PRIMITIVITY OF SOME FULL GROUP C^*-ALGEBRAS

ERIK BÉDOS[1*] AND TRON Å. OMLAND[2]

ABSTRACT. We show that the full group C^*-algebra of the free product of two nontrivial countable amenable discrete groups, where at least one of them has more than two elements, is primitive. We also show that in many cases, this C^*-algebra is antiliminary and has an uncountable family of pairwise inequivalent, faithful irreducible representations.

1. INTRODUCTION

Let G denote a countable discrete group. It is known that $C^*(G)$, the full group C^*-algebra of G, is primitive in a number of cases [17, 3, 11, 8, 10, 1]. Especially, this is true for many groups which have a free product decomposition satisfying various conditions: see [8, 10, 1]. These results suggest that $C^*(G)$ should be primitive whenever G is the free product of two nontrivial countable discrete groups G_1 and G_2, where at least one of them has more than two elements. In this note, we show that this is indeed the case when both G_1 and G_2 are also assumed to be amenable.

This applies for example when G_1 and G_2 are both finite with $|G_1| \geq 2$ and $|G_2| \geq 3$. This case is not covered by any of the papers cited above, except when $G_1 = \mathbb{Z}_2$ and $G_2 = \mathbb{Z}_3$, i.e. G is the modular group $PSL(2, \mathbb{Z})$, for which primitivity of $C^*(G)$ was shown in [1]. The reader should consult [10] and [1] for more information around the problem of determining when the full group C^*-algebra of a countable discrete group is primitive.

Our proof will rely on the following result from [1]:

Theorem 1.1. *Assume that a group G has a normal subgroup H such that*

   i) $C^*(H)$ *is primitive,*
   ii) $K = G/H$ *is amenable,*
   iii) *the natural action of K on* $\widehat{C^*(H)}^{\,o}$ *has a free point.*

Then $C^*(G)$ *is primitive.*

We recall here what condition iii) means. Set $A = C^*(H)$. Then the set

$$\widehat{A}^o = \{[\pi] \in \widehat{A} \mid \pi \text{ is faithful}\}$$

is nonempty since A is assumed to be primitive. The natural action of $K = G/H$ on $\widehat{A}^o$ is defined as follows.

Let $n\colon K \to G$ be a normalized section for the canonical homomorphism p from G onto K. Let $\alpha\colon K \to \mathrm{Aut}\,(A)$ and $u\colon K \times K \to A$ be given by

$$\alpha_k\big(i_H(h)\big) = i_H\big(n(k)\,h\,n(k)^{-1}\big), \quad k \in K, h \in H,$$
$$u(k,l) = i_H\big(n(k)\,n(l)\,n(kl)^{-1}\big), \quad k, l \in K,$$

where i_H denotes the canonical injection of H into A.

Then (α, u) is a twisted action of K on A (cf. [12]), which induces an action of K on $\widehat{A}^o$ given by

$$k \cdot [\pi] = [\pi \circ \alpha_{k^{-1}}].$$

This action is independent of the choice of normalized section for p and called the natural action of K on $\widehat{A}^o$. Finally, we recall that $[\pi] \in \widehat{A}^o$ is a free point for this action whenever we have $k \cdot [\pi] \neq [\pi]$ for all $k \in K, k \neq e$.

Throughout this paper, we let G_1 and G_2 be two nontrivial countable discrete groups and assume that at least one of them has more than two elements. Further we let $G = G_1 * G_2$ denote the free product of G_1 and G_2. It is well known that G is icc and nonamenable. Section 2 is devoted to the proof of our main result in this paper:

Theorem 1.2. *Assume moreover that G_1 and G_2 are both amenable. Then $C^*(G)$ is primitive.*

In the final section (Section 3), we discuss the problem of deciding when $C^*(G)$ is antiliminary and has an uncountable family of pairwise inequivalent, faithful irreducible representations.

As will be evident from its proof, the annoying amenability assumption in Theorem 1.2 is due to the amenability assumption on K in Theorem 1.1. Now, if one replaces this assumption on K by requiring that the twisted action of K on $C^*(H)$ is amenable in the sense that the full and the reduced crossed products of $C^*(H)$ by this action agree, then Theorem 1.1 still holds. An interesting problem is whether one can find condition(s) other than the amenability of K ensuring that this more general requirement is satisfied.

2. Proof of Theorem 1.2

We let e_1 (resp. e_2) denote the unit of G_1 (resp. G_2) and set $G_1' = G_1 \setminus \{e_1\}$, $G_2' = G_2 \setminus \{e_2\}$. We let $X \subset G$ denote the set of commutators given by

$$X = \{ [a, b] = a\, b\, a^{-1} b^{-1} \in G \mid a \in G_1',\, b \in G_2' \}.$$

As is well known (see e.g. [14]), X is free and generates the kernel H of the canonical homomorphism p from the free product $G = G_1 * G_2$ onto the direct product $K = G_1 \times G_2$. The map $(a, b) \to [a, b]$ is then a bijection between $G_1' \times G_2'$ and X, and H is isomorphic to the free group $\mathbb{F}_{|X|}$ with $|X|$ generators.

As $|X| = |G_1'| \cdot |G_2'| \geq 2$, $A = C^*(H)$ is primitive (cf. [17, 3]). Further, as G_1 and G_2 are both assumed to be amenable, K is amenable.

Now let π be a faithful irreducible representation of A acting on a (necessarily separable) Hilbert space $\mathcal{H}_\pi$. For each function $\lambda \colon X \to \mathbb{T}$, we let γ_λ denote the $*$-automorphism of A determined by

$$\gamma_\lambda(i_H(x)) = \lambda(x) i_H(x), \quad x \in X,$$

and set $\pi_\lambda = \pi \circ \gamma_\lambda$. Clearly, each π_λ is also faithful and irreducible, i.e. $[\pi_\lambda] \in \widehat{A}^o$.

The burden of the proof is to establish the following:

Proposition 2.1. *There exist $[\pi] \in \widehat{A}^o$ and $\lambda \colon X \to \mathbb{T}$ such that $[\pi_\lambda]$ is a free point for the natural action of K on $\widehat{A}^o$.*

Once we have proven this proposition, the primitivity of $C^*(G)$ then clearly follows from Theorem 1.1 and the proof of Theorem 1.2 will therefore be finished.

Proof of Proposition 2.1. As a normalized section $n : K \to G$ for p, we choose

$$n(a, b) = a\, b, \quad a \in G_1,\, b \in G_2.$$

We have to show that some faithful irreducible representation π of A and some $\lambda \colon X \to \mathbb{T}$ may be chosen so that

$$\pi_\lambda \circ \alpha_k \not\simeq \pi_\lambda$$

for all nontrivial $k \in K$.

Clearly, to show that this condition holds, it suffices to show that for each nontrivial $k \in K$, there exists some $x \in X$ (depending on k) such that

$$(\pi_\lambda \circ \alpha_k)(i_H(x)) \not\simeq \pi_\lambda\big(i_H(x)\big). \tag{2.1}$$

To show this, we will use following fact:

Assume $x_0 \in X$ is fixed. Then, as follows from Choi's proof [3] (see [10, Proof of Theorem 3.2]), we may choose a faithful irreducible representation $\pi = \pi_{x_0}$ of A such that for each $x \neq x_0$ in X the unitary operator $\pi(i_H(x))$ is diagonal relative to some orthonormal basis of $\mathcal{H}_\pi$ (which depends on x). We will call such a representation for a *Choi representation of A associated to x_0*.

Our choice of x_0, and thereby of $\pi = \pi_{x_0}$, will depend on the possible existence of elements of order 2 in G_1 or G_2.

We will also use repeatedly the following elementary fact (already used in [10] and in [1]):

Assume $\mathcal{H}$ is a separable Hilbert space. Let U and V be unitary operators on $\mathcal{H}$ and assume that U is diagonal relative to some orthonormal basis of $\mathcal{H}$. Then the sets

$$\{\mu \in \mathbb{T} \mid \mu U \simeq V\} \text{ and } \{\mu \in \mathbb{T} \mid \mu U \simeq (\mu U)^*\}$$

are both countable.

Consider some faithful irreducible representation π of A and $\lambda \colon X \to \mathbb{T}$.

When $a \in G_1'$, $b \in G_2'$, so $[a,b] \in X$, we let $U(a,b)$ $(= U_\pi(a,b))$ denote the unitary operator on $\mathcal{H}_\pi$ given by $U(a,b) = \pi\big(i_H([a,b])\big)$. Further, we set $\lambda(a,b) = \lambda([a,b])$. Thus we have

$$\pi_\lambda\big(i_H([a,b])\big) = \lambda(a,b)U(a,b) . \tag{2.2}$$

Some straightforward calculations give the following identities which we will use in the sequel:

$$\pi_\lambda\big(\alpha_{(a,b)}(i_H([a^{-1},b^{-1}]))\big) = \lambda(a,b)U(a,b)$$
$$\pi_\lambda\big(\alpha_{(a,e_2)}(i_H([a^{-1},b]))\big) = \big(\lambda(a,b)U(a,b)\big)^* \tag{2.3}$$
$$\pi_\lambda\big(\alpha_{(e_1,b)}(i_H([a,b^{-1}]))\big) = \big(\lambda(a,b)U(a,b)\big)^*$$

$$\pi_\lambda\big(\alpha_{(a,b)}(i_H([a^{-1}c,b^{-1}]))\big) = \lambda(a,b)U(a,b)\big(\lambda(c,b)U(c,b)\big)^*$$
$$\pi_\lambda\big(\alpha_{(a,b)}(i_H([c,b^{-1}]))\big) = \lambda(a,b)U(a,b)\big(\lambda(ac,b)U(ac,b)\big)^* \tag{2.4}$$

$$\pi_\lambda\big(\alpha_{(a,e_2)}(i_H([a^{-1}c,b]))\big) = \lambda(c,b)U(c,b)\big(\lambda(a,b)U(a,b)\big)^*$$
$$\pi_\lambda\big(\alpha_{(a,e_2)}(i_H([c,b]))\big) = \lambda(ac,b)U(ac,b)\big(\lambda(a,b)U(a,b)\big)^* \tag{2.5}$$

whenever $a \in G_1'$, $b \in G_2'$ and $c \in G_1' \setminus \{a,a^{-1}\}$.

We will show how to pick π and λ such that (2.1) holds. It turns out that the possible existence of elements of order 2 in G_1 or G_2 complicates the argument.

Set $P = \{s \in G_1' \mid s^2 \neq e_1\}$, $S = G_1' \setminus P$,

and $Q = \{t \in G_2' \mid t^2 \neq e_2\}$, $T = G_2' \setminus Q$.

Hence we have

$$G_1 = \{e_1\} \sqcup P \sqcup S, \quad G_2 = \{e_2\} \sqcup Q \sqcup T .$$

We divide our discussion into three separate cases.

Case 1. *Both P and Q are nonempty.*

We pick $p_0 \in P$, $q_0 \in Q$ and set $x_0 = [p_0^{-1}, q_0^{-1}] \in X$.

Then we let $\pi = \pi_{x_0}$ be a Choi representation of A associated to x_0, and set $U(a,b) = U_\pi(a,b)$ for each $x = [a,b] \in X$.

It remains to define $\lambda \colon X \to \mathbb{T}$ so that (2.1) holds for each nontrivial $k \in K$.

We introduce the following notation.

Assume that $a \in G'_1$, $b \in G'_2$, $p \in P$, $q \in Q$, $s \in S, t \in T$. Then we set

$$\Omega(a,b) = \{\mu \in \mathbb{T} \mid \mu\, U(a,b) \simeq U(a^{-1}, b^{-1})\}\,,$$
$$\Omega_1(p) = \{\mu \in \mathbb{T} \mid \mu\, U(p, q_0) \simeq U(p^{-1}, q_0)^*\}\,,$$
$$\Omega_2(q) = \{\mu \in \mathbb{T} \mid \mu\, U(p_0, q) \simeq U(p_0, q^{-1})^*\}\,,$$
$$\Omega_1(s) = \{\mu \in \mathbb{T} \mid \mu\, U(s, q_0) \simeq \big(\mu\, U(s, q_0)\big)^*\}\,,$$
$$\Omega_2(t) = \{\mu \in \mathbb{T} \mid \mu\, U(p_0, t) \simeq \big(\mu\, U(p_0, t)\big)^*\}\,.$$

Note that if $(a,b) \neq (p_0^{-1}, q_0^{-1})$, then $\Omega(a,b)$ is countable (as $U(a,b)$ is then diagonalisable). Similarly, $\Omega_1(p)$, $\Omega_2(q)$, $\Omega_1(s)$ and $\Omega_2(t)$ are countable.

To ease our notation, we will define λ on $G'_1 \times G'_2$ and identify it with the function on X given by $\lambda([a, b]) = \lambda(a, b)$, $a \in G'_1$, $b \in G'_2$.

We will first define λ on $P \times Q$.

Let $P = \sqcup_{i \in I} \{p_i, p_i^{-1}\}$, $Q = \sqcup_{j \in J} \{q_j, q_j^{-1}\}$ be enumerations of P and Q, where the index set I (resp. J) is a (finite or infinite) set of successive integers starting from 0.

For each $i \in I$ and $j \in J$, we set

$$\lambda(p_i^{-1}, q_j) = \lambda(p_i^{-1}, q_j^{-1}) = 1\,.$$

Now let $i \in I, j \in J$. Using (2.2) and (2.3), we see that (2.1) will hold for

$$k = (p_i, q_j^{-1}) \text{ and } k = (p_i^{-1}, q_j) \text{ if } \lambda(p_i, q_j^{-1})U(p_i, q_j^{-1}) \not\simeq U(p_i^{-1}, q_j)\,;$$
$$k = (p_i, q_j) \text{ and } k = (p_i^{-1}, q_j^{-1}) \text{ if } \lambda(p_i, q_j)U(p_i, q_j) \not\simeq U(p_i^{-1}, q_j^{-1})\,;$$
$$k = (p_i, e_2) \text{ and } k = (p_i^{-1}, e_2) \text{ if } \lambda(p_i, q_0)U(p_i, q_0) \not\simeq U(p_i^{-1}, q_0)^*\,;$$
$$k = (e_1, q_j) \text{ and } k = (e_1, q_j^{-1}) \text{ if } \lambda(p_0, q_j)U(p_0, q_j) \not\simeq \big(\lambda(p_0, q_j^{-1})U(p_0, q_j^{-1})\big)^*\,.$$

For each $i \in I$ and $j \in J$, we therefore pick

$$\lambda(p_i, q_j^{-1}) \in \mathbb{T} \setminus \Omega(p_i, q_j^{-1}).$$

Next, for each $i \in I$, $i \neq 0$, and $j \in J$, $j \neq 0$, we pick

$$\lambda(p_i, q_j) \in \mathbb{T} \setminus \Omega(p_i, q_j)\,,$$
$$\lambda(p_i, q_0) \in \mathbb{T} \setminus \big(\Omega(p_i, q_0) \cup \Omega_1(p_i)\big)\,,$$
$$\lambda(p_0, q_j) \in \mathbb{T} \setminus \big(\Omega(p_0, q_j) \cup \overline{\lambda(p_0, q_j^{-1})}\, \Omega_2(q_j)\big)\,.$$

Finally, we pick $\lambda(p_0, q_0) \in \mathbb{T} \setminus \big(\Omega(p_0, q_0) \cup \Omega_1(p_0) \cup \overline{\lambda(p_0, q_0^{-1})}\, \Omega_2(q_0)\big)\,.$

All these choices are possible as all the involved Ω's are countable. After having done this, λ is defined on $P \times Q$ and we know that (2.1) will hold for all $k \in (P \times Q) \cup (P \times \{e_2\}) \cup (\{e_1\} \times Q)\,.$

This means that if both S and T happen to be empty, then λ is defined on the whole of X and (2.1) holds for every nontrivial k in K, as desired.

We assume from now on and until the end of Case 1 that S is nonempty.

Consider $s \in S$. For each $j \in J$ we set $\lambda(s, q_j^{-1}) = 1$.

Using (2.2) and (2.3), we see that (2.1) will hold for

$k = (s, q_j)$ and $k = (s, q_j^{-1})$ if $\lambda(s, q_j)U(s, q_j) \not\simeq U(s, q_j^{-1})$;

$k = (s, e_2)$ if $\lambda(s, q_0)U(s, q_0) \not\simeq (\lambda(s, q_0)\, U(s, q_0))^*$.

For each $j \in J$, $j \neq 0$, we therefore pick $\lambda(s, q_j) \in \mathbb{T} \setminus \Omega(s, q_j)$.

We also pick $\lambda(s, q_0) \in \mathbb{T} \setminus (\Omega(s, q_0) \cup \Omega_1(s))$.

Again, these choices are possible as all the involved Ω's are countable. Following this procedure for every $s \in S$, we achieve that λ is defined on $G_1' \times Q$ in such a way that (2.1) will hold for all

$$k \in \big(G_1' \times (\{e_2\} \cup Q)\big) \cup (\{e_1\} \times Q).$$

If T happens to be empty, this means that λ is defined on the whole of X and (2.1) holds for every nontrivial k in K, as desired.

Finally, we assume from now on and until the end of Case 1 that T is also nonempty.

Consider $t \in T$. For each $i \in I$ we set $\lambda(p_i^{-1}, t) = 1$.

Using (2.2) and (2.3), we see that (2.1) will hold for

$k = (p_i, t)$ and $k = (p_i^{-1}, t)$ if $\lambda(p_i, t)U(p_i, t) \not\simeq U(p_i^{-1}, t)$;

$k = (e_1, t)$ if $\lambda(p_0, t)U(p_0, t) \not\simeq (\lambda(p_0, t)U(p_0, t))^*$.

For each $i \in I$, $i \neq 0$, we pick $\lambda(p_i, t) \in \mathbb{T} \setminus \Omega(p_i, t)$.

We also pick $\lambda(p_0, t) \in \mathbb{T} \setminus (\Omega(p_0, t) \cup \Omega_2(t))$.

Once again, these choices are possible as all the involved Ω's are countable.

By doing this for every $t \in T$, we achieve that λ is defined on $(G_1' \times G_2') \setminus (S \times T)$ and (2.1) will hold for all

$$k \in \big(G_1' \times (\{e_2\} \cup Q)\big) \cup \big((\{e_1\} \cup P) \times G_2'\big).$$

It remains to define λ on $S \times T$ in a way which ensures that (2.1) also will hold for all $k \in S \times T$.

Let $t \in T$. We will below describe how to define λ on $S \times \{t\}$ in a way which ensures that (2.1) will hold for all $k \in S \times \{t\}$. By following this procedure for each $t \in T$, the proof in Case 1 will then be finished.

It is now appropriate to partition S as $S = S' \sqcup S''$, where

$$S' = \{s \in S \mid sp_0 \in P\}, \ S'' = \{s \in S \mid sp_0 \in S\}.$$

Assume that $s \in S'$.

Using (2.2) and (2.4), we see that (2.1) will hold for

$k = (s, t)$ if $\lambda(s, t)U(s, t)(\lambda(p_0, t)U(p_0, t))^* \not\simeq \lambda(sp_0, t)U(sp_0, t)$.

Note that $\lambda(sp_0, t)$ is already defined since $sp_0 \in P$. Further, as $\lambda(sp_0, t)U(sp_0, t)$ is diagonalisable, the set

$$\Omega'(s, t) = \{\mu \in \mathbb{T} \mid \mu\left(\lambda(sp_0, t)U(sp_0, t)\right) \simeq U(s, t)\left(\lambda(p_0, t)U(p_0, t)\right)^*\}$$

is countable. We can therefore pick $\lambda(s, t) \in \mathbb{T} \setminus \overline{\Omega'(s, t)}$.

If S' is nonempty, we can do this for each $s \in S'$ and λ will then be defined on $S' \times \{t\}$ in such a way that (2.1) will hold for every $k \in S' \times \{t\}$.

If S'' is empty, then S' has to be nonempty and the proof of Case 1 is then finished.

Assume now that S'' is nonempty and consider $s \in S''$, so $(sp_0)^2 = e_1$.

One easily checks that this implies that $s\,p_0{}^n = p_0{}^{-n}\,s$ for all $n \in \mathbb{Z}$. It is then almost immediate that $S''(s) = \{sp_0{}^n \mid n \in \mathbb{Z}\}$ is a subset of S''.

Furthermore, if $\tilde{s} \in S'' \setminus S''(s)$, then $S''(s)$ and $S''(\tilde{s})$ are disjoint.

Hence, as S'' is countable, we may pick a countable family $\{s_l\}_{l \in L}$ of distinct elements in S'' such that $S'' = \sqcup_{l \in L} S''(s_l)$.

Consider $l \in L$. To ease notation we write $s = s_l$.

We are going to define λ on $S''(s) \times \{t\}$ in such a way that (2.1) will hold for every $k \in S''(s) \times \{t\}$. By doing this for each $l \in L$, λ will then be defined on $S'' \times \{t\}$ and (2.1) will hold for every $k \in S'' \times \{t\}$.

Since $S \times \{t\} = (S' \times \{t\}) \sqcup (S'' \times \{t\})$, the proof of Case 1 will then be finished.

For each $n \in \mathbb{Z}$, using (2.2) and (2.4) (with $a = sp_0{}^n, b = t$ and $c = sp_0{}^{n\pm 1}$), we see that (2.1) will hold for

$$k = (sp_0{}^n, t) \text{ if } \lambda(sp_0{}^n, t)U(sp_0{}^n, t)\big(\lambda(sp_0{}^{n+1}, t)U(sp_0{}^{n+1}, t)\big)^* \not\simeq \lambda(p_0, t)U(p_0, t)$$
$$\text{or } \lambda(sp_0{}^n, t)U(sp_0{}^n, t)\big(\lambda(sp_0{}^{n-1}, t)U(sp_0{}^{n-1}, t)\big)^* \not\simeq \lambda(p_0^{-1}, t)U(p_0^{-1}, t)\,.$$

Suppose first that p_0 is aperiodic, so $S''(s) = \sqcup_{n \in \mathbb{Z}}\{sp_0{}^n\}$.

We first set $\lambda(s, t) = 1$. Then, for each $m \in \mathbb{N}$, we do inductively the following two steps:

i) Define

$$\Omega^m(s, t)$$
$$= \{\mu \in \mathbb{T} \mid \mu\big(\lambda(p_0, t)U(p_0, t)\big) \simeq \lambda(sp_0^{m-1}, t)U(sp_0^{m-1}, t)U(sp_0{}^m, t)^*\}$$

(which is countable) and pick $\lambda(sp_0{}^m, t) \in \mathbb{T} \setminus \Omega^m(s, t)$.

ii) Define

$$\Omega^{-m}(s, t)$$
$$= \{\mu \in \mathbb{T} \mid \mu\big(\lambda(p_0{}^{-1}, t)U(p_0{}^{-1}, t)\big) \simeq \lambda(sp_0{}^{-m+1}, t)U(sp_0{}^{-m+1}, t)\,U(sp_0{}^{-m}, t)^*\}$$

(which is countable) and pick $\lambda(sp_0{}^{-m}, t) \in \mathbb{T} \setminus \Omega^{-m}(s, t)$.

Once this inductive process is finished, λ is defined on $S''(s) \times \{t\}$ and we know that (2.1) holds for every $k = (sp_0{}^{\pm(m-1)}, t)$, $m \in \mathbb{N}$, i.e. for every $k \in S''(s) \times \{t\}$, as desired.

Assume now that p_0 is periodic with period N. Note that $N \geq 3$ since $p_0 \in P$. The aperiodic case has to be modified as follows.

Again, we first set $\lambda(s,t) = 1$. Then, for each $m = 1, \cdots, N-2$, we define inductively

$$\Omega^m(s,t) = \{\mu \in \mathbb{T} \mid \mu\left(\lambda(p_0,t)U(p_0,t)\right) \simeq \lambda(sp_0^{m-1},t)U(sp_0^{m-1},t)U(sp_0^m,t)^*\}$$

(which is countable) and pick $\lambda(sp_0{}^m, t) \in \mathbb{T} \setminus \Omega^m(s,t)$.

This ensures that (2.1) holds for each $k = (sp_0{}^{m-1}, t)$, $m = 1, \cdots, N-2$.

We also define

$$\Omega^{N-1}(s,t) = \{\mu \in \mathbb{T} \mid \mu\left(\lambda(p_0,t)U(p_0,t)\right) \simeq \lambda(sp_0^{N-2},t)U(sp_0^{N-2},t)U(sp_0^{N-1},t)^*\}$$

(which is countable). If we pick $\lambda(sp_0{}^{N-1}, t)$ outside $\Omega^{N-1}(s,t)$, then (2.1) will hold for $k = (sp_0{}^{N-2}, t)$. However, we want to pick $\lambda(sp_0{}^{N-1}, t)$ so that (2.1) also holds for $k = (sp_0{}^{N-1}, t)$.

Now, using (2.2) and (2.4) (with $a = sp_0{}^{N-1}$, $b = t$ and $c = s$), we see that (2.1) will hold for $k = (sp_0{}^{N-1}, t)$ if

$$\lambda(p_0,t)U(p_0,t) \not\simeq \lambda(sp_0^{N-1},t)U(sp_0^{N-1},t)U(s,t)^*.$$

Hence we define

$$\Omega_N(s,t) = \{\mu \in \mathbb{T} \mid \mu\left(\lambda(p_0,t)U(p_0,t)\right) \simeq U(sp_0^{N-1},t)U(s,t)^*\}$$

(which is countable) and pick

$$\lambda(sp_0^{N-1},t) \in \mathbb{T} \setminus \left(\Omega^{N-1}(s,t) \cup \overline{\Omega_N(s,t)}\,\right).$$

This choice does ensure that (2.1) holds for $k = (sp_0{}^{N-2}, t)$ and $k = (sp_0{}^{N-1}, t)$.

Hence, λ is defined on $S''(s) \times \{t\}$ and (2.1) holds for every $k \in S''(s) \times \{t\}$. This finishes the proof of Case 1.

Case 2. *Either P is nonempty and Q is empty, or P is empty and Q is nonempty.*

Clearly, it suffices to consider the first alternative. We then pick $p_0 \in P$, $t_0 \in T$ and set $x_0 = [p_0{}^{-1}, t_0] \in X$.

We let $\pi = \pi_{x_0}$ be a Choi representation of A associated to x_0 and set $U(a,b) = U_\pi(a,b)$ for each $x = [a,b] \in X$.

Our proof that $\lambda : X \to \mathbb{T}$ may be defined so that (2.1) holds for each nontrivial $k \in K$ is quite similar to our proof of Case 1, but some care is required and some repetitions seem unavoidable in our presentation.

For $p \in P$, $s \in S$, $t \in T$, we now set

$$\Omega(p,t) = \{\mu \in \mathbb{T} \mid \mu\, U(p,t) \simeq U(p^{-1},t)\},$$
$$\Omega_1(p) = \{\mu \in \mathbb{T} \mid \mu\, U(p,t_0) \simeq U(p^{-1},t_0)^*\},$$
$$\Omega_1(s) = \{\mu \in \mathbb{T} \mid \mu\, U(s,t_0) \simeq \left(\mu\, U(s,t_0)\right)^*\},$$
$$\Omega_2(t) = \{\mu \in \mathbb{T} \mid \mu\, U(p_0,t) \simeq \left(\mu\, U(p_0,t)\right)^*\}.$$

Note that if $(p,t) \neq (p_0^{-1}, t_0)$, then $\Omega(p,t)$ is countable. On the other hand, $\Omega_1(p)$ is countable when $a \neq p_0^{-1}$, while $\Omega_1(s)$ and $\Omega_2(t)$ are always countable.

Let $P = \sqcup_{i \in I} \{p_i, p_i^{-1}\}$ be an enumeration of P, where I is a (finite or infinite) set of successive integers starting from 0.

First, we set $\lambda(p_i^{-1}, t) = 1$ for all $i \in I$ and $t \in T$.

Let $i \in I, t \in T$. Using (2.2) and (2.3), we see that (2.1) will hold for

$$k = (p_i, t) \text{ and } k = (p_i^{-1}, t) \text{ if } \lambda(p_i, t)U(p_i, t) \not\simeq U(p_i^{-1}, t);$$

$$k = (p_i, e_2) \text{ and } k = (p_i^{-1}, e_2) \text{ if } \lambda(p_i, t_0)U(p_i, t_0) \not\simeq U(p_i^{-1}, t_0)^*;$$

$$k = (e_1, t) \text{ if } \lambda(p_0, t)U(p_0, t) \not\simeq \big(\lambda(p_0, t)U(p_0, t)\big)^*.$$

Therefore, for each $i \in I, i \neq 0$, and $t \in T, t \neq t_0$, we pick

$$\lambda(p_i, t) \in \mathbb{T} \setminus \Omega(p_i, t),$$
$$\lambda(p_i, t_0) \in \mathbb{T} \setminus \big(\Omega(p_i, t_0) \cup \Omega_1(p_i)\big),$$
$$\lambda(p_0, t) \in \mathbb{T} \setminus \big(\Omega(p_0, t) \cup \Omega_2(t)\big).$$

Finally, we pick

$$\lambda(p_0, t_0) \in \mathbb{T} \setminus \big(\Omega(p_0, t_0) \cup \Omega_1(p_0) \cup \Omega_2(t_0)\big).$$

These choices ensure that λ is defined on $P \times T$ and (2.1) will hold for all $k \in (P \times (T \cup \{e_2\})) \cup (\{e_1\} \times T)$.

This means that if S happens to be empty, λ is defined on the whole of X and (2.1) holds for every nontrivial k in K, as desired.

We assume from now on and until the end of Case 2 that S is nonempty.

Consider $s \in S$. Using (2.2) and (2.3), we see that (2.1) will hold for

$$k = (s, e_2) \text{ if } \lambda(s, t_0)U(s, t_0) \not\simeq \big(\lambda(s, t_0)U(s, t_0)\big)^*.$$

We will therefore pick $\lambda(s, t_0)$ in a subset of $\mathbb{T} \setminus \Omega_1(s)$. But which subset will depend on whether s belongs to S' or S'', where

$$S' = \{s \in S \mid sp_0 \in P\} \text{ and } S'' = \{s \in S \mid sp_0 \in S\}$$

(using the same notation as in Case 1).

Assume that $s \in S', t \in T$. As in Case 1, (2.1) will hold for

$$k = (s, t) \text{ if } \lambda(s, t)U(s, t)\big(\lambda(p_0, t)U(p_0, t)\big)^* \not\simeq \lambda(sp_0, t)U(sp_0, t).$$

Again, we set

$$\Omega'(s, t)$$
$$= \{\mu \in \mathbb{T} \mid \mu \big(\lambda(sp_0, t)U(sp_0, t)\big) \simeq U(s, t)(\lambda(p_0, t)U(p_0, t))^*\}.$$

If $t = t_0$, then we pick $\lambda(s, t_0) \in \mathbb{T} \setminus \big(\Omega_1(s) \cup \overline{\Omega'(s, t_0)}\big)$.

Otherwise, we pick $\lambda(s, t) \in \mathbb{T} \setminus \overline{\Omega'(s, t)}$.

If S' is nonempty, we can do this for every $s \in S'$ and every $t \in T$. This ensures that λ is defined on $S' \times T$ and that (2.1) will hold for every $k \in (S' \times (T \cup \{e_2\})$. Hence, if S'' is empty, then S' has to be nonempty and the proof of Case 2 is finished.

Assume now that S'' is nonempty. As in Case 1, we then pick a countable family $\{s_l\}_{l \in L}$ of distinct elements in S'' such that $S'' = \bigsqcup_{l \in L} S''(s_l)$, where $S''(s) = \{sp_0^n \mid n \in \mathbb{Z}\}$ for $s \in S''$.

Consider $l \in L$, $t \in T$ and set $s = s_l$.

If $t = t_0$, then we pick $\lambda(s, t_0) \in \mathbb{T} \setminus \Omega_1(s)$. Otherwise, we set $\lambda(s, t) = 1$.

Let $n \in \mathbb{Z}$. As in Case 1, (2.1) will hold for

$$k = (sp_0^n, t) \text{ if } \lambda(sp_0^n, t)U(sp_0^n, t)\big(\lambda(sp_0^{n+1}, t)U(sp_0^{n+1}, t)\big)^* \not\simeq \lambda(p_0, t)U(p_0, t)$$
$$\text{or } \lambda(sp_0^n, t)U(sp_0^n, t)\big(\lambda(sp_0^{n-1}, t)U(sp_0^{n-1}, t)\big)^* \not\simeq \lambda(p_0^{-1}, t)U(p_0^{-1}, t).$$

Suppose first that p_0 is aperiodic, so $S''(s) = \bigsqcup_{n \in \mathbb{Z}}\{sp_0^n\}$.

Then, for each $m \in \mathbb{N}$, we proceed inductively and do the following two steps:

i) Define

$$\Omega^m(s, t) = \{\mu \in \mathbb{T} \mid \mu\big(\lambda(p_0, t)U(p_0, t)\big) \simeq \lambda(sp_0^{m-1}, t)U(sp_0^{m-1}, t)U(sp_0^m, t)^*\}.$$

If $t = t_0$, then we pick $\lambda(sp_0^m, t_0) \in \mathbb{T} \setminus \big(\Omega_1(sp_0^m) \cup \Omega^m(s, t_0)\big)$. Otherwise, we pick $\lambda(sp_0^m, t) \in \mathbb{T} \setminus \Omega^m(s, t)$.

ii) Define

$$\Omega^{-m}(s, t)$$
$$= \{\mu \in \mathbb{T} \mid \mu\big(\lambda(p_0^{-1}, t)U(p_0^{-1}, t)\big) \simeq \lambda(sp_0^{-m+1}, t)U(sp_0^{-m+1}, t)\,U(sp_0^{-m}, t)^*\}.$$

If $t = t_0$, then we pick $\lambda(sp_0^{-m}, t_0) \in \mathbb{T} \setminus \big(\Omega_1(sp_0^{-m}) \cup \Omega^{-m}(s, t_0)\big)$. Otherwise, we pick $\lambda(sp_0^{-m}, t) \in \mathbb{T} \setminus \Omega^{-m}(s, t)$.

Assume next that p_0 is periodic with period $N \geq 3$.

Then for each $m = 1, \cdots, N - 2$, proceeding inductively, we define

$$\Omega^m(s, t) = \{\mu \in \mathbb{T} \mid \mu\big(\lambda(p_0, t)U(p_0, t)\big) \simeq \lambda(sp_0^{m-1}, t)U(sp_0^{m-1}, t)U(sp_0^m, t)^*\}.$$

If $t = t_0$, we pick $\lambda(sp_0^m, t_0) \in \mathbb{T} \setminus \big(\Omega_1(sp_0^m) \cup \Omega^m(s, t_0)\big)$. Otherwise, we pick $\lambda(sp_0^m, t) \in \mathbb{T} \setminus \Omega^m(s, t)$.

We also define

$$\Omega^{N-1}(s, t) = \{\mu \in \mathbb{T} \mid \mu\big(\lambda(p_0, t)U(p_0, t)\big) \simeq \lambda(sp_0^{N-2}, t)U(sp_0^{N-2}, t)U(sp_0^{N-1}, t)^*\}.$$

As in Case 1, (2.1) will hold for $k = (sp_0^{N-1}, t)$ if

$$\lambda(p_0, t)U(p_0, t) \not\simeq \lambda(p_0^{N-1}, t)U(sp_0^{N-1}, t)U(s, t)^*.$$

So we define

$$\Omega_N(s, t) = \{\mu \in \mathbb{T} \mid \mu\big(\lambda(p_0, t)U(p_0, t)\big) \simeq U(sp_0^{N-1}, t)U(s, t)^*\}.$$

Now, if $t = t_0$, then we pick

$$\lambda(sp_0^{N-1}, t_0) \in \mathbb{T} \setminus \big(\Omega_1(sp_0^{N-1}) \cup \Omega^{N-1}(s, t_0) \cup \overline{\Omega_N(s, t_0)}\big).$$

Otherwise, we pick

$$\lambda(sp_0^{N-1}, t) \in \mathbb{T} \setminus \left(\Omega^{N-1}(s,t) \cup \overline{\Omega_N(s,t)} \right).$$

Under both alternatives (p_0 being aperiodic or not), these processes ensure that λ is defined on $S''(s) \times \{t\}$ and that (2.1) will hold for every $k \in S''(s) \times (\{t\} \cup \{e_2\})$.

After having done this for every $s = s_l$ ($l \in L$) and every $t \in T$, λ is defined on $S'' \times T$ and we know that (2.1) will hold for every $k \in S'' \times (T \cup \{e_2\})$.

Altogether, this means that λ is defined on the whole of $G_1' \times G_2'$ and (2.1) holds for every nontrivial $k \in K$. This finishes the proof of Case 2.

Case 3. *Both P and Q are empty.*

This means that $G_1' = S$ and $G_2' = T$, i.e. all nontrivial elements in G_1 and G_2 have order 2, so both groups are abelian. As one of them is assumed to have more than two elements, we may assume that $|G_1| \geq 4$ and $|G_2| \geq 2$.

We pick $s_0 \in S$, $t_0 \in T$ and set $x_0 = [s_0, t_0] \in X$.

Next, we let $\pi = \pi_{x_0}$ be a Choi representation of A associated to x_0 and set $U(a, b) = U_\pi(a, b)$ for each $(a, b) \in S \times T = G_1' \times G_2'$.

Now, since S is countable, it is not difficult to see that we may find a family $\{s_l\}_{l \in L}$ of distinct elements in $S \setminus \{s_0\}$ such that

$$S = \{s_0\} \sqcup \left(\sqcup_{l \in L} \{s_l, s_0 s_l\} \right),$$

where L is a (finite or infinite) set of successive integers starting from 1.

Let $t \in T$. Set $\lambda(s_0, t) = 1$ and $\lambda(s_l, t) = 1$ for each $l \in L$, $l \geq 2$.

Using (2.2) and (2.3), we see that (2.1) will hold for

$$k = (e_1, t) \text{ if } \lambda(s_1, t) U(s_1, t) \not\simeq (\lambda(s_1, t) U(s_1, t))^* \,.$$

Hence we set $\Omega(t) = \{\mu \in \mathbb{T} \mid \mu\, U(s_1, t) \simeq (\mu\, U(s_1, t))^*\}$, which is countable, and pick

$$\lambda(s_1, t) \in \mathbb{T} \setminus \Omega(t).$$

Consider now $l \in L$. Using (2.2), (2.3), (2.4) and (2.5), we see that (2.1) will hold for

$$k = (s_0, t) \text{ and } k = (s_l, e_2) \text{ if } \lambda(s_0 s_l, t) U(s_0 s_l, t) \not\simeq U(s_0, t)(\lambda(s_l, t) U(s_l, t))^* \,;$$

$$k = (s_0, e_2) \text{ and } k = (s_l, t) \text{ if } \lambda(s_0 s_l, t) U(s_0 s_l, t) \not\simeq \lambda(s_l, t) U(s_l, t) U(s_0, t)^* \,;$$

$$k = (s_0, t) \text{ and } k = (s_0 s_l, e_2) \text{ if } \lambda(s_l, t) U(s_l, t) \not\simeq U(s_0, t)(\lambda(s_0 s_l, t) U(s_0 s_l, t))^* \,;$$

$$k = (s_0 s_l, t) \text{ and } k = (s_0, e_2) \text{ if } \lambda(s_l, t) U(s_l, t) \not\simeq \lambda(s_0 s_l, t) U(s_0 s_l, t) U(s_0, t)^* \,.$$

For each $l \in L$, we therefore set

$$\Omega_1(l, t) = \{\mu \in \mathbb{T} \mid \mu\, U(s_l s_0, t) \simeq U(s_0, t)\left(\lambda(s_l, t) U(s_l, t)\right)^*\} \,,$$
$$\Omega_2(l, t) = \{\mu \in \mathbb{T} \mid \mu\, U(s_0 s_l, t) \simeq \lambda(s_l, t) U(s_l, t) U(s_0, t)^*\} \,,$$
$$\Omega_3(l, t) = \{\mu \in \mathbb{T} \mid \mu\left(\lambda(s_l, t) U(s_l, t)\right) \simeq U(s_0, t) U(s_0 s_l, t)^*\} \,,$$
$$\Omega_4(l, t) = \{\mu \in \mathbb{T} \mid \mu\left(\lambda(s_l, t) U(s_l, t)\right) \simeq U(s_l s_0, t) U(s_0, t)^*\} \,.$$

All these sets are countable. Hence, for each $l \in L$, we can pick

$$\lambda(s_0 s_l, t) \in \mathbb{T} \setminus \left(\Omega_1(l, t) \cup \Omega_2(l, t) \cup \Omega_3(l, t) \cup \overline{\Omega_4(l, t)} \right).$$

We have thereby defined λ on $S \times \{t\}$ in such a way that (2.1) will hold for every $k \in \left(G_1 \times \{t\} \right) \sqcup \left(S \times \{e_2\} \right)$. By doing this for each $t \in T$, λ is defined on $S \times T = G'_1 \times G'_2$ and (2.1) holds for every nontrivial $k \in K$. This finishes the proof of Case 3 (and thereby the proofs of Proposition 1 and Theorem 2).

$\square$

3. Some further aspects

We believe that if G is a countable group such that $C^*(G)$ is primitive, then $C^*(G)$ is antiliminary and has an uncountable family of pairwise inequivalent, irreducible faithful representations. It is not difficult to see that this true in the case where G is nontrivial, icc and amenable (see below). As pointed out in [1], this also holds when $G = \mathbb{Z}_2 * \mathbb{Z}_3$. The argument was based on the following observation, which goes back to the work of J. Glimm and J. Dixmier in the sixties. We recall that a representation of a C^*-algebra is called *essential* whenever its range contains no compact operators other than zero.

Proposition 3.1. *Let A be a primitive separable C^*-algebra and consider the nonempty set $\widehat{A}^o = \{ [\pi] \in \widehat{A} \mid \pi \text{ is faithful} \}$. Then the following conditions are equivalent:*

  i) *$| \widehat{A}^o | > 1$.*
 ii) *Every faithful irreducible representation of A is essential.*
iii) *A has a faithful irreducible representation which is essential.*
 iv) *$\widehat{A}^o$ is uncountable.*

Moreover, if A satisfies any of these conditions, then A is antiliminary.

Proof. The implications ii) $\Rightarrow$ iii) and iv) $\Rightarrow$ i) are trivial. The implication i) $\Rightarrow$ ii) follows from [4, Corollaire 4.1.10], while iii) $\Rightarrow$ iv) follows from [4, Compléments 4.7.2]. The final assertion follows from [4, Compléments 9.5.4].

$\square$

For completeness we mention that there is another way to show that a unital separable C^*-algebra is primitive and antiliminary. Indeed, using that primitivity and primeness are equivalent notions for separable C^*-algebras (see e.g. [13]), one deduces that a separable unital C^*-algebra A is primitive and antiliminary if and only if the pure state space of A is weak*-dense in the state space of A (cf. [4, Lemme 11.2.4 and Compléments 11.6.6]). H. Yoshizawa showed in [17] that the right-hand side of this equivalence holds when $A = C^*(\mathbb{F}_2)$.

Now let $G = G_1 * G_2$ be as in Theorem 1.2. It is conceivable that one might be able to check that condition i) in Proposition 3.1 holds for $A = C^*(G)$ by following the line of proof used in [1] when $G = \mathbb{Z}_2 * \mathbb{Z}_3$. However, in light of our proof of Theorem 1.2, the necessary combinatorics will certainly be very messy.

We will instead use the following well known lemma to check that condition ii) holds for $A = C^*(G)$ in many cases.

Lemma 3.2. *Let A be a primitive, unital, infinite-dimensional C^*-algebra. Assume that A contains no nontrivial projections or that A has a faithful tracial state. Then A satisfies condition ii) in Proposition 3.1.*

Proof. For completeness, we give the proof. Let π be a faithful irreducible representation of A acting on a Hilbert space $\mathcal{H}$ and let $\mathcal{K}$ denote the compact operators on $\mathcal{H}$. Note that $\mathcal{H}$ is infinite-dimensional since $\pi(A)$ is infinite-dimensional.

Assume first that A contains no nontrivial projections. Since π is faithful, $\pi(A)$ contains no nontrivial projections. Hence $\pi(A) \cap \mathcal{K} = \{0\}$ (otherwise we would have $\mathcal{K} \subset \pi(A)$ by irreducibility, and $\pi(A)$ would contain all finite-dimensional projections), so π is essential.

Assume now that A has a faithful tracial state τ. Assume (for contradiction) that $\pi(A) \cap \mathcal{K} \neq \{0\}$. Then $\mathcal{K} \subset \pi(A)$. As is well known, when $\mathcal{H}$ is infinite-dimensional, the only bounded trace on $\mathcal{K}$ is the zero map. Hence the restriction of $\tau \circ \pi^{-1}$ to $\mathcal{K}$ must be zero. But $\mathcal{K}$ contains nontrivial projections and evaluation of $\tau \circ \pi^{-1}$ on any of these does not give zero since $\tau \circ \pi^{-1}$ is faithful. This gives a contradiction, and it follows that π is essential. $\square$

Corollary 3.3. *Let $G = G_1 * G_2$ satisfy the assumptions of Theorem 1.2. Assume also that G_1 and G_2 are both torsion-free. Then $C^*(G)$ has no nontrivial projections. Moreover, it is antiliminary and has an uncountable family of of pairwise inequivalent, irreducible faithful representations.*

Proof. The first assertion is mentioned by G.J. Murphy [10, p. 703], where he refers to [5] and [9] for a proof. It seems to us that this is somewhat unprecise. We therefore provide an alternative way to prove this assertion:

Since G_1 and G_2 are amenable, G has the Haagerup property ([2, Proposition 6.2.3]). Hence, as shown by N. Higson and G. Kasparov in [7], G satisfies the Baum–Connes conjecture. As G is easily seen to be torsion-free, G also satisfies the Kadison–Kaplansky conjecture (see e.g. [16]), i.e. the reduced group C^*-algebra $C_r^*(G)$ contains no nontrivial projections.

Moreover, as shown by J.L. Tu in [15], any group having the Haagerup property is K-amenable. It follows that the homomorphism λ_* from $K_0(C^*(G))$ to $K_0(C_r^*(G))$ induced by the canonical map $\lambda : C^*(G) \to C_r^*(G)$ is an isomorphism. It is then straightforward to check that this implies that $C^*(G)$ has no nontrivial projections.

Now, Theorem 1.2 says that $C^*(G)$ is primitive. The second assertion follows therefore from Proposition 3.1 in combination with the first assertion and Lemma 3.2. $\square$

To our knowledge, the class of countable discrete groups which are such that their full group C^*-algebras have a *faithful* tracial state has not been much studied. Clearly, it does contain all countable amenable groups (as the full and the reduced

group C^*-algebras agree for such groups, and the canonical tracial state on the reduced algebra is always faithful). Hence, if a group H is nontrivial, icc and amenable, then $C^*(H)$ is primitive (cf. [10, 11]) and Lemma 3.2 may be applied. Our assertion at the beginning of this section follows then from Proposition 3.1. On the other hand, this class also contains all free groups with countably many generators. This fact is due to Choi [3, Corollary 9] and may be put in a somewhat more general framework as follows.

We first recall that a C^*-algebra is called *residually finite-dimensional* (RFD) if it has a separating family of finite-dimensional representations (see e.g. [6]). Clearly, any abelian or finite-dimensional C^*-algebra is RFD. If F is a free group on countably many generators, then $C^*(F)$ is RFD (cf. [3, Theorem 7]). Moreover, the class of RFD C^*-algebras is closed under free products (see [6, Theorem 3.2]). Finally, any unital RFD C^*-algebra has a faithful tracial state (see the proof of [3, Corollary 9]). Hence we get:

Corollary 3.4. *Consider $G = G_1 * G_2$, where at least one of the G_i's has more than two elements, and assume that G_1 (resp. G_2) is abelian or finite. Then $C^*(G)$ is RFD, antiliminary and has an uncountable family of of pairwise inequivalent, irreducible faithful representations.*

Proof. It follows from Theorem 1.2 that $C^*(G)$ is primitive. Moreover, $C^*(G) = C^*(G_1) * C^*(G_2)$ is RFD since $C^*(G_1)$ and $C^*(G_2)$ are RFD. Hence $C^*(G)$ has a faithful tracial state, and the assertion follows from Proposition 3.1 combined with Lemma 3.2. $\square$

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[1] INSTITUTE OF MATHEMATICS, UNIVERSITY OF OSLO, P.B. 1053 BLINDERN, 0316 OSLO, NORWAY.

E-mail address: bedos@math.uio.no

[2] DEPARTMENT OF MATHEMATICAL SCIENCES, NTNU, 7491 TRONDHEIM, NORWAY.

E-mail address: tronanen@math.ntnu.no