations $u \to w$ which are physically relevant. For instance, when $b_1 \equiv 0$ (resp., -1), $b_2 \equiv 1$, and $h \equiv 0$, the relation assigning to a function $u(t)$ the solution $w(t)$ of (1.2) is called a play (resp., stop) operator (cf. [5, 6, 7]). These operators are typical examples of hysteresis input-output relations, and are used to describe irreversible phenomena such as solid-liquid phase transition with supercooling effect and martensite-austenite phase transition in shape memory alloys (cf. [2, 8]).

In a particular case when $a_1 \equiv 1$, $a_2 \equiv 1$, $b_1 \equiv 0$, $b_2 \equiv 1$, $g \equiv 0$, and $h \equiv 0$, the system (1.1), (1.2) was studied in detail by Visintin [7] in a very general framework; the idea for uniqueness proof is based on the so-called L^1-theory of nonlinear semigroups (cf. [1, 4]), and the same idea was applied to the uniqueness proof of the Cauchy problem for (1.1), (1.2) with nonzero right-hand sides and diffusion effects in [3].

Our main objective of this paper is to give an existence-uniqueness result of the Cauchy problem for system (1.1), (1.2) under some restrictions on coefficients a_i, b_i, $i = 1, 2$, general enough. One of the main points in our proof is to eliminate the term u' from (1.2) to get

$$w'(t) + \partial I_{u(t)}(w(t)) \ni \tilde{h}(u(t), w(t)), \quad 0 < t < T, \qquad (1.5)$$

with a function $\tilde{h}$ satisfying the same property as h, and then consider the coupling of (1.1) and (1.5). Another objective of this paper is to show by some numerical simulations that our nonlinear system covers many of physically relevant relations $u \to w$ arising in the mathematical descriptions of phase transition phenomena. In fact, a suitable choice of the set of coefficients a_i, b_i and forcing terms g, h, can create the clockwise or anti-clockwise trend of the orbit $(u(t), w(t))$ as t increases.

2. Theoretical results

In this section, we mention the precise assumptions on the functions a_i, b_i, $i = 1, 2, g, h$, and theoretical results on the existence and uniqueness of a solution of (1.1), (1.2), and (1.3).

Let f_* and f^* be functions on $\mathbb{R}$ such that

$$f_*, \ f^* \text{ are nondecreasing and Lipschitz continuous on } \mathbb{R},$$

$$\text{the derivatives } f_*' := \frac{df_*}{du}, \ f^{*'} := \frac{df^*}{du} \text{ are Lipschitz continuous on } \mathbb{R},$$

$$f_* \leq f^* \quad \text{on } \mathbb{R},$$

$$f_* = f^* \quad \text{on } (-\infty, -k_0] \cup [k_0, +\infty) \text{ for a positive number } k_0,$$

$$(2.1)$$

and put

$$\mathcal{F} := \{(u, w) \in \mathbb{R}^2; \ f_*(u) \leq w \leq f^*(u)\}. \tag{2.2}$$

For these functions f_*, f^*, we consider the indicator function

$$I_u(w) := \begin{cases} 0 & \text{for } f_*(u) \leq w \leq f^*(u), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.3}$$

associated with the interval $[f_*(u), f^*(u)]$ for every $u \in \mathbb{R}$, and denote its subdifferential by $\partial I_u(\cdot)$ given by (1.4). Next, let $a_i, b_i, i = 1, 2, g, h$ be functions on $\mathcal{F}$ such that

$$a_i, \ b_i, \ i = 1, 2, \text{ are Lipschitz continuous on } \mathcal{F},$$

$$a_1 \geq c_0 \quad \text{on } \mathcal{F}, \qquad b_2 \geq c_0 \quad \text{on } \mathcal{F},$$

$$a_1 b_2 - a_2 b_1 \geq c_0 \quad \text{on } \mathcal{F} \text{ for a positive constant } c_0,$$

$$(2.4)$$

$$g, \ h \text{ are Lipschitz continuous on } \mathcal{F}. \tag{2.5}$$

Now we give the definition of a solution of (1.1), (1.2).

Definition 2.1. A pair of (scalar) functions $\{u, w\}$ is called a solution of system (1.1), (1.2), if $u, w \in W^{1,2}(0, T)$ and they satisfy (1.1) and (1.2) a.e. on $[0, T]$; hence

$$(u(t), w(t)) \in \mathcal{F} \quad \forall t \in [0, T]. \tag{2.6}$$

Definition 2.2. A pair of functions $\{u, w\}$ is called a solution of (1.1), (1.2), and (1.3), if it is a solution of (1.1), (1.2) and the initial conditions $u(0) = u_0, w(0) = w_0$ are satisfied.

The existence of a solution is proved under an additional condition.

THEOREM 2.3. *Suppose that (2.1), (2.2), (2.3), (2.4), and (2.5) hold, and moreover suppose that there is a positive constant δ_0 such that*

$$
\begin{aligned}
a_1(u, f_*(u)) + a_2(u, f_*(u)) f_*'(u) &\geq \delta_0, \\
a_1(u, f^*(u)) + a_2(u, f^*(u)) f^{*\prime}(u) &\geq \delta_0 \quad \text{for any } u \in \mathbb{R}.
\end{aligned}
\tag{2.7}
$$

Then

(1) *for any initial data u_0, w_0 satisfying $(u_0, w_0) \in \mathcal{F}$, there exists at least one solution $\{u, w\}$ of system (1.1), (1.2), and (1.3);*

(2) *there is a positive constant K_0, depending only on the quantities in assumptions (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), and (2.7), such that*

$$
\begin{aligned}
|u(t)| + |w(t)| &\leq e^{K_0(1+T)}(1 + |u_0|) \quad \forall t \in [0, T], \\
|u'(t)| + |w'(t)| &\leq e^{K_0(1+T)}\left(1 + |u_0|^3\right) \quad \text{for a.e. } t \in [0, T],
\end{aligned}
\tag{2.8}
$$

for any solution $\{u, w\}$ of (1.1), (1.2), and (1.3) with initial data u_0, w_0.

Our second result is concerned with the uniqueness of a solution of (1.1), (1.2), and (1.3).

THEOREM 2.4. *Suppose that (2.1), (2.2), (2.3), (2.4), and (2.5) hold as well as*

$$
a_2 \geq c_0 \quad \text{on } \mathcal{F},
\tag{2.9}
$$

where c_0 is the same positive constant as in (2.4). Then system (1.1), (1.2), and (1.3) admits at most one solution. More precisely, there is a positive constant M_0, depending only on the quantities in assumptions (2.1), (2.2), (2.3), (2.4), (2.5), and (2.9), and the length T of the time interval, such that

$$
\begin{aligned}
|u_1(t) - u_2(t)| &+ |w_1(t) - w_2(t)| \\
&\leq e^{M_0(1+|u_{01}|^7+|u_{02}|^7)}(|u_{01} - u_{02}| + |w_{01} - w_{02}|) \quad \forall t \in [0, T],
\end{aligned}
\tag{2.10}
$$

for any two solutions $\{u_i, w_i\}$ of (1.1), (1.2) with initial values u_{0i}, w_{0i}, $i = 1, 2$.

Remark 2.5. We see from the proof of Theorem 2.4 given in Section 4 that the constant M_0 is of the form $e^{K_1(1+T)}$ for a positive constant depending only on the quantities in assumptions (2.1), (2.2), (2.3), (2.4), (2.5), and (2.9).

Remark 2.6. As is easily understood, for the construction of a local in time solution of (1.1), (1.2), and (1.3) it is enough to assume that the functions a_i, b_i, g, h, f_*, f^* are locally Lipschitz continuous and the inequalities in (2.4) and (2.7) are satisfied as well in a neighborhood of the initial point (u_0, w_0).

As is easily checked, without loss of generality, we may assume that the functions a_i, b_i, $i = 1, 2$, g, and h are globally Lipschitz continuous on $\mathbb{R}^2$ and all the inequalities in (2.4) hold on $\mathbb{R}^2$. In fact, it is enough to extend them outside $\mathcal{F}$,

for instance, in the following manner:

$$
\begin{aligned}
a_i(u,w) &= a_i(u, f^*(u)), & b_i(u,w) &= b_i(u, f^*(u)), & \text{for } w > f^*(u), \\
a_i(u,w) &= a_i(u, f_*(u)), & b_i(u,w) &= b_i(u, f_*(u)), & \text{for } w < f_*(u), \\
g(u,w) &= g(u, f^*(u)), & h(u,w) &= h(u, f^*(u)), & \text{for } w > f^*(u), \\
g(u,w) &= g(u, f_*(u)), & h(u,w) &= h(u, f_*(u)), & \text{for } w < f_*(u).
\end{aligned}
\tag{2.11}
$$

In the rest of this paper, we always assume such extended conditions for functions a_i, b_i, $i = 1, 2$, g, and h.

The key for the proofs of our theorems is found in the following lemma.

LEMMA 2.7. *Under conditions* (2.1), (2.2), (2.3), (2.4), *and* (2.5), *system* (1.1), (1.2) *is equivalent to the coupling of* (1.1) *and the following inclusion:*

$$
w'(t) + \partial I_{u(t)}(w(t)) \ni \tilde{h}(u(t), w(t)) \quad \textit{for a.e. } t \in [0, T], \tag{2.12}
$$

where

$$
\tilde{h}(u,w) := \frac{a_1(u,w)h(u,w) - b_1(u,w)g(u,w)}{a_1(u,w)b_2(u,w) - a_2(u,w)b_1(u,w)}. \tag{2.13}
$$

Proof. In order to eliminate the term u' from (1.2), compute $((1.2) \times a_1 - (1.1) \times b_1)/(a_1 b_2 - a_2 b_1)$. Then we have

$$
w' + \frac{a_1}{a_1 b_2 - a_2 b_1} \partial I_u(w) \ni \tilde{h} \quad \text{a.e. on } [0, T], \tag{2.14}
$$

where $\tilde{h}$ is the same as given by (2.13). Here we note the invariance of $\partial I_u(w)$ under multiplication by positive numbers, namely $\partial I_u(w) = k\partial I_u(w)$ for every positive k. In fact, this property is immediately seen from (1.4). Therefore, by (2.4), we have $(a_1/(a_1 b_2 - a_2 b_1))\partial I_u(w) = \partial I_u(w)$. Hence (2.12) is obtained. □

This lemma shows that it is enough to prove Theorems 2.3 and 2.4 to the system $\{(1.1), (2.12)\}$ instead of $\{(1.1), (1.2)\}$.

3. A priori bounds of solutions

In this section, we give a priori bounds of the form (2.8) for solutions under the same assumptions as Theorem 2.3. For the sake of simplicity of notation we denote by L_0, chosen so that $L_0 > 1$, a common Lipschitz constant of functions a_i, b_i, $i = 1, 2$, g, h on $\mathbb{R}^2$; note the extended condition (2.11).

Let $\{u, w\}$ be any solution of (1.1), (1.2) with given initial data u_0, w_0; of course, the relation $f_*(u_0) \leq w_0 \leq f^*(u_0)$ is satisfied. Then we prove the following lemma.

LEMMA 3.1. *Put $N_0 := k_0 + |u_0|$, where k_0 is the same number as in (2.1). Then,*

$$
\begin{aligned}
-N_0 - &\frac{|g(-N_0, f_*(-N_0))|}{2L_0^2} \exp\left(\frac{2L_0^2}{\delta_0}T\right) \\
&\leq u(t) \leq N_0 + \frac{|g(N_0, f^*(N_0))|}{2L_0^2} \exp\left(\frac{2L_0^2}{\delta_0}T\right)
\end{aligned}
\tag{3.1}
$$

for all $t \in [0, T]$.

Proof. We multiply (1.1) by $(u - N_0)^+$ to get

$$
a_1 u'(u - N_0)^+ + a_2 w'(u - N_0)^+ = g(u - N_0)^+.
\tag{3.2}
$$

Since $w = f^*(u)$ for $u \geq N_0$ by (2.1), it follows from the above equality that

$$
\frac{1}{2}(a_1 + a_2 f^{*\prime}(u)) \frac{d}{dt} |(u - N_0)^+|^2 = g(u - N_0)^+.
\tag{3.3}
$$

Also, we note that

$$
\begin{aligned}
g(u - N_0)^+ &= (g(u, f^*(u)) - g(N_0, f^*(N_0)))(u - N_0)^+ \\
&\quad + g(N_0, f^*(N_0))(u - N_0)^+ \\
&\leq 2L_0^2 |(u - N_0)^+|^2 + |g(N_0, f^*(N_0))|(u - N_0)^+.
\end{aligned}
\tag{3.4}
$$

By (3.3) and (3.4) with assumption (2.7) we have

$$
\frac{d}{dt}(u - N_0)^+ \leq \frac{2L_0^2}{\delta_0}(u - N_0)^+ + \frac{1}{\delta_0}|g(N_0, f^*(N_0))|,
\tag{3.5}
$$

and the second inequality of (3.1) is obtained. The first inequality is similarly obtained, too. □

COROLLARY 3.2. *There is a positive constant $K_0^{(1)}$, depending only on the quantities in (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), and (2.7), such that*

$$
|u(t)| + |w(t)| \leq e^{K_0^{(1)}(1+T)}(1 + |u_0|), \quad 0 \leq \forall t \leq T.
\tag{3.6}
$$

Proof. We note that $|w|$ is bounded by a linear function of $|u|$ and hence so are $|g(N_0, f^*(N_0))|$ and $|g(-N_0, f_*(-N_0))|$. Therefore, from (3.1) together with this fact we immediately derive (3.6) for a certain positive constant $K_0^{(1)}$. □

LEMMA 3.3. *There is a positive constant $K_0^{(2)}$, depending only on the quantities in (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), and (2.7), such that*

$$|u'(t)| + |w'(t)| \leq e^{K_0^{(2)}(1+T)}(1 + |u_0|^3) \quad \text{for a.e. } t \in [0, T]. \qquad (3.7)$$

Proof. We have $a_1|u'|^2 + a_2 w' u' = g u'$ by multiplying (1.1) by u'. Here, since u, w satisfy (2.12), we observe that

$$w' = \begin{cases} \tilde{h} & \text{a.e. on } \{t; f_*(u) < w < f^*(u)\}, \\ f_*'(u)u' & \text{a.e. on } \{t; w = f_*(u)\}, \\ f^{*'}(u)u' & \text{a.e. on } \{t; w = f^*(u)\}, \end{cases} \qquad (3.8)$$

so that

$$|g||u'| \geq a_1|u'|^2 + a_2 w' u'$$
$$= \begin{cases} a_1|u'|^2 + a_2 \tilde{h} u' & \text{a.e. on } \{t; f_*(u) < w < f^*(u)\}, \\ (a_1 + a_2 f_*'(u))|u'|^2 & \text{a.e. on } \{t; w = f_*(u)\}, \\ (a_1 + a_2 f^{*'}(u))|u'|^2 & \text{a.e. on } \{t; w = f^*(u)\}. \end{cases} \qquad (3.9)$$

Accordingly, using our assumptions (2.4) and (2.7), we see from the above equality that

$$|u'|^2 \leq \begin{cases} \dfrac{2}{c_0^2}(|a_2\tilde{h}|^2 + |g|^2) & \text{a.e. on } \{t; f_*(u) < w < f^*(u)\}, \\[2mm] \dfrac{|g|^2}{\delta_0^2} & \text{a.e. on } \{t; w = f_*(u) \text{ or } f^*(u)\}. \end{cases} \qquad (3.10)$$

Moreover, note that $|\tilde{h}| \leq \text{const}(1 + |u|^2)$ and hence $|a_2\tilde{h}| \leq \text{const}(1 + |u|^3)$. Therefore, it follows from (3.10) with (3.6) and (3.8) that an estimate of the form (2.8) holds for a certain positive constant $K_0^{(2)}$. $\qquad \square$

Now, putting $K_0 := \max\{K_0^{(1)}, K_0^{(2)}\}$, we see the estimate (2.8) for all solutions $\{u, w\}$ with initial data u_0, w_0.

4. Proof of uniqueness

In this section, we prove (2.10) for two solutions $\{u_k, w_k\}$, $k = 1, 2$, of system (1.1), (1.2), and (1.3) for given initial data u_{0k}, w_{0k}, assuming always that conditions (2.1), (2.2), (2.3), (2.4), (2.5), and (2.9) are satisfied.

For simplicity we put

$$
\begin{aligned}
a_i^{(k)} &:= a_i(u_k, w_k), \quad b_i^{(k)} := b_i(u_k, w_k), \quad g^{(k)} := g(u_k, w_k), \\
h^{(k)} &:= h(u_k, w_k), \quad \tilde{h}^{(k)} := \tilde{h}(u_k, w_k), \quad i, k = 1, 2, \\
\bar{u} &:= u_1 - u_2, \quad \bar{w} := w_1 - w_2, \\
\bar{u}_0 &:= u_{01} - u_{02}, \quad \bar{w}_0 := w_{01} - w_{02}.
\end{aligned}
\tag{4.1}
$$

With these notations, by taking the difference of two solutions, we have

$$
a_1^{(1)} \bar{u}' + a_2^{(1)} \bar{w}' = \left(g^{(1)} - g^{(2)}\right) - \left(a_1^{(1)} - a_1^{(2)}\right) u_2' - \left(a_2^{(1)} - a_2^{(2)}\right) w_2'.
\tag{4.2}
$$

Now, take any measurable function $s_{\bar{u}}$ in time so that

$$
s_{\bar{u}} \in \operatorname{sign}(\bar{u}) =
\begin{cases}
1 & \text{if } \bar{u} > 0, \\
[-1, 1] & \text{if } \bar{u} = 0, \\
-1 & \text{if } \bar{u} < 0,
\end{cases}
\tag{4.3}
$$

and multiply (4.2) by $s_{\bar{u}}$ and use the Lipschitz continuity of functions a_i, g. Then, since $\bar{u}' s_{\bar{u}} = (d/dt)|\bar{u}|$, we have

$$
a_1^{(1)} \frac{d}{dt}|\bar{u}| + a_2^{(1)} \bar{w}' s_{\bar{u}} \le L_0 \left(1 + |u_2'| + |w_2'|\right) \left(|\bar{u}| + |\bar{w}|\right).
\tag{4.4}
$$

We arrange this inequality in the following form:

$$
\frac{d}{dt}\left\{a_1^{(1)}|\bar{u}|\right\} + a_2^{(1)} \bar{w}' s_{\bar{u}} \le L_0 \left(1 + |u_1'| + |u_2'| + |w_1'| + |w_2'|\right) \left(|\bar{u}| + |\bar{w}|\right).
\tag{4.5}
$$

Next, we show that

$$
\bar{w}' s_{\bar{u}} \ge \frac{d}{dt}|\bar{w}| - 2L_1\left(|\bar{u}| + |\bar{w}|\right) \quad \text{a.e. on } [0, T],
\tag{4.6}
$$

where L_1 is a Lipschitz constant of $\tilde{h}$ on $\mathbb{R}^2$; note that L_1 is dominated by a positive number of the form $\operatorname{const}(1 + |u_{01}|^3 + |u_{02}|^3)$. We show (4.6) on the following four subsets of time $t \in [0, T]$: (a) $E_1 := \{t; \bar{u} \ge 0, \ \bar{w} < 0\}$, (b) $E_2 := \{t; \bar{u} \le 0, \ \bar{w} > 0\}$, (c) $E_3 := \{t; \bar{u} \ge 0, \ \bar{w} \ge 0\}$, (d) $E_4 := \{t; \bar{u} \le 0, \ \bar{w} \le 0\}$. On the set $E_3 \cup E_4$, by the definition of the function $\operatorname{sign}(\cdot)$, we can choose $s_{\bar{w}}$ so that $s_{\bar{w}} = s_{\bar{u}}$. Therefore, (4.6) trivially holds. Next, we consider the case (a). In this case observe that

$$
w_1 < f^*(u_1), \quad w_2 > f_*(u_2) \quad \text{on } E_1.
\tag{4.7}
$$

In fact, if not, then $w_1 = f^*(u_1) \geq f^*(u_2) \geq w_2$ or $w_2 = f_*(u_2) \leq f_*(u_1) \leq w_1$, which contradicts our assumption. By the definition of $\partial I_{u_1}(w_1)$ we see that

$$(\tilde{h}^{(1)} - w_1')(z - w_1) \leq 0, \quad f_*(u_1) \leq \forall z \leq f^*(u_1). \tag{4.8}$$

Taking $f^*(u_1)$ as z in (4.8) yields that $\tilde{h}^{(1)} \leq w_1'$ on E_1 and similarly $\tilde{h}^{(2)} \geq w_2'$ on E_1. Consequently, it follows that

$$\bar{w}' - (\tilde{h}^{(1)} - \tilde{h}^{(2)}) \geq 0 \quad \text{on } E_1. \tag{4.9}$$

Since $s_{\bar{w}} = -1 \leq s_{\bar{u}}$ on E_1, we derive by multiplying the above inequality by $s_{\bar{u}} - s_{\bar{w}} \; (\geq 0)$ that

$$\bar{w}' s_{\bar{u}} \geq \frac{d}{dt}|\bar{w}| + (\tilde{h}^{(1)} - \tilde{h}^{(2)})(s_{\bar{u}} - s_{\bar{w}}) \quad \text{a.e. on } E_1. \tag{4.10}$$

This is easily arranged to the form (4.6). Just as (a), we see that inequality (4.6) holds a.e. on E_2, too. Thus (4.6) holds a.e. on $[0, T]$.

Moreover, multiply (4.6) by $a_2^{(1)} \; (\geq c_0)$ to get

$$\begin{aligned}
a_2^{(1)} \bar{w}' s_{\bar{u}} &\geq a_2^{(1)} \frac{d}{dt}|\bar{w}| - 2a_2^{(1)} L_1(|\bar{u}| + |\bar{w}|) \\
&\geq \frac{d}{dt}\{a_2^{(1)}|\bar{w}|\} \\
&\quad - (L_0 + 2a_2^{(1)} L_1)(1 + |u_1'| + |w_1'|)(|\bar{u}| + |\bar{w}|) \quad \text{a.e. on } [0, T].
\end{aligned} \tag{4.11}$$

We then obtain from (4.5) with the above inequality that

$$\begin{aligned}
\frac{d}{dt}\{a_1^{(1)}|\bar{u}| &+ a_2^{(1)}|\bar{w}|\} \\
&\leq 2(L_0 + a_2^{(1)} L_1)(1 + |u_1'| + |u_2'| + |w_1'| + |w_2'|)(|\bar{u}| + |\bar{w}|) \\
&\leq \frac{2(L_0 + a_2^{(1)} L_1)}{c_0}(1 + |u_1'| + |u_2'| \\
&\quad + |w_1'| + |w_2'|)(a_1^{(1)}|\bar{u}| + a_2^{(1)}|\bar{w}|) \quad \text{a.e. on } [0, T].
\end{aligned} \tag{4.12}$$

Here, on account of estimates (2.8) with the fact that $a_2^{(1)} L_1$ is dominated by a positive number of the form $\mathrm{const}(1 + |u_{01}|^4 + |u_{02}|^4)$, the last inequality implies that

$$\begin{aligned}
\frac{d}{dt}\{a_1^{(1)}|\bar{u}| &+ a_2^{(1)}|\bar{w}|\} \\
&\leq e^{K_1'(1+T)}\left(1 + |u_{01}|^7 + |u_{02}|^7\right)(a_1^{(1)}|\bar{u}| + a_2^{(1)}|\bar{w}|) \quad \text{a.e. on } [0, T],
\end{aligned} \tag{4.13}$$

where K_1' is a positive constant depending only on the quantities in assumptions (2.1), (2.2), (2.3), (2.4), (2.5), and (2.9). Therefore,

$$
\begin{aligned}
a_1^{(1)}(t)\,|\,\bar{u}(t)\,| &+ a_2^{(1)}(t)\,|\,\bar{w}(t)\,| \\
&\le \exp\left\{e^{K_1'(1+T)}\left(1+|\,u_{01}\,|^7 + |\,u_{02}\,|^7\right)T\right\} \\
&\quad \cdot \left(a_1^{(1)}(0)\,|\,\bar{u}_0\,| + a_2^{(1)}(0)\,|\,\bar{w}_0\,|\right) \quad \forall t \in [0,T].
\end{aligned}
\tag{4.14}
$$

Now, noting that $a_1^{(1)}(0)$ and $a_2^{(1)}(0)$ are dominated by a number of the form $\text{const}(1+|u_{01}|)$, from (4.14) we infer the required inequality (2.10) for a positive number M_0 of the form $e^{K_1(1+T)}$, where K_1 is a positive constant depending only on the quantities in (2.1), (2.2), (2.3), (2.4), (2.5), and (2.9).

5. Proof of existence

Throughout this section assume that the extended assumptions (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), and (2.7) hold and (2.12) as well. Let u_0, w_0 be a pair of initial data such that $(u_0, w_0) \in \mathcal{F}$. According to the a priori estimates (2.8) for solutions, we may assume, without loss of generality, that the functions a_i, b_i, $i = 1, 2$, g and $\tilde{h}$ are bounded on $\mathbb{R}^2$.

Consider approximate problems (P_λ), which consist of (5.1), (5.2), and (5.3), with parameter $\lambda \in (0, \lambda_0]$ for a positive number λ_0 small enough, to find a pair of functions $\{u, w\}$ satisfying

$$
a_1(u,w)u' + a_2(u,w)\frac{d}{dt}J(u,w) = g(u,w) \quad \text{a.e. on } [0,T],
\tag{5.1}
$$

$$
w' + \partial I_u^\lambda(w) = \tilde{h}(u,w) \quad \text{a.e. on } [0,T],
\tag{5.2}
$$

subject to the initial conditions

$$
u(0) = u_0, \qquad w(0) = w_0,
\tag{5.3}
$$

where $J(u, \cdot)$ is the projection mapping from $\mathbb{R}$ onto $[f_*(u), f^*(u)]$, namely

$$
J(u,w) = w - \left(w - f^*(u)\right)^+ + \left(f_*(u) - w\right)^+ \quad \forall w \in \mathbb{R}
\tag{5.4}
$$

and ∂I_u^λ is the Yosida approximation of ∂I_u, namely

$$
\partial I_u^\lambda(w) = \frac{\left(w - f^*(u)\right)^+}{\lambda} - \frac{\left(f_*(u) - w\right)^+}{\lambda} \quad \forall w \in \mathbb{R}.
\tag{5.5}
$$

Clearly, J is Lipschitz continuous on $\mathbb{R}^2$ and nondecreasing in both variables, and ∂I_u^λ is Lipschitz continuous on $\mathbb{R}^2$ and nondecreasing in w. Also, $\partial I_u^\lambda(\cdot)$ is the subdifferential of the convex function

$$I_u^\lambda(w) := \frac{|(w - f^*(u))^+|^2}{2\lambda} + \frac{|(f_*(u) - w)^+|^2}{2\lambda} \qquad \forall w \in \mathbb{R}. \qquad (5.6)$$

Lemma 5.1. *For each parameter $\lambda \in (0, \lambda_0]$, problem (P_λ) has a solution $\{u_\lambda, w_\lambda\}$ such that u_λ and w_λ are Lipschitz continuous on $[0, T]$. Moreover, the following uniform estimates hold: there is a positive constant R_1 independent of $\lambda \in (0, \lambda_0]$ such that*

$$|u_\lambda(t)| + |w_\lambda(t)| \le R_1 \quad \forall t \in [0, T],$$
$$|u_\lambda'(t)| \le R_1 \quad \text{for a.e. } t \in [0, T], \qquad |w_\lambda'|_{L^2(0,T)} \le R_1. \qquad (5.7)$$

Proof. Let J_ε be the regularization J by means of the usual mollifier ρ_ε, $0 < \varepsilon \le \varepsilon_0$ (ε_0 is a positive number close enough to 0), namely

$$J_\varepsilon(u, w) := \int_{\mathbb{R}^2} J(\zeta, \xi) \rho_\varepsilon(u - \zeta, w - \xi) \, d\zeta \, d\xi \qquad \forall (u, w) \in \mathbb{R}^2. \qquad (5.8)$$

Consider the further regularized approximate equation of (5.1)

$$a_1(u, w)u' + a_2(u, w)\frac{d}{dt}J_\varepsilon(u, w) = g(u, w) \quad \text{on } [0, T]. \qquad (5.9)$$

This is written in the form

$$\left\{ a_1(u, w) + a_2(u, w)\frac{\partial}{\partial u}J_\varepsilon(u, w) \right\} u' + a_2(u, w)\frac{\partial}{\partial w}J_\varepsilon(u, w)w' = g(u, w). \qquad (5.10)$$

Note here that

$$\frac{\partial}{\partial u}J(u, w)\left(\text{resp., } \frac{\partial}{\partial w}J(u, w)\right) = \begin{cases} f^{*\prime}(u) \text{ (resp., 0)} & \text{if } w > f^*(u), \\ 0 \text{ (resp., 1)} & \text{if } f_*(u) < w < f^*(u), \\ f_*'(u) \text{ (resp., 0)} & \text{if } w < f_*(u), \end{cases} \qquad (5.11)$$

which shows that

$$0 \le \frac{\partial}{\partial u}J_\varepsilon(u, w) \le f^{*\prime}(u) \text{ (resp., } f_*'(u)\text{)} + \tau_\varepsilon \qquad (5.12)$$

in the ε-neighborhood of $\{(u, w); w \ge f^*(u) \text{ (resp., } w \le f_*(u))\}$, where

$$\tau_\varepsilon := \sup\left\{ |J(u_1, w_1) - J(u_2, w_2)|; \left(|u_1 - u_2|^2 + |w_1 - w_2|^2\right)^{1/2} \le \varepsilon \right\}; \qquad (5.13)$$

clearly $\tau_\varepsilon \to 0$ as $\varepsilon \to 0$. Therefore, on account of (2.4), (2.7), and (2.11), we see that the coefficient $a_1(u, w) + a_2(u, w)(\partial/\partial u)J_\varepsilon(u, w)$ of u' in (5.10) is bounded

from below by a positive constant δ_0' on $\mathbb{R}^2$, where δ_0' is independent of $\lambda \in (0, \lambda_0]$ and $\varepsilon \in (0, \varepsilon_0]$. Therefore, system (5.9) and (5.2) is written in the form

$$
\begin{cases}
u' = \dfrac{g(u, w) - a_2(u, w)(\partial/\partial w)J_\varepsilon(u, w)(\tilde{h}(u, w) - \partial I_u^\lambda(w))}{a_1(u, w) + a_2(u, w)(\partial/\partial u)J_\varepsilon(u, w)}, \\
w' = \tilde{h}(u, w) - \partial I_u^\lambda(w)
\end{cases} \tag{5.14}
$$

and the right-hand sides are Lipschitz continuous in (u, w). By the general existence-uniqueness theorem for ODEs, problem (5.9) and (5.2) with initial condition (5.3) admits a unique solution, denoted by $\{u_{\lambda\varepsilon}, w_{\lambda\varepsilon}\}$.

Next, we give uniform estimates for $\{u_{\lambda\varepsilon}, w_{\lambda\varepsilon}\}$ in ε and λ. To do so, multiply (5.9) by $(u_{\lambda\varepsilon} - N_0')^+$, with $N_0' := k_0 + 1 + |u_0|$. Then, just as (3.1) of Lemma 3.1, we see that

$$
|u_{\lambda\varepsilon}(t)| \le K_2, \quad |J_\varepsilon(u_{\lambda\varepsilon}(t), w_{\lambda\varepsilon}(t))| \le K_2, \quad 0 \le \forall t \le T, \tag{5.15}
$$

for a positive constant K_2 independent of λ and ε. From these uniform estimates and (5.14) with (5.11) we derive easily that for each $\lambda > 0$ small enough, there is a positive constant $K_3(\lambda)$ independent of ε such that

$$
|u_{\lambda\varepsilon}'| \le K_3(\lambda), \quad |w_{\lambda\varepsilon}'| \le K_3(\lambda), \quad \left| \frac{d}{dt} J_\varepsilon(u_{\lambda\varepsilon}, w_{\lambda\varepsilon}) \right| \le K_3(\lambda), \quad \text{a.e. on } [0, T]. \tag{5.16}
$$

Therefore, it is possible to extract a null sequence $\{\varepsilon_n\}$ such that $u_{\lambda\varepsilon_n}$ and $w_{\lambda\varepsilon_n}$ converge to some Lipschitz continuous functions u_λ and w_λ uniformly on $[0, T]$, respectively, as $n \to +\infty$. Moreover, it is easy to see that the pair $\{u_\lambda, w_\lambda\}$ is a solution of (P_λ); note from (5.15) that

$$
|u_\lambda(t)| \le K_2, \quad 0 \le \forall t \le T. \tag{5.17}
$$

Taking account of (5.17) and the expression

$$
\frac{d}{dt} J(u_\lambda, w_\lambda) = \begin{cases}
\tilde{h}(u_\lambda, w_\lambda) & \text{a.e. on } \{t; f_*(u_\lambda) \le w_\lambda \le f^*(u_\lambda)\}, \\
f^{*\prime}(u_\lambda)u_\lambda' & \text{a.e. on } \{t; w > f^*(u_\lambda)\}, \\
f_*'(u_\lambda)u_\lambda' & \text{a.e. on } \{t; w < f^*(u_\lambda)\},
\end{cases} \tag{5.18}
$$

just as (3.10) in the proof of Lemma 3.3, we obtain by condition (2.7) that

$$
|u_\lambda'| \le K_4 \quad \text{a.e. on } [0, T] \tag{5.19}
$$

for a positive constant K_4 independent of $\lambda > 0$.

Finally, multiplying (5.2) by w_λ', we have

$$
|w_\lambda'|^2 + \partial I_{u_\lambda}^\lambda(w_\lambda) \cdot w_\lambda' = \tilde{h}w_\lambda'. \tag{5.20}
$$

Here, use the following inequality which is derived directly from the expressions (5.4), (5.5), and (5.6):

$$\frac{d}{dt}I_{u_\lambda}^\lambda(w_\lambda) - \partial I_{u_\lambda}^\lambda(w_\lambda) \cdot w_\lambda' \le L_0 \left| u_\lambda' \right| \left| \partial I_{u_\lambda}^\lambda(w_\lambda) \right| \quad \text{a.e. on } [0, T]. \tag{5.21}$$

Then, noting this with (5.19), we immediately obtain

$$\left| w_\lambda' \right|^2 + \frac{d}{dt}I_{u_\lambda}^\lambda(w_\lambda) \le L_0 K_4 \left(\left| w_\lambda' \right| + \left| \tilde{h} \right| \right) \quad \text{a.e. on } [0, T]. \tag{5.22}$$

From this inequality it is easy to get a uniform estimate for $|w_\lambda'|_{L^2(0,T)}$ with respect to λ, since $\tilde{h} := \tilde{h}(u_\lambda, w_\lambda)$ is uniformly bounded and $I_{u_0}^\lambda(w_0) = 0$. Thus uniform estimates of the form (5.7) hold for a positive constant R_1 independent of λ. ☐

By virtue of Lemma 5.1 there is a null sequence $\{\lambda_n\}$ such that the solution $u_n := u_{\lambda_n}$, $w_n := w_{\lambda_n}$ of (P_{λ_n}) constructed above converge to some functions u, w uniformly on $[0, T]$ as $n \to +\infty$. Simultaneously, since $\partial I_{u_n}^{\lambda_n}(w_n)$ $(= -w_n' + \tilde{h}(u_n, w_n))$ is bounded in $L^2(0, T)$, we see from (5.5) that

$$\begin{aligned}(w_n - f^*(u_n))^+ &- (f_*(u_n) - w_n)^+ \\ &= \lambda_n \partial I_{u_n}^{\lambda_n}(w_n) \longrightarrow 0 \quad \text{in } L^2(0, T) \text{ (as } n \longrightarrow +\infty),\end{aligned} \tag{5.23}$$

so that $f_*(u) \le w \le f^*(u)$ on $[0, T]$ and

$$\begin{aligned}J(u_n, w_n) &\longrightarrow w \quad \text{uniformly on } [0, T], \\ u_n' &\longrightarrow u' \quad \text{weakly in } L^2(0, T)\end{aligned} \tag{5.24}$$

as well as

$$\frac{d}{dt}J(u_n, w_n) \longrightarrow w' \quad \text{weakly in } L^2(0, T). \tag{5.25}$$

Therefore, the limit $\{u, w\}$ gives a solution of the original problem. Thus the existence proof is now complete.

6. Some numerical simulations

Another objective of this paper is to verify by some numerical simulations that our system (1.1), (1.2) is useful enough as the mathematical description of various input-output relations $u \to w$ of hysteresis type appearing in many nonlinear phenomena, and what is the influence of the choice of the coefficients a_i, b_i, $i = 1, 2$, and of functions g, h on the behaviour of the orbit $(u(t), w(t))$.

In order to catch the main trends of the input-output relation $u \to w$ we simply take constants as the coefficients $a_i(u, w)$, $b_i(u, w)$, $i = 1, 2$, and mostly linear

functions as $g(u, w)$, $h(u, w)$. The constants are denoted by the same notations a_i, b_i, and they are chosen so as to satisfy that

$$a_1 > 0, \quad b_2 > 0, \quad a_1 b_2 - a_2 b_1 > 0. \tag{6.1}$$

Also, in our simulation the functions f_* and f^* are fixed as follows:

$$f^*(u) := \begin{cases} -1 & \text{if } u < -1.6, \\ 5u^2 + 16u + 11.8 & \text{if } -1.6 \le u < -1.4, \\ 2u + 2 & \text{if } -1.4 \le u < -0.6, \\ -5u^2 - 4u + 0.2 & \text{if } -0.6 \le u < -0.4, \\ 1 & \text{if } -0.4 \le u, \end{cases}$$

$$\tag{6.2}$$

$$f_*(u) := \begin{cases} -1 & \text{if } u < 0.4, \\ 5u^2 - 4u - 0.2 & \text{if } 0.4 \le u < 0.6, \\ 2u - 2 & \text{if } 0.6 \le u < 1.4, \\ -5u^2 + 16u - 11.8 & \text{if } 1.4 \le u < 1.6, \\ 1 & \text{if } 1.6 \le u. \end{cases}$$

As was mentioned above, the following system is considered in this section:

$$a_1 u' + a_2 w' = g(u, w), \quad 0 < t < T, \tag{6.3}$$

$$w' + \partial I_u(w) \ni \tilde{h}(u, w) := \frac{a_1 g(u, w) - b_1 h(u, w)}{a_1 b_2 - a_2 b_1}, \quad 0 < t < T, \tag{6.4}$$

$$u(0) = u_0, \quad w(0) = w_0, \tag{6.5}$$

where $\partial I_u(\cdot)$ is the subdifferential of $I_u(\cdot)$ associated with functions f^* and f_* given by (6.2). Note that (6.4) is equivalent to the original form $b_1 u' + b_2 w' + \partial I_u(w) \ni h(u, w)$.

Now let λ and Δt be positive numbers small enough, and put $t^k = k\Delta t$ ($k = 1, 2, \ldots$). Then the difference scheme for our numerical simulation is of the form

$$\frac{a_1 (u^{k+1} - u^k)}{\Delta t} + \frac{a_2 (w^{k+1} - w^k)}{\Delta t} = g(u^k, w^k),$$

$$\frac{w^{k+1} - w^k}{\Delta t} + \partial I_{u^k}^\lambda (w^{k+1}) = \tilde{h}(u^k, w^k), \quad k = 0, 1, 2, \ldots, \tag{6.6}$$

$$u^0 = u_0, \quad w^0 = w_0,$$

where

$$\partial I_{u^k}^\lambda (w^{k+1}) = \frac{[w^{k+1} - f^*(u^k)]^+}{\lambda} - \frac{[f_*(u^k) - w^{k+1}]^+}{\lambda}. \tag{6.7}$$

The graphs of I_u^λ and ∂I_u^λ are illustrated in Figures 6.1 and 6.2, respectively.
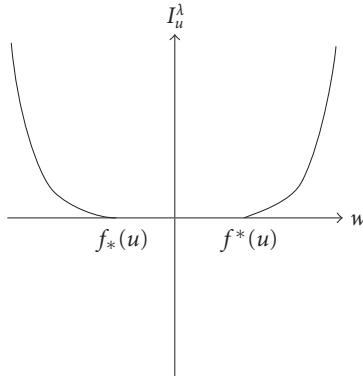
Figure 6.1
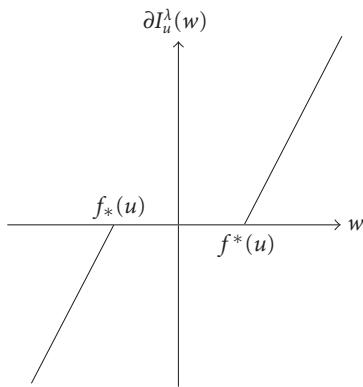
Figure 6.2

In the numerical computations which are performed below, we take

$$\Delta t = \frac{1}{1000}, \qquad \lambda = \frac{1}{10000}, \tag{6.8}$$

and we examine the following items.

(i) By choice of coefficients a_i, b_i and functions g, h, system (6.3), (6.4) creates various behaviours of the solution pair $(u(t), w(t))$. In particular, it is possible to control by them the trend of clockwise or anti-clockwise behaviour of $(u(t), w(t))$.

(ii) Fixing the coefficients a_i, b_i and the initial data, we investigate the influence of the functions g, h on the behaviour of $(u(t), w(t))$.

(iii) We investigate the asymptotic behaviour of $(u(t), w(t))$ as $t \to +\infty$ for various initial data when any other data are fixed.
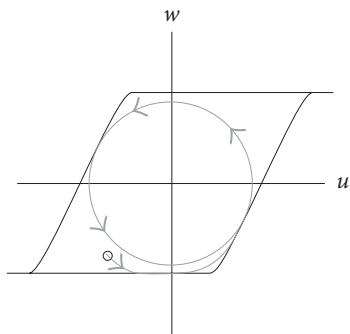
Figure 6.3
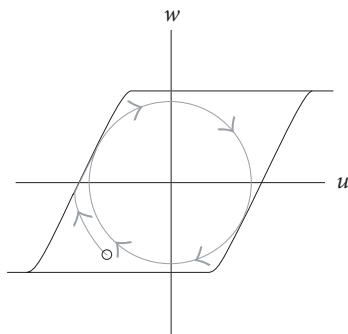
Figure 6.4
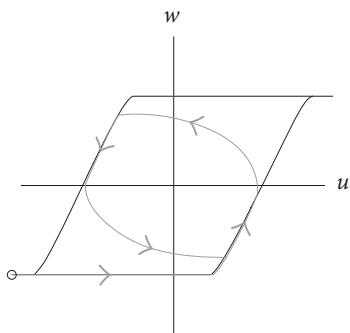
Figure 6.5
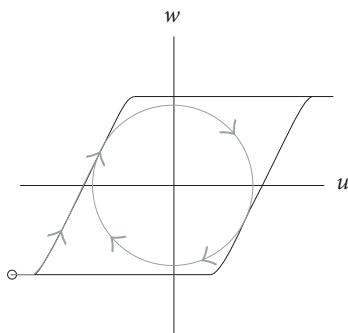
Figure 6.6

Simulation 1.   These experiments show that the clockwise and anti-clockwise behaviour of the orbit $(u(t), w(t))$ is created by choosing suitable coefficients for given initial data. It seems that the orbit is periodic in time after a certain time.

| Data table	a_1	a_2	b_1	b_2	$h(u, w)$	$g(u, w)$	u_0, w_0
Figure 6.3	1	1	−1	1	$u + w$	$u - w$	$u_0 = -0.7, w_0 = -0.8$
Figure 6.4	1	1	−1	1	$-u - w$	$-u + w$	$u_0 = -0.7, w_0 = -0.8$
Figure 6.5	1	1	−1	1	$u + 3w$	$u - 2w$	$u_0 = -1.8, w_0 = -1.0$
Figure 6.6	1	1	−3	1	$-u - w$	$-u + w$	$u_0 = -1.8, w_0 = -1.0$
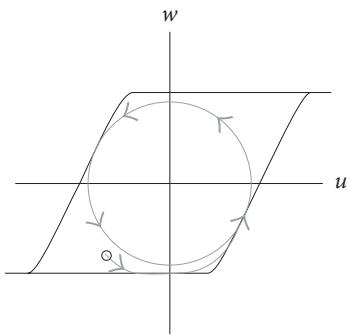
Figure 6.7
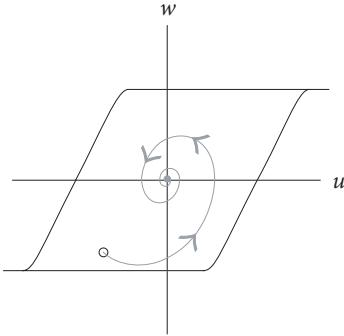
Figure 6.8
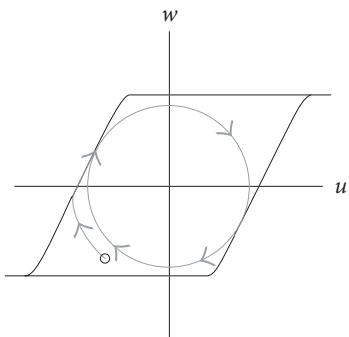
Figure 6.9
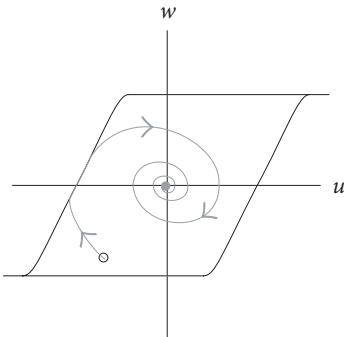
Figure 6.10

Simulation 2.   These experiments show that $b_1 = -1$ gives an anti-clockwise (resp., a clockwise) periodic behaviour of the orbit $(u(t), w(t))$ in time. When the value of b_1 becomes smaller (resp., larger), the orbit asymptotically converges anti-clockwise (resp., clockwise) to a stable point along a spiral curve.

| Data table	a_1	a_2	b_1	b_2	$h(u,w)$	$g(u,w)$	u_0, w_0
Figure 6.7	1	1	-1	1	$u+w$	$u-w$	$u_0 = -0.7, w_0 = -0.8$
Figure 6.8	1	1	-2	1	$u+w$	$u-w$	$u_0 = -0.7, w_0 = -0.8$
Figure 6.9	1	1	-1	1	$-u-w$	$-u+w$	$u_0 = -0.7, w_0 = -0.8$
Figure 6.10	1	1	-0.5	1	$-u-w$	$-u+w$	$u_0 = -0.7, w_0 = -0.8$
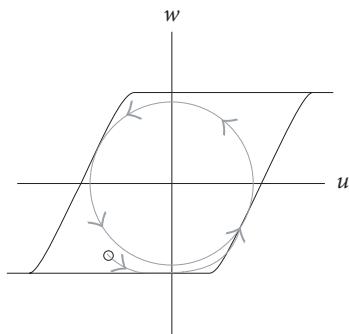
Figure 6.11
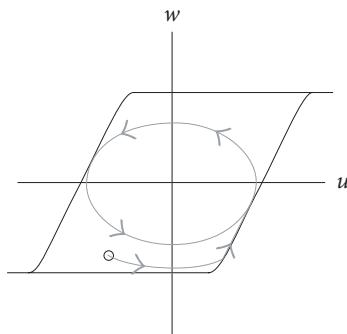
Figure 6.12
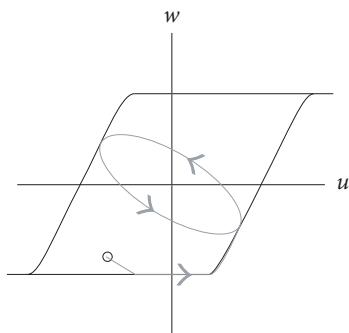
Figure 6.13
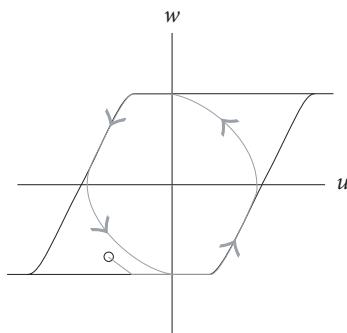
Figure 6.14

Simulation 3. These experiments show that for fixed coefficients a_i, b_i, $i = 1, 2$ and initial data as in the table, we can create various (asymptotically) anti-clockwise periodic orbits only by the choice of linear functions g, h. Also we observe the similar behaviours of orbits in the clockwise case.

| Data table	a_1	a_2	b_1	b_2	$h(u, w)$	$g(u, w)$	u_0, w_0
Figure 6.11	1	1	−1	1	$u + w$	$u - w$	$u_0 = -0.7, w_0 = -0.8$
Figure 6.12	1	1	−1	1	$u + 2w$	$u - 2w$	$u_0 = -0.7, w_0 = -0.8$
Figure 6.13	1	1	−1	1	$2u + 3w$	$-w$	$u_0 = -0.7, w_0 = -0.8$
Figure 6.14	1	1	−1	1	$u + 2w$	$u - w$	$u_0 = -0.7, w_0 = -0.8$
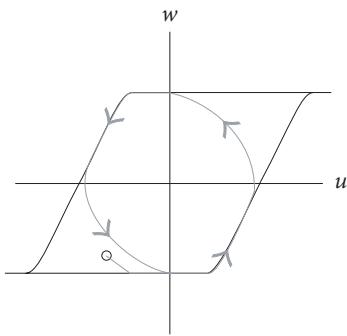
Figure 6.15
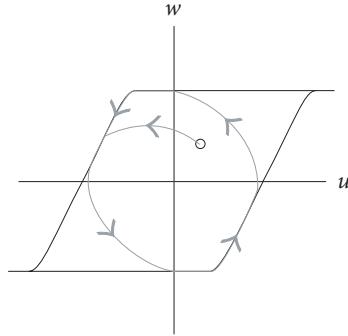

Figure 6.16
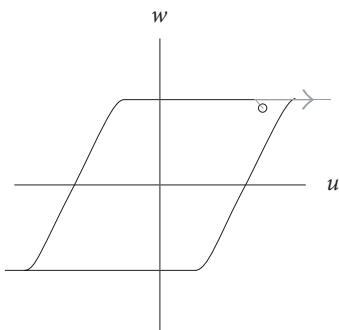
Figure 6.17
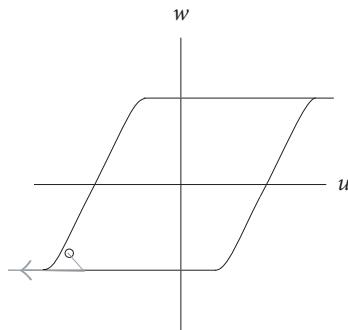
Figure 6.18

Simulation 4.   These experiments suggest that the large-time behaviour of orbits $(u(t), w(t))$ is one of the following two cases: (1) the orbit converges to a periodic one as $t \to +\infty$; (2) the orbit diverges to the point $(-\infty, -1)$ or $(1, +\infty)$ as $t \to +\infty$.

| Data table	a_1	a_2	b_1	b_2	$h(u, w)$	$g(u, w)$	u_0, w_0
Figure 6.15	1	1	-1	1	$u + 2w$	$u - w$	$u_0 = -0.7, w_0 = -0.8$
Figure 6.16	1	1	-1	1	$u + 2w$	$u - w$	$u_0 = 0.4, w_0 = 0.3$
Figure 6.17	1	1	-1	1	$u + 2w$	$u - w$	$u_0 = 1.2, w_0 = 0.9$
Figure 6.18	1	1	-1	1	$u + 2w$	$u - w$	$u_0 = -1.3, w_0 = -0.8$
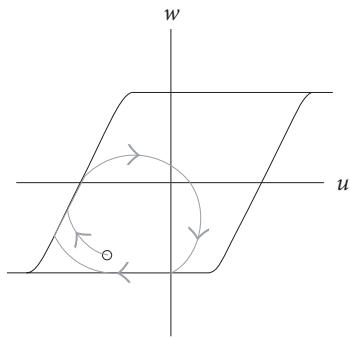
Figure 6.19
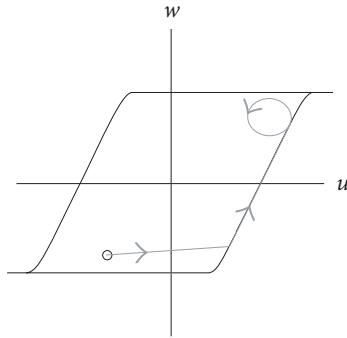
Figure 6.20
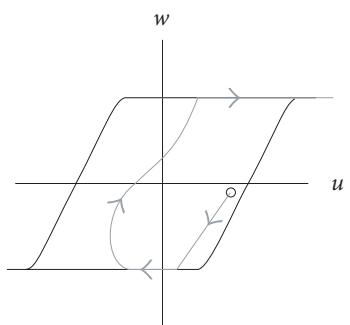
Figure 6.21
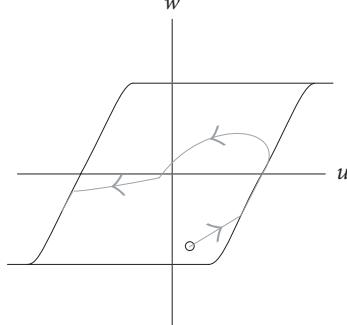
Figure 6.22

Simulation 5 (nonlinear case of h and g).   These experiments show that for fixed coefficients a_i, b_i, $i = 1, 2$ as in the table, we can create various behaviours different from the linear case of h and g. Figures 6.19 and 6.20 are periodic orbits of $(u(t), w(t))$ in time. Figure 6.21 shows that the orbit diverges to the point $(1, +\infty)$ as $t \to \infty$. Figure 6.22 shows that the orbit converges to the point $(-1.2, -0.4)$ as $t \to \infty$.

Data table	a_1	a_2	b_1	b_2	$h(u, w)$	$g(u, w)$	u_0, w_0
Figure 6.19	1	1	-1	1	$-\sin(u+w)+\cos(u+w)$	$-u+1.5w$	$u_0 = -0.7,$ $w_0 = -0.8$
Figure 6.20	1	1	-1	1	$\sin(uw)-\cos(uw)$	$u-1.5w$	$u_0 = -0.7,$ $w_0 = -0.8$
Figure 6.21	1	1	-1	1	uw	$uw-u+w$	$u_0 = 0.8,$ $w_0 = -0.1$
Figure 6.22	1	1	-1	1	$2uw$	$u-3w$	$u_0 = 0.2,$ $w_0 = -0.8$

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