A THREE-POINT BOUNDARY VALUE PROBLEM WITH AN INTEGRAL CONDITION FOR A THIRD-ORDER PARTIAL DIFFERENTIAL EQUATION

C. LATROUS AND A. MEMOU

Received 9 February 2004

We prove the existence and uniqueness of a strong solution for a linear third-order equation with integral boundary conditions. The proof uses energy inequalities and the density of the range of the operator generated.

1. Introduction

In the rectangle $\Omega = (0,1) \times (0,T)$, we consider the equation

$$f(x,t) = \frac{\partial^3 u}{\partial t^3} + \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) \tag{1.1}$$

with the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in (0,1), \tag{1.2}$$

the final condition

$$\frac{\partial^2 u}{\partial t^2}(x,T) = 0, \quad x \in (0,1), \tag{1.3}$$

the Dirichlet condition

$$u(0,t) = 0 \quad \forall t \in (0,T), \tag{1.4}$$

and the integral condition

$$\int_l^1 u(x,t)dx = 0, \quad 0 \le l < 1, \ t \in (0,T). \tag{1.5}$$

Abstract and Applied Analysis 2005:1 (2005) 33–43
DOI: 10.1155/AAA.2005.33

In addition, we assume that the function $a(x,t)$ and its derivatives satisfy the conditions

$$
\begin{gathered}
0 < a_0 < a(x,t) < a_1 \quad \forall x,t \in \Omega, \\
\left|\frac{\partial a}{\partial x}\right| \leq b \quad \forall x,t \in \Omega, \\
c_k' < \frac{\partial^k u}{\partial t^k}(x,t) < c_k \quad \forall x,t \in \Omega,\ k = \overline{1,3},\ \text{with } c_1' > 0.
\end{gathered}
\tag{1.6}
$$

Over the last few years, many physical phenomena were formulated into nonlocal mathematical models with integral boundary conditions [1, 9, 10, 11]. The reader should refer to [13, 14] and the references therein. The importance of these kinds of problems has also been pointed out by Samarskii [22]. This type of boundary value problems has been investigated in [2, 3, 4, 6, 7, 8, 12, 18, 19, 20, 23, 25] for parabolic equations, in [21, 24] for hyperbolic equations, and in [15, 16, 17] for mixed-type equations. The basic tool in [5, 15, 16, 17, 20, 25] is the energy inequality method which, of course, requires appropriate multipliers and functional spaces. In this paper, we extend this method to the study of a linear third-order partial differential equation.

2. Preliminairies

In this paper, we prove the existence and uniqueness of a strong solution of the problem (1.1)–(1.5). For this, we consider the solution of problem (1.1)–(1.5) as a solution of the operator equation

$$
Lu = \mathcal{F}, \tag{2.1}
$$

where the operator L has domain of definition $D(L)$ consisting of functions $u \in L^2(\Omega)$ such that $(\partial^{k+1}u/\partial t^k\partial x)(x,t) \in L^2(\Omega)$, $k = \overline{1,3}$ and satisfing the conditions (1.4)-(1.5).

The operator L is considered from E to F, where E is the Banach space consisting of function $u \in L^2(\Omega)$, with the finite norm

$$
\begin{aligned}
\|u\|_E^2 = &\int_\Omega \Theta(x)\left[\left|\frac{\partial^3 u}{\partial t^3}\right|^2 + \left|\frac{\partial^2 u}{\partial x^2}\right|^2\right]dx\,dt \\
&+ \int_\Omega \Theta(x)\left[\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial^2 u}{\partial t \partial x}\right|^2\right]dx\,dt \\
&+ \int_\Omega \Phi(x)\left[\left|\frac{\partial u}{\partial t}\right|^2 + |u|^2\right]dx\,dt.
\end{aligned}
\tag{2.2}
$$

F is the Hilbert space of functions $\mathcal{F} = (f,0,0,0)$, $f \in L^2(\Omega)$, with the finite norm

$$
\|\mathcal{F}\|_F^2 = \int_\Omega \Theta(x)\left|f(x,t)\right|^2 dx\,dt, \tag{2.3}
$$

where

$$\Theta(x)=\begin{cases}(1-l)^2, & 0<x\le l,\\ (1-x)^2, & l\le x<1,\end{cases}$$
$$\Phi(x)=\begin{cases}0, & 0<x<l,\\ 1, & l\le x<1.\end{cases} \tag{2.4}$$

3. An energy inequality and its application

THEOREM 3.1. *For any function $u\in D(L)$, the a priori estimate*

$$\|u\|_E\le k\|Lu\|_F \quad \text{for } u\in D(L), \tag{3.1}$$

where $k^2=40\exp(cT)/k_1$ with $k_1=\inf\{1/4,(c_3'-3cc_1'+3c^2c_1'-c^3a_1-b^2)/2,\ a_0^2/2,\ (3/2)(ca_0-c_1)\}$. The constant c satisfies

$$\sup_{(x,t)\in\Omega}\left(\frac{1}{a}\frac{\partial a}{\partial t}\right)<c<\inf_{(x,t)\in\Omega}\left(\frac{1}{a}\frac{\partial a}{\partial t}+1\right),$$
$$c_3'-3cc_1'+3c^2c_1'-c^3a_1-b^2>0, \tag{3.2}$$
$$c_2'-2cc_1'+c^2a_1^2+ca_0-c_1>0.$$

Proof. Let

$$Mu=\begin{cases}(1-l)^2\dfrac{\partial^3 u}{\partial t^3}, & 0<x<l,\\[2mm] (1-x)^2\dfrac{\partial^3 u}{\partial t^3}+2(1-x)J_x\dfrac{\partial^3 u}{\partial t^3}, & l<x<1,\end{cases} \tag{3.3}$$

where $J_xu=\int_l^x u(x,t)dx$.

We consider the quadratic form obtained by multiplying (1.1) by $\exp(-ct)\overline{Mu}$, with the constant c satisfying (3.2), integrating over $\Omega=(0,1)\times(0,T)$, and taking the real part:

$$\Phi(u,u)=\operatorname{Re}\int_\Omega \exp(-ct)f(x,t)\overline{Mu}\,dx\,dt. \tag{3.4}$$

By substituting the expression of Mu in (3.4), integrating with respect to x, and using the Dirichlet and integral conditions, we obtain

$$\begin{aligned}
&\operatorname{Re}\int_{\Omega}\exp(-ct)f(x,t)\overline{Mu}\,dx\,dt \\
&= \int_0^T\int_0^1 \Theta(x)\exp(-ct)\left|\frac{\partial^3 u}{\partial t^3}\right|^2 dx\,dt \\
&\quad -\frac{3}{2}\int_0^T\int_0^1 \Theta(x)\exp(-ct)\left[\frac{\partial a}{\partial t}-ca\right]\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dx\,dt \\
&\quad +\int_0^T\int_0^1 \frac{\Theta(x)}{2}\exp(-ct)\left[\frac{\partial^3 a}{\partial t^3}-3c\frac{\partial^2 a}{\partial t^2}+3c\frac{\partial a}{\partial t}-c^3 a\right]\left|\frac{\partial u}{\partial x}\right|^2 dx\,dt \\
&\quad +\int_0^T\int_l^1 \exp(-ct)\left|J_x\frac{\partial^3 u}{\partial t^3}\right|^2 dx\,dt \\
&\quad -2\operatorname{Re}\int_0^T\int_l^1 \exp(-ct)a(x,t)u\overline{\frac{\partial^3 u}{\partial t^3}}\,dx\,dt \\
&\quad +\int_0^1 \Theta(x)\exp(-ct)a(x,t)\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dx|_{t=T} \\
&\quad -\int_0^1 \Theta(x)\exp(-ct)\left(\frac{\partial a}{\partial t}-ca\right)\frac{\partial u}{\partial x}\overline{\frac{\partial^2 u}{\partial x\partial t}}\,dx|_{t=T} \\
&\quad -\int_0^1 \frac{\Theta(x)}{2}\exp(-ct)\left[\frac{\partial^2 a}{\partial t^2}-2c\frac{\partial a}{\partial t}+c^2 a\right]\left|\frac{\partial u}{\partial x}\right|^2 dx|_{t=T} \\
&\quad -2\operatorname{Re}\int_0^T\int_l^1 \exp(-ct)\frac{\partial a}{\partial x}uJ_x\overline{\frac{\partial^3 u}{\partial t^3}}\,dx\,dt.
\end{aligned} \tag{3.5}$$

Integrating by parts $-2\operatorname{Re}\int_0^T\int_l^1\exp(-ct)a(x,t)u(\overline{\partial^3 u/\partial t^3})\,dx\,dt$ with respect to t, and using the initial conditions, the final conditions, and the elementary inequalities, we obtain

$$\begin{aligned}
&\int_0^T\int_0^1 \frac{\Theta(x)}{2}\exp(-ct)\left|\frac{\partial^3 u}{\partial t^3}\right|^2 dx\,dt \\
&\quad -\frac{3}{2}\int_0^T\int_0^1 \Theta(x)\exp(-ct)\left[\frac{\partial a}{\partial t}-ca\right]\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dx\,dt \\
&\quad +\int_0^T\int_0^1 \frac{\Theta(x)}{2}\exp(-ct)\left[\frac{\partial^3 a}{\partial t^3}-3c\frac{\partial^2 a}{\partial t^2}+3c\frac{\partial a}{\partial t}-c^3 a\right]\left|\frac{\partial u}{\partial x}\right|^2 dx\,dt \\
&\quad +\int_0^T\int_l^1 \exp(-ct)\left|J_x\frac{\partial^3 u}{\partial t^3}\right|^2 dx\,dt \\
&\quad +\int_0^T\int_l^1 \exp(-ct)\left[\frac{\partial^3 a}{\partial t^3}-3c\frac{\partial^2 a}{\partial t^2}+3c\frac{\partial a}{\partial t}-c^3 a\right]|u|^2 dx\,dt \\
&\quad -\frac{3}{2}\int_0^T\int_l^1 \exp(-ct)\left[\frac{\partial a}{\partial t}-ca\right]\left|\frac{\partial u}{\partial t}\right|^2 dx\,dt \\
&\quad +\int_0^1 \frac{\Theta(x)}{2}\exp(-ct)\left[a-\left|\frac{\partial a}{\partial t}-ca\right|\right]\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dx|_{t=T}
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^1 \frac{\Theta(x)}{2}\exp(-ct)\left[\frac{\partial^2 a}{\partial t^2}-2c\frac{\partial a}{\partial t}+c^2a+\left|\frac{\partial a}{\partial t}-ca\right|\right]\left|\frac{\partial u}{\partial x}\right|^2 dx|_{t=T}\\
&+\int_0^1 \Phi(x)\exp(-ct)\left[a-\left|\frac{\partial a}{\partial t}-ca\right|\right]\left|\frac{\partial u}{\partial t}\right|^2 dx|_{t=T}\\
&-\int_0^1 \Phi(x)\exp(-ct)\left[\frac{\partial^2 a}{\partial t^2}-2c\frac{\partial a}{\partial t}+c^2a+\left|\frac{\partial a}{\partial t}-ca\right|\right]|u|^2 dx|_{t=T}\\
\le{}& 17\int_0^T\int_l^1 \Theta(x)\exp(-ct)|f|^2 dx\,dt.
\end{aligned}
\tag{3.6}
$$

From (1.1), we get

$$
\begin{aligned}
&\int_\Omega \Theta(x)a^2\left|\frac{\partial^2 u}{\partial x^2}\right|^2 dx\,dt\\
&\quad\le 2\int_\Omega \Theta(x)\left|\frac{\partial^3 u}{\partial t^3}\right|^2 dx\,dt+2\int_\Omega \Theta(x)\left(\frac{\partial a}{\partial x}\right)^2\left|\frac{\partial u}{\partial x}\right|^2 dx\,dt\\
&\qquad+4\int_\Omega \Theta(x)|f|^2 dx\,dt.
\end{aligned}
\tag{3.7}
$$

Combining this last inequality with (3.6) and using the conditions (3.2) yield

$$
\begin{aligned}
&\int_\Omega \Theta(x)\left[\left|\frac{\partial^3 u}{\partial t^3}\right|^2+\left|\frac{\partial^2 u}{\partial x^2}\right|^2\right]dx\,dt\\
&\quad+\int_\Omega \Theta(x)\left[\left|\frac{\partial u}{\partial x}\right|^2+\left|\frac{\partial^2 u}{\partial t\partial x}\right|^2\right]dx\,dt+\int_\Omega \Phi(x)\left[\left|\frac{\partial u}{\partial t}\right|^2+|u|^2\right]dx\,dt\\
&\le k\int_\Omega \Theta(x)\,|f(x,t)|^2 dx\,dt,
\end{aligned}
\tag{3.8}
$$

which is the desired inequality. □

It can be proved in a standard way that the operator $L: E\to F$ is closable. Let $\overline{L}$ be the closure of this operator, with the domain of definition $D(\overline{L})$.

Definition 3.2. A solution of the operator equation $\overline{L}u=\mathcal{F}$ is called a strong solution of problem (1.1)–(1.5).

The a priori estimate (3.1) can be extended to strong solutions, that is, we have the estimate

$$
\|u\|_E\le c\|\overline{L}u\|_F \quad \forall u\in D(\overline{L}).
\tag{3.9}
$$

This last inequality implies the following corollaries.

Corollary 3.3. *A strong solution of (1.1)–(1.5) is unique and depends continuously on $\mathcal{F}$.*

Corollary 3.4. *The range $R(\overline{L})$ of $\overline{L}$ is closed in F and $\overline{R(L)}=R(\overline{L})$.*

Corollary 3.4 shows that to prove that problem (1.1)–(1.5) has a strong solution for arbitrary $\mathcal{F}$, it suffices to prove that set $R(L)$ is dense in F.

4. Solvability of problem (1.1)–(1.5)

To prove the solvability of problem (1.1)–(1.5) it is sufficient to show that $R(L)$ is dense in F. The proof is based on the following lemma.

Lemma 4.1. *Suppose that the function $a(x,t)$ and its derivatives are bounded. Let $u \in D_0(L) = \{u \in D(L),\ u(x,0) = 0,\ (\partial u/\partial t)(x,0) = 0,\ (\partial^2 u/\partial t^2)(x,T) = 0\}$. If for $u \in D_0(L)$ and some functions $w(x,t) \in L^2(\Omega)$,*

$$\int_\Omega h(x) f \overline{w} dx\, dt = 0, \tag{4.1}$$

where

$$h(x) = \begin{cases} 1-l, & 0 < x < l, \\ 1-x, & l < x < 1, \end{cases} \tag{4.2}$$

holds, for arbitrary $u \in D_0(L)$, and then $w = 0$.

Proof. The equality (4.1) can be written as follows:

$$\int_\Omega h(x) \frac{\partial^3 u}{\partial t^3} \overline{w} dx\, dt = \int_\Omega A(t) u \overline{v} dx\, dt, \tag{4.3}$$

for a given $w(x,t)$, where

$$v = \begin{cases} (1-l)w, & 0 < x < l, \\ w - \displaystyle\int_l^x \frac{w}{1-\zeta} d\zeta, & l < x < 1, \end{cases}$$
$$A(t)u = \frac{\partial}{\partial x}\left(h(x) a(x,t) \frac{\partial u}{\partial x}\right), \tag{4.4}$$
$$Nv = \begin{cases} (1-l)v, & 0 < x < l, \\ (1-x)v + J_x v, & l < x < 1. \end{cases}$$

For $v = w - \int_l^x (w/(1-\zeta)) d\zeta$, $l < x < 1$ we deduce $\int_l^x v(\zeta,t) d\zeta = (1-x)\int_l^x (w/(1-\zeta)) d\zeta$, then $\int_l^1 v(\zeta,t) d\zeta = 0$.

Following [25], we introduce the smoothing operators with respect to t, $(J_\epsilon^{-1}) = (I - \epsilon(\partial^3/\partial t^3))^{-1}$, and $(J_\epsilon^{-1})^* = (I + \epsilon(\partial^3/\partial t^3))^{-1}$ which provide the solution of the respective problems:

$$\begin{aligned} & u_\epsilon - \epsilon \frac{\partial^3 u_\epsilon}{\partial t^3} = u, \quad && u_\epsilon(x,0) = 0, \quad && \frac{\partial u_\epsilon}{\partial t}(x,0) = 0, \quad && \frac{\partial^2 u_\epsilon}{\partial t^2}(x,T) = 0, \\ & v_\epsilon^* + \epsilon \frac{\partial^3 v_\epsilon^*}{\partial t^3} = v, \quad && v_\epsilon^*(x,0) = 0, \quad && \frac{\partial v_\epsilon^*}{\partial t}(x,T) = 0, \quad && \frac{\partial^2 v_\epsilon^*}{\partial t^2}(x,T) = 0. \end{aligned} \tag{4.5}$$

And also, we have the following properties: for any $u \in L^2(0,T)$, the function $J_\epsilon^{-1}u \in W_2^3(0,T)$, $(J_\epsilon^{-1})^* u \in W_2^3(0,T)$. If $u \in D(L)$, $J_\epsilon^{-1}u \in D(L)$.

$$\lim_{\epsilon\to 0} \|J_\epsilon^{-1}u - u\|_{L^2(0,T)} = 0, \qquad \lim_{\epsilon\to 0} \|(J_\epsilon^{-1})^* u - u\|_{L^2(0,T)} = 0. \tag{4.6}$$

Substituting the function u in (4.3) by the smoothing function u_ϵ and using the relation $A(t)u_\epsilon = J_\epsilon^{-1}A(t)u + \epsilon J_\epsilon^{-1}B_\epsilon(t)u$, where $B_\epsilon(t) = (3\partial/\partial t)((\partial A(t)/\partial t)(\partial u_\epsilon/\partial t)) + (\partial^3 A(t)/\partial t^3)u_\epsilon$, we obtain

$$\int_\Omega uN\overline{\frac{\partial^3 v_\epsilon^*}{\partial t^3}}\,dx\,dt = \int_\Omega A(t)u\overline{v_\epsilon^*}\,dx\,dt - \epsilon\int_\Omega B_\epsilon(t)u\overline{v_\epsilon^*}\,dx\,dt. \tag{4.7}$$

The operator $A(t)$ has a continuous inverse in $L^2(0,1)$ defined by

$$A^{-1}(t)g = \begin{cases} -\dfrac{1}{1-l}\displaystyle\int_0^x \frac{d\zeta}{a(\zeta,t)}\int_0^\zeta g(\eta)d\eta + \frac{C_1(t)}{1-l}\int_0^x \frac{d\zeta}{a(\zeta,t)}, & 0<x<l,\\ \displaystyle\int_l^x \frac{-d\zeta}{(1-\zeta)a(\zeta,t)}\int_l^\zeta g(\eta)d\eta + C_2(t)\int_l^x \frac{d\zeta}{(1-\zeta)a(\zeta,t)} + u(l), & l<x<1, \end{cases} \tag{4.8}$$

where

$$\begin{aligned} C_1(t) &= \frac{(1-l)u(l) + \int_0^l (d\zeta/a(\zeta,t))\int_0^\zeta g(\eta)d\eta}{\int_0^l (d\zeta/a(\zeta,t))},\\ C_2(t) &= \frac{-(1-l)u(l) + \int_l^1 (d\zeta/a(\zeta,t))\int_l^\zeta g(\eta)d\eta}{\int_l^1 (d\zeta/a(\zeta,t))}. \end{aligned} \tag{4.9}$$

Then we have $\int_l^1 A^{-1}(t)u = 0$, hence, the function $J_\epsilon^{-1}u = u_\varepsilon$ can be represented in the form

$$u_\varepsilon = J_\epsilon^{-1}A^{-1}(t)A(t)u. \tag{4.10}$$

The adjoint of $B_\epsilon(t)$ has the form

$$\begin{aligned} B_\epsilon^*(t)v = {} & \frac{1}{a}(J_\epsilon^{-1})^*\frac{\partial^3 a}{\partial t^3}v + \frac{3}{a}(J_\epsilon^{-1})^*\frac{\partial}{\partial t}\left(\frac{\partial a}{\partial t}\frac{\partial v}{\partial t}\right) - G_\epsilon(v)(x)\\ & + \frac{\int_0^x (d\zeta/a(\zeta,t))}{\int_0^1 (d\zeta/a(\zeta,t))}G_\epsilon(v)(1), \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} G_\epsilon(v)(x) = {} & \int_0^x \left[\frac{3}{a}(J_\epsilon^{-1})^*\frac{\partial}{\partial t}\left(\frac{\partial^2 a}{\partial t\partial\zeta}\frac{\partial v}{\partial t}\right) - \frac{3}{a^2}\frac{\partial a}{\partial\zeta}(J_\epsilon^{-1})^*\frac{\partial}{\partial t}\left(\frac{\partial a}{\partial t}\frac{\partial v}{\partial t}\right)\right.\\ & \left. + \frac{1}{a}(J_\epsilon^{-1})^*\frac{\partial}{\partial t}\left(\frac{\partial^4 a}{\partial t^3\partial\zeta}v\right) - \frac{1}{a^2}\frac{\partial a}{\partial\zeta}(J_\epsilon^{-1})^*\left(\frac{\partial^3 a}{\partial t^3}v\right)\right]d\zeta. \end{aligned} \tag{4.12}$$

Consequently, equality (4.7) becomes

$$\int_{\Omega} uN\overline{\frac{\partial^3 v_\epsilon^*}{\partial t^3}}\,dx\,dt = \int_{\Omega} A(t)u\overline{h_\epsilon}\,dx\,dt, \tag{4.13}$$

where $h_\epsilon = v_\epsilon^* - \epsilon B_\epsilon^*(t)v_\epsilon^*$.

The left-hand side of (4.13) is a continuous linear functional of u, hence the function h_ϵ has the derivatives $\partial h_\epsilon/\partial x$, $(1-x)(\partial h_\epsilon/\partial x) \in L^2(\Omega)$, and the condition $h_\epsilon(0,t) = 0$ is satisfied.

From the equality

$$(1-x)\frac{\partial h_\epsilon}{\partial x} = \left[I - \epsilon\frac{1}{a}(J_\epsilon^{-1})^*\left(\frac{\partial^3 a}{\partial t^3}\right)\right](1-x)\frac{\partial v_\epsilon^*}{\partial x} - 3\epsilon\frac{1}{a}(J_\epsilon^{-1})^*\frac{\partial}{\partial t}\left(\frac{\partial a}{\partial t}\frac{\partial}{\partial t}(1-x)\frac{\partial v_\epsilon^*}{\partial x}\right), \tag{4.14}$$

and since the operator $(J_\epsilon^{-1})^*$ is bounded in $L^2(\Omega)$, for sufficiently small ϵ, we have $\|\epsilon(1/a)(J_\epsilon^{-1})^*(\partial^3 a/\partial t^3)\| < 1$. Hence, the operator $I - \epsilon(1/a)(J_\epsilon^{-1})^*(\partial^3 a/\partial t^3)$ has a bounded inverse in $L^2(\Omega)$. We conclude that $(1-x)(\partial v_\epsilon^*/\partial x) \in L^2(\Omega)$. Similarly, we conclude that $(\partial/\partial x)((1-x)(\partial v_\epsilon^*/\partial x))$ exists and belongs to $L^2(\Omega)$, and the condition $v_\epsilon^*(0,t) = 0$ is satisfied.

Putting $u = \int_0^t \int_0^\zeta \int_\eta^T \exp(c\tau)v_\epsilon^*\,d\tau\,d\eta\,d\zeta$ in (4.3), where the constant c satisfies (3.2) and using the proprieties of smoothing operator, we obtain

$$\int_{\Omega} \exp(ct)v_\varepsilon^*\overline{Nv}\,dx\,dt = -\int_{\Omega} A(t)u\overline{v_\varepsilon^*}\,dx\,dt - \varepsilon\int_{\Omega} A(t)u\overline{\frac{\partial^3 v_\epsilon^*}{\partial t^3}}\,dx\,dt, \tag{4.15}$$

and from

$$\begin{aligned}
&-\varepsilon\int_{\Omega} A(t)u\overline{\frac{\partial^3 v_\epsilon^*}{\partial t^3}}\,dx\,dt \\
&\quad= 3\int_{\Omega} h(x)\exp(-ct)\frac{\partial^2 a}{\partial t^2}\left|\frac{\partial^3 u}{\partial t^2\partial x}\right|^2 dx\,dt \\
&\qquad- 3\int_{\Omega} h(x)\exp(-ct)\left[\frac{\partial^3 a}{\partial t^3} - c\frac{\partial^2 a}{\partial t^2}\right]\frac{\partial^3 u}{\partial t^2\partial x}\overline{\frac{\partial^2 u}{\partial t\partial x}}\,dx\,dt \\
&\qquad+ 3\int_0^1 \frac{h(x)}{2}\exp(-ct)\frac{\partial a}{\partial t}\left|\frac{\partial^3 u}{\partial t^2\partial x}\right|^2 dx|_{t=T} \\
&\qquad+ 3\int_0^1 \frac{h(x)}{2}\exp(-ct)\left[\frac{\partial^2 a}{\partial t^2} - c\frac{\partial a}{\partial t}\right]\left|\frac{\partial^2 u}{\partial t\partial x}\right|^2 dx|_{t=T} \\
&\qquad- \int_{\Omega} h(x)\exp(-ct)a\left|\frac{\partial^3 v_\epsilon^*}{\partial t^3}\right|^2 dx\,dt \\
&\qquad- \int_{\Omega} h(x)\exp(-ct)\frac{\partial^3 a}{\partial t^3}\frac{\partial u}{\partial x}\overline{\frac{\partial^3 u}{\partial t^2\partial x}}\,dx\,dt,
\end{aligned} \tag{4.16}$$

we have

$$
\begin{aligned}
&-\varepsilon\operatorname{Re}\int_\Omega A(t)u\overline{\frac{\partial^3 v_\epsilon^*}{\partial t^3}}\,dx\,dt\\
&\quad\le \varepsilon\Bigg\{3\int_\Omega h(x)\exp(-ct)\left[\frac{\partial^2 a}{\partial t^2}+\frac{1}{2}\left|\frac{\partial^3 a}{\partial t^3}-c\frac{\partial^2 a}{\partial t^2}\right|\right]\left|\frac{\partial^3 u}{\partial t^2\partial x}\right|^2 dx\,dt\\
&\qquad+\frac{3}{2}\int_\Omega h(x)\exp(-ct)\left[\frac{\partial^2 a}{\partial t^2}-c\frac{\partial a}{\partial t}+\left|\frac{\partial^3 a}{\partial t^3}-c\frac{\partial^2 a}{\partial t^2}\right|\right]\left|\frac{\partial^2 u}{\partial t\partial x}\right|^2 dx\,dt\\
&\qquad-\int_\Omega h(x)\exp(-ct)a\left|\frac{\partial^3 v_\epsilon^*}{\partial t^3}\right|^2 dx\,dt\\
&\qquad+\frac{3}{2}\int_\Omega h(x)\exp(-ct)\left|\frac{\partial^3 a}{\partial t^3}\right|\left|\frac{\partial u}{\partial x}\right|^2 dx\,dt\\
&\qquad+\frac{1}{2}\int_\Omega h(x)\exp(-ct)\left|\frac{\partial^3 a}{\partial t^3}\right|\left|\frac{\partial^4 u}{\partial t^3\partial x}\right|^2 dx\,dt\\
&\qquad+\frac{1}{2}\int_\Omega h(x)\exp(-ct)\frac{\partial a}{\partial t}\left|\frac{\partial^3 u}{\partial t^2\partial x}\right|^2 dx\,dt\Bigg\}.
\end{aligned}
\tag{4.17}
$$

Integrating the first term on the right-hand side by parts in (4.15), we obtain

$$
\begin{aligned}
&-\varepsilon\operatorname{Re}\int_\Omega A(t)u\overline{v_\varepsilon^*}\,dx\,dt\\
&\quad=\frac{3}{2}\int_\Omega h(x)\exp(-ct)\left[\frac{\partial a}{\partial t}-ca\right]\left|\frac{\partial^2 u}{\partial t\partial x}\right|^2 dx\,dt\\
&\qquad-\int_\Omega h(x)\exp(-ct)\left\{\frac{\partial^3 a}{\partial t^3}-3c\frac{\partial^2 a}{\partial t^2}+3c^2\frac{\partial a}{\partial t}-c^3 a\right\}\left|\frac{\partial u}{\partial x}\right|^2 dx\,dt\\
&\qquad-\int_0^1\frac{1}{2}h(x)\exp(-ct)a\left|\frac{\partial^2 u}{\partial t\partial x}\right|^2 dx|_{t=T}\\
&\qquad+\int_0^1\frac{1}{2}h(x)\exp(-ct)\left\{\frac{\partial^2 a}{\partial t^2}-2c\frac{\partial a}{\partial t}+c^2 a\right\}\left|\frac{\partial u}{\partial x}\right|^2 dx|_{t=T}\\
&\qquad-\int_0^1 h(x)\exp(-ct)\left\{\frac{\partial a}{\partial t}-ca\right\}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t\partial x}dx|_{t=T}.
\end{aligned}
\tag{4.18}
$$

This last equality gives

$$
\begin{aligned}
&-\varepsilon\operatorname{Re}\int_\Omega A(t)u\overline{v_\varepsilon^*}\,dx\,dt\\
&\quad\le-\int_0^1 h(x)\exp(-ct)\left|\frac{\partial a}{\partial t}+a-ca\right|\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dx|_{t=T}\\
&\qquad+\int_0^1\frac{1}{2}h(x)\exp(-ct)\left\{\frac{\partial^2 a}{\partial t^2}-2c\frac{\partial a}{\partial t}+c^2 a+ca-\frac{\partial a}{\partial t}\right\}\left|\frac{\partial u}{\partial x}\right|^2 dx|_{t=T}.
\end{aligned}
\tag{4.19}
$$

By using the conditions (3.2), inequalities (4.17) and (4.19), we obtain

$$
\operatorname{Re}\int_\Omega \exp(ct)v_\varepsilon^*\overline{Nv}\,dx\,dt\le 0\quad\text{as }\epsilon\longrightarrow 0.
\tag{4.20}
$$

This implies $\operatorname{Re}\int_\Omega \exp(ct)(v_\varepsilon^* - v)\overline{Nv}\,dx\,dt + \operatorname{Re}\int_\Omega \exp(ct)v\overline{Nv}\,dx\,dt \le 0$, that is,

$$\begin{aligned}&\int_0^T\int_0^l \exp(-ct)(1-l)|v|^2dx\,dt\\&\quad+\int_0^T\int_l^1\int_0^l \exp(-ct)(1-x)|v|^2dx\,dt+\int_0^T\int_l^1 \exp(-ct)\left|J_xv\right|^2dx\,dt\\&\quad+\int_0^T\int_0^l \frac{1-l}{2l}\exp(-ct)\left|J_xv\right|^2dx\,dt\le 0.\end{aligned}\tag{4.21}$$

Then $v = 0$.

Finally from (4.4), we conclude $w = 0$. □

Theorem 4.2. *The range $R(\overline{L})$ of $\overline{L}$ coincides with F.*

Proof. Since F is Hilbert space, then $R(\overline{L}) = F$ if and only if the relation

$$\int_\Omega \Theta(x)f\overline{g}\,dx\,dt = 0\tag{4.22}$$

holds.

Arbitrary $u \in D_0(L)$ and $\mathcal{F} = (f,0,0,0) \in F$ implies $f = 0$. Taking in (4.22), $u \in D_0(L)$, and using Lemma 4.1, we obtain

$$w = \begin{cases}(1-l)g, & 0 < x < l,\\(1-x)g, & l < x < 1,\end{cases}\tag{4.23}$$

then $g = 0$. □

References

[1] W. Allegretto, Y. Lin, and A. Zhou, *A box scheme for coupled systems resulting from microsensor thermistor problems*, Dynam. Contin. Discrete Impuls. Systems **5** (1999), no. 1–4, 209–223.

[2] G. W. Batten, Jr., *Second-order correct boundary conditions for the numerical solution of the mixed boundary problem for parabolic equations*, Math. Comp. **17** (1963), 405–413.

[3] S. A. Beilin, *Existence of solutions for one-dimensional wave equations with nonlocal conditions*, Electron. J. Differential Equations **2001** (2001), no. 76, 1–8.

[4] N.-E. Benouar and N. I. Yurchuk, *Mixed problem with an integral condition for parabolic equations with the Bessel operator*, Differ. Equ. **27** (1991), no. 12, 1482–1487.

[5] A. Bouziani and N.-E. Benouar, *Mixed problem with integral conditions for a third order parabolic equation*, Kobe J. Math. **15** (1998), no. 1, 47–58.

[6] B. Cahlon, D. M. Kulkarni, and P. Shi, *Stepwise stability for the heat equation with a nonlocal constraint*, SIAM J. Numer. Anal. **32** (1995), no. 2, 571–593.

[7] J. R. Cannon, *The solution of the heat equation subject to the specification of energy*, Quart. Appl. Math. **21** (1963), 155–160.

[8] ———, *The One-Dimensional Heat Equation*, Encyclopedia of Mathematics and its Applications, vol. 23, Addison-Wesley Publishing, Massachusetts, 1984.

[9] J. R. Cannon, Y. Lin, and S. Wang, *An implicit finite difference scheme for the diffusion equation subject to mass specification*, Internat. J. Engrg. Sci. **28** (1990), no. 7, 573–578.

[10] J. R. Cannon and A. L. Matheson, *A numerical procedure for diffusion subject to the specification of mass*, Internat. J. Engrg. Sci. **31** (1993), no. 3, 347–355.

[11] V. Capasso and K. Kunisch, *A reaction-diffusion system arising in modelling man-environment diseases*, Quart. Appl. Math. **46** (1988), no. 3, 431–450.

[12] Y. S. Choi and K.-Y. Chan, *A parabolic equation with nonlocal boundary conditions arising from electrochemistry*, Nonlinear Anal. **18** (1992), no. 4, 317–331.

[13] J. H. Cushman and T. R. Ginn, *Nonlocal dispersion in porous media with continuously evolving scales of heterogeneity*, J. Transport in Porous Media **13** (1993), no. 1, 123–138.

[14] J. H. Cushman, H. Xu, and F. Deng, *Nonlocal reactive transport with physical and chemical heterogeneity: localization error*, Water Resources Res. **31** (1995), no. 9, 2219–2237.

[15] M. Denche and A. L. Marhoune, *High-order mixed-type differential equations with weighted integral boundary conditions*, Electron. J. Differential Equations **2000** (2000), no. 60, 1–10.

[16] ———, *Mixed problem with nonlocal boundary conditions for a third-order partial differential equation of mixed type*, Int. J. Math. Math. Sci. **26** (2001), no. 7, 417–426.

[17] ———, *Mixed problem with integral boundary condition for a high order mixed type partial differential equation*, J. Appl. Math. Stochastic Anal. **16** (2003), no. 1, 69–79.

[18] N. I. Ionkin, *The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition*, Differ. Uravn. **13** (1977), no. 2, 294–304 (Russian).

[19] L. I. Kamynin, *A boundary value problem in the theory of heat conduction with a nonclassical boundary condition*, Comput. Math. Math. Phys. **4** (1964), no. 6, 33–59.

[20] A. V. Kartynnik, *Three-point boundary-value problem with an integral space-variable condition for a second-order parabolic equation*, Differ. Equ. **26** (1990), no. 9, 1160–1166.

[21] L. S. Pulkina, *A non-local problem with integral conditions for hyperbolic equations*, Electron. J. Differential Equations **1999** (1999), no. 45, 1–6.

[22] A. A. Samarski, *Some problems in the modern theory of differential equations*, Differ. Uravn. **16** (1980), 1925–1935 (Russian).

[23] P. Shi, *Weak solution to an evolution problem with a nonlocal constraint*, SIAM J. Math. Anal. **24** (1993), no. 1, 46–58.

[24] V. F. Volkodavov and V. E. Zhukov, *Two problems for the string vibration equation with integral conditions and special matching conditions on the characteristic*, Differ. Equ. **34** (1998), no. 4, 501–505.

[25] N. I. Yurchuk, *Mixed problem with an integral condition for certain parabolic equations*, Differ. Equ. **22** (1986), 1457–1463.

C. Latrous: Laboratoire Equations Differentielles, Département de Mathematiques, Université Mentouri Constantine, 25000 Constantine, Algeria

E-mail address: clatrous@wissal.dz

A. Memou: Laboratoire Equations Differentielles, Département de Mathematiques, Université Mentouri Constantine, 25000 Constantine, Algeria

E-mail address: memoua@wissal.dz

Mathematical Problems in Engineering

Special Issue on Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil ; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru