INVERTIBILITY-PRESERVING MAPS OF C^*-ALGEBRAS WITH REAL RANK ZERO

ISTVAN KOVACS

In 1996, Harris and Kadison posed the following problem: show that a linear bijection between C^*-algebras that preserves the identity and the set of invertible elements is a Jordan isomorphism. In this paper, we show that if A and B are semisimple Banach algebras and $\Phi : A \to B$ is a linear map onto B that preserves the spectrum of elements, then Φ is a Jordan isomorphism if either A or B is a C^*-algebra of real rank zero. We also generalize a theorem of Russo.

1. Notation

In what follows, the term Banach algebra will mean a unital complex Banach algebra and a C^*-algebra will mean a unital complex C^*-algebra. The unit is denoted by 1 and the spectrum of an element x by $\sigma(x)$. The set of invertible elements of a Banach algebra A is denoted by A_{inv} and the closed unit ball of A by A_1. The density of a subset of a Banach algebra in another subset is meant to be in the norm topology. A linear map Φ from a Banach algebra A to a normed algebra B is a Jordan homomorphism if $\Phi(a^2) = \Phi(a)^2$ for every $a \in A$. Properties of Jordan homomorphisms are given in [7] or [9]. For C^*-algebras A and B, a C^*-homomorphism in the sense of Kadison is a selfadjoint linear mapping of A into B which is a Jordan homomorphism, that is, $\Phi(a^*) = \Phi(a)^*$ and $\Phi(a^2) = \Phi(a)^2$ for all $a \in A$ [13].

2. Introduction

There are many results on the conjecture of Harris and Kadison. A summary of these results can be found in [7]. One of the most important results is [2, Theorem 1.3] of Aupetit.

THEOREM 2.1. *Let A and B be two von Neumann algebras and let Φ be a spectrum-preserving linear mapping from A onto B. Then Φ is a Jordan isomorphism.*

Among other theorems, Russo proved the following [12, Theorem 2] in 1996.

THEOREM 2.2. *Let Φ be a linear mapping from a von Neumann algebra M into a C^*-algebra B such that $\Phi(M_{\mathrm{inv}} \cap M_1) \subset B_{\mathrm{inv}} \cap B_1$ and $\Phi(1) = 1$. Then Φ is a C^*-homomorphism.*

The definition of a C^*-algebra with real rank zero was given by Brown and Pedersen [3].

Definition 2.3. A C^*-algebra A has real rank zero if the set of invertible selfadjoint elements of A is dense in the set of selfadjoint elements of A.

Also in [3, Theorem 2.6] Brown and Pedersen prove the following.

THEOREM 2.4. *A C^*-algebra A has real rank zero exactly when the set of selfadjoint elements of A with finite spectra is dense in the set of selfadjoint elements of A.*

Theorem 2.4 enables us to generalize Theorems 2.1 and 2.2 and thus obtain our main results.

THEOREM 2.5. *Suppose A is a C^*-algebra with real rank zero and B is a semisimple Banach algebra. If Φ is a spectrum-preserving linear map from A onto B, then Φ is a Jordan isomorphism.*

THEOREM 2.6. *Let Φ be a linear mapping from a C^*-algebra A with real rank zero into a C^*-algebra B such that $\Phi(A_{\mathrm{inv}} \cap A_1) \subset B_{\mathrm{inv}} \cap B_1$ and $\Phi(1) = 1$. Then Φ is a C^*-homomorphism.*

3. Proofs

We use the following lemma to complete the proofs of both Theorems 2.5 and 2.6.

LEMMA 3.1. *Let Φ be a continuous linear mapping from a C^*-algebra A with real rank zero into a normed algebra B such that if p and q are mutually orthogonal projections in A, then $\Phi(p)$ and $\Phi(q)$ are mutually orthogonal idempotents in B. Then Φ is a Jordan homomorphism.*

Proof of Lemma 3.1. Let a be a selfadjoint element of A with finite spectrum and write $\sigma(a) = \{\lambda_1, \dots, \lambda_n\}$ where $\lambda_i \in \mathbb{R}$. Let further

$$p_j(\lambda) = \prod_{k \neq j} \frac{\lambda - \lambda_k}{\lambda_j - \lambda_k}, \qquad p(\lambda) = \sum_{j=1}^{n} \lambda_j p_j(\lambda). \tag{3.1}$$

Let $e_j = p_j(a)$ for all j. We show that $\{e_1, \dots, e_n\}$ is a set of mutually orthogonal idempotents in A and $a = \sum_{j=1}^{n} \lambda_j e_j$. Each e_j is selfadjoint and

$$e_j^2 - e_j = (p_j^2 - p_j)(a). \tag{3.2}$$

By the spectral mapping theorem, if $i \neq j$,

$$\sigma(e_j^2 - e_j) = (p_j^2 - p_j)(\sigma(a)) = \{0\},$$
$$\sigma(e_i e_j) = p_i p_j(\sigma(a)) = \{0\}, \tag{3.3}$$
$$\sigma(a - p(a)) = (\mathrm{id} - p)(\sigma(a)) = \{0\}.$$

Hence, $e_j^2 - e_j = 0$, $e_i e_j = 0$ for $i \neq j$ and $a - p(a) = 0$.

Now put $f_j = \Phi(e_j)$ for all j. By assumption $\{f_1, \ldots, f_n\}$ is a set of mutually orthogonal idempotents in B (containing possibly the zero idempotent). Then

$$a = \sum_{j=1}^{n} \lambda_j e_j, \qquad \Phi(a) = \sum_{j=1}^{n} \lambda_j f_j,$$
$$a^2 = \sum_{j=1}^{n} \lambda_j^2 e_j, \qquad \Phi(a)^2 = \sum_{j=1}^{n} \lambda_j^2 f_j. \tag{3.4}$$

Hence, $\Phi(a^2) = \Phi(a)^2$.

Theorem 2.4 ensures that for any selfadjoint $a \in A$, there is a sequence a_n of selfadjoint elements of A with finite spectra such that $a_n \to a$ in norm. Then $a_n^2 \to a^2$. Hence, $\Phi(a_n) \to \Phi(a)$ and $\Phi(a_n^2) \to \Phi(a^2)$ by the continuity of Φ. Also

$$\Phi(a_n)^2 \longrightarrow \Phi(a)^2, \qquad \Phi(a_n^2) = \Phi(a_n)^2, \tag{3.5}$$

so $\Phi(a^2) = \Phi(a)^2$. It follows that $\Phi(x^2) = \Phi(x)^2$ for all $x \in A$ since $x = a + ib$ for some selfadjoint elements $a, b \in A$ and

$$(a + ib)^2 = a^2 - b^2 + i[(a+b)^2 - a^2 - b^2]. \tag{3.6}$$

This proves Lemma 3.1. □

The mapping Φ of Theorem 2.5 has the following properties given by Aupetit in [2].

PROPOSITION 3.2. *Suppose A and B are semisimple Banach algebras and Φ is a spectrum-preserving linear map from A into B. Then Φ is injective, and if in addition Φ is onto, then $\Phi(1) = 1$ and Φ is continuous.*

Proof. To prove that Φ is injective, suppose $a \in A$ and $\Phi(a) = 0$. Then

$$\sigma(a + x) = \sigma(\Phi(a + x)) = \sigma(\Phi(x)) = \sigma(x) \tag{3.7}$$

for every $x \in A$. Hence, $a = 0$ by [8, Corollary 2.4].

To show that Φ preserves the identity write $\Phi(1) = 1 + q$ where $q \in B$. As Φ is spectrum-preserving, if $x \in A$, then

$$1 + \sigma(\Phi(x)) = 1 + \sigma(x) = \sigma(1 + x),$$
$$\sigma(\Phi(1 + x)) = \sigma(1 + q + \Phi(x)) = 1 + \sigma(q + \Phi(x)), \tag{3.8}$$

so $\sigma(\Phi(x)) = \sigma(q + \Phi(x))$. Then $q = 0$ again by [8, Corollary 2.4].

The continuity of Φ is proven in [1, Theorem 1].

The mappings of Theorems 2.5 and 2.6 both satisfy the assumptions of Lemma 3.1.

To prove Theorem 2.5, we need the next theorem of Aupetit [2, Theorem 1.2]. □

THEOREM 3.3. *If A and B are semisimple Banach algebras and if Φ is a spectrum-preserving operator from A onto B, then Φ transforms a set of mutually orthogonal idempotents of A to a set of mutually orthogonal idempotents of B.*

Lemma 3.1 completes the proof of Theorem 2.5.

Remarks 3.4. (a) Note that Φ is onto, so Proposition 3.2 implies that Φ is a homeomorphism and Φ^{-1} is spectrum-preserving. Hence, A and B are interchangeable in Theorem 2.5.

(b) The spectral resolution theorem [10, Theorem 5.5.2] ensures that in a von Neumann algebra a selfadjoint element is the norm limit of real linear combinations of orthogonal projections. Hence, von Neumann algebras have real rank zero.

Proof of Theorem 2.6. Let U denote the set of unitaries of A. In [6, Corollary 1], Harris gives an elegant proof of the fact that the open unit ball of A is the convex hull of U. A more elementary proof of Gardner can be found in [11, Proposition 3.2.23]. It follows easily that $\|a\|_u = \|a\|$ for $a \in A$ where

$$\|a\|_u := \inf \left\{ \sum_{i=1}^{n} |\lambda_i| : a = \sum_{i=1}^{n} \lambda_i u_i, \ \lambda_i \in \mathbb{C}, \ u \in U, \ n \in \mathbb{N} \right\}. \tag{3.9}$$

(See [13, Lemma 2].) For Φ satisfying the conditions of Theorem 2.6, we have that if $a \in A$ and

$$a = \sum_{j=1}^{n} \lambda_j u_j \tag{3.10}$$

then

$$\|\Phi(a)\| \leq \sum_{j=1}^{n} |\lambda_j|. \tag{3.11}$$

Hence, $\|\Phi(a)\| \leq \|a\|_u = \|a\|$ for every $a \in A$ and $\|\Phi\| = 1$.

As B is a C^*-algebra, this is enough to ensure $\Phi \geq 0$ by [13, Corollary 1], that is, $\Phi(a) \geq 0$ whenever $a \in A$ and $a \geq 0$.

Since Φ is an invertibility-preserving selfadjoint map from A into B, by [12, Lemma 3] Φ maps mutually orthogonal projections of A into mutually orthogonal idempotents of B. Hence, we can apply Lemma 3.1 and $\Phi(a^2) = \Phi(a)^2$ follows for $a \in A$. This proves Theorem 2.6. □

Remarks 3.5. (a) It follows from [4, Theorem 2] that the assumption that A has real rank zero can not be omitted in Theorem 2.6 even when A is commutative.

(b) It is known that if Φ is a linear bijection between C^*-algebras with $\Phi(A_{\text{inv}}) \subset B_{\text{inv}}$ and $\|\Phi\| \leq 1$, then Φ is a Jordan isomorphism (see [4, Theorem 6] and [7, Corollary 8]). Theorem 2.6 does not require bijectivity of the mapping.

(c) If in Theorem 2.6 we require only that $\Phi(1)$ is unitary, then Φ becomes a Jordan homomorphism followed by multiplication by $\Phi(1)$.

(d) The C^*-algebra generated by the compact operators $\mathcal{K}$ and the identity on an infinite-dimensional Hilbert space $\mathcal{H}$ has real rank zero, though it is not a von Neumann algebra. The Calkin algebra, which is the factor C^*-algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$, has real rank zero, though it is not a von Neumann algebra. All the Bunce-Deddens algebras, the Cuntz algebras, AF-algebras, and irrational rotation algebras have real rank zero (see [5]). The class of C^*-algebras with real rank zero is considerably wider than the class of von Neumann algebras. Thus Theorems 2.5 and 2.6 are nontrivial extensions of Theorems 2.1 and 2.2.

Acknowledgment

References

[1] B. Aupetit, *The uniqueness of the complete norm topology in Banach algebras and Banach Jordan algebras*, J. Funct. Anal. **47** (1982), no. 1, 1–6.

[2] ———, *Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras*, J. London Math. Soc. (2) **62** (2000), no. 3, 917–924.

[3] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), no. 1, 131–149.

[4] M. D. Choi, D. Hadwin, E. Nordgren, H. Radjavi, and P. Rosenthal, *On positive linear maps preserving invertibility*, J. Funct. Anal. **59** (1984), no. 3, 462–469.

[5] K. R. Davidson, *C^*-Algebras by Example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Rhode Island, 1996.

[6] L. A. Harris, *Banach algebras with involution and Möbius transformations*, J. Funct. Anal. **11** (1972), 1–16.

[7] ———, *Invertibility preserving linear maps of Banach algebras*, Complex Analysis and Dynamical Systems, Contemp. Math., vol. 364, American Mathematical Society, Rhode Island, 2004, pp. 59–66.

[8] L. A. Harris and R. V. Kadison, *Affine mappings of invertible operators*, Proc. Amer. Math. Soc. **124** (1996), no. 8, 2415–2422.

[9] N. Jacobson and C. E. Rickart, *Jordan homomorphisms of rings*, Trans. Amer. Math. Soc. **69** (1950), 479–502.

[10] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Vol. I*, Graduate Studies in Mathematics, vol. 15, American Mathematical Society, Rhode Island, 1997.

[11] G. K. Pedersen, *Analysis Now*, Graduate Texts in Mathematics, vol. 118, Springer-Verlag, New York, 1989.

[12] B. Russo, *Linear mappings of operator algebras*, Proc. Amer. Math. Soc. **17** (1966), 1019–1022.

[13] B. Russo and H. A. Dye, *A note on unitary operators in C^*-algebras*, Duke Math. J. **33** (1966), 413–416.

Istvan Kovacs: Department of Mathematics, Case Western Reserve University, Cleveland, OH 44106, USA

E-mail address: istvan.kovacs@case.edu

Special Issue on
Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning

- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk