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CHARACTERIZATIONS OF INNER PRODUCT SPACES BY STRONGLY CONVEX FUNCTIONS

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ABSTRACT. New characterizations of inner product spaces among normed spaces involving the notion of strong convexity are given. In particular, it is shown that the following conditions are equivalent: (1) $(X, \|\cdot\|)$ is an inner product space; (2) $f : X \to \mathbb{R}$ is strongly convex with modulus $c > 0$ if and only if $f - c\|\cdot\|^2$ is convex; (3) $\|\cdot\|^2$ is strongly convex with modulus 1.

1. INTRODUCTION

It is well known that in a normed space $(X, \|\cdot\|)$ the following Jordan–von Neumann parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in X,$$

holds if and only if the norm $\|\cdot\|$ is derivable from an inner product (cf.[8], [5]). In the literature one can find many other conditions characterizing inner product spaces among normed spaces. A rich collection of such characterizations is contained in the celebrated book of Amir [5] (cf. also [1, Chpt. 11], [2], [3], [4], [6], [11]). The aim of this note is to present some new results of this type involving strongly convex and strongly midconvex functions.

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In what follows $(X, \|\cdot\|)$ is a real normed space, D stands for a convex subset of X and c is a positive constant. A function $f : D \to \mathbb{R}$ is called *strongly convex with modulus* c if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2, \tag{1.1}$$

for all $x, y \in D$ and $t \in (0,1)$. We say that f is *strongly midconvex with modulus* c if (1.1) is assumed only for $t = 1/2$, that is

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2, \qquad x, y \in D. \tag{1.2}$$

Recall also that f is convex (midconvex) if it satisfies (1.1) ((1.2), respectively) with $c = 0$. Strongly convex functions have been introduced by Polyak [10] and they play an important role in optimization theory. Many properties of them can be found, among other, in [7], [9], [12], [13]. The following result gives relationships between strongly convex (strongly midconvex) and convex (midconvex) functions. In the case where $X = \mathbb{R}^n$ the first part of this result can be found in [7, Prop. 1.1.2].

2. Main result

We start this section with a useful lemma.

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a real inner product space, D be a convex subset of X and c be a positive constant.*

1. *A function $f : D \to \mathbb{R}$ is strongly convex with modulus c if and only if the function $g = f - c\|\cdot\|^2$ is convex.*
2. *A function $f : D \to \mathbb{R}$ is strongly midconvex with modulus c if and only if the function $g = f - c\|\cdot\|^2$ is midconvex.*

Proof. 1. Assume that f is strongly convex with modulus c. Using elementary properties of the inner product and the fact that $\|x\|^2 = \langle x|x\rangle$, we get

$$\begin{aligned}
g(tx + (1-t)y) &= f(tx + (1-t)y) - c\|tx + (1-t)y\|^2 \\
&\le tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2 - c\|tx + (1-t)y\|^2 \\
&\le tf(x) + (1-t)f(y) - c\Big(t(1-t)\big(\|x\|^2 - 2\langle x|y\rangle + \|y\|^2\big) \\
&\qquad + t^2\|x\|^2 + 2t(1-t)\langle x|y\rangle + (1-t)^2\|y\|^2\Big) \\
&= tf(x) + (1-t)f(y) - ct\|x\|^2 - c(1-t)\|y\|^2 \\
&= tg(x) + (1-t)g(y),
\end{aligned}$$

which proves that g is convex.
Conversely, if g is convex, then

$$\begin{aligned} f(tx+(1-t)y) &= g(tx+(1-t)y) + c\|tx+(1-t)y\|^2 \\ &\le tg(x)+(1-t)g(y)+c\big(t^2\|x\|^2+2t(1-t)\langle x|y\rangle+(1-t)^2\|y\|^2\big) \\ &= t\left(g(x)+c\|x\|^2\right)+(1-t)\left(g(y)+c\|y\|^2\right) \\ &\quad - ct(1-t)\left(\|x\|^2-2\langle x|y\rangle+\|y\|^2\right) \\ &= f(x)+(1-t)f(y)-ct(1-t)\|x-y\|^2, \end{aligned}$$

which shows that f is strongly convex with modulus c.

2. Assume now that f is strongly midconvex with modulus c. Using the parallelogram law we get

$$\begin{aligned} g\Big(\frac{x+y}{2}\Big) &= f\Big(\frac{x+y}{2}\Big) - c\Big\|\frac{x+y}{2}\Big\|^2 \\ &\le \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2 - \frac{c}{4}\|x+y\|^2 \\ &= \frac{f(x)+f(y)}{2} - \frac{c}{4}\big(2\|x\|^2+2\|y\|^2\big) = \frac{g(x)+g(y)}{2}\ . \end{aligned}$$

Similarly, if g is midconvex, then

$$\begin{aligned} f\Big(\frac{x+y}{2}\Big) &= g\Big(\frac{x+y}{2}\Big) + c\Big\|\frac{x+y}{2}\Big\|^2 \le \frac{g(x)+g(y)}{2} + \frac{c}{4}\|x+y\|^2 \\ &= \frac{g(x)+\|x\|^2}{2} + \frac{g(y)+\|y\|^2}{2} + \frac{c}{4}\big(\|x+y\|^2-2\|x\|^2-2\|y\|^2\big) \\ &= \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2. \end{aligned}$$

□

The following example shows that the assumption that X is an inner product space is essential in the above lemma.

Example 2.2. Let $X=\mathbb{R}^2$ and $\|x\|=|x_1|+|x_2|$, for $x=(x_1,x_2)$. Take $f=\|\cdot\|^2$. Then $g=f-\|\cdot\|^2$ is convex being the zero function. However, f is neither strongly convex nor strongly midconvex with modulus 1. Indeed, for $x=(1,0)$ and $y=(0,1)$ we have

$$f\Big(\frac{x+y}{2}\Big) = 1 > 0 = \frac{f(x)+f(y)}{2} - \frac{1}{4}\|x-y\|^2,$$

which contradicts (1.2).

It appears that something stronger can be proved: the assumption that X is an inner product space is necessary in Lemma 2.1. Namely, the following characterizations of inner product spaces hold.

Theorem 2.3. *Let $(X,\|\cdot\|)$ be a real normed space. The following conditions are equivalent to each other:*

1. *For all $c>0$ and for all functions $f : D\to\mathbb{R}$, f is strongly convex with modulus c if and only if $g=f-c\|\cdot\|^2$ is convex;*

2. *For all $c > 0$ and for all functions $f : D \to \mathbb{R}$, f is strongly midconvex with modulus c if and only if $g = f - c\|\cdot\|^2$ is midconvex;*
3. *There exists $c > 0$ such that, for all functions $f : D \to \mathbb{R}$, g is convex if and only if $f = g + c\|\cdot\|^2$ is strongly convex with modulus c;*
4. *There exists $c > 0$ such that, for all functions $f : D \to \mathbb{R}$, g is midconvex if and only if $f = g + c\|\cdot\|^2$ is strongly midconvex with modulus c;*
5. *$\|\cdot\|^2 : X \to \mathbb{R}$ is strongly convex with modulus 1;*
6. *$\|\cdot\|^2 : X \to \mathbb{R}$ is strongly midconvex with modulus 1;*
7. *$(X, \|\cdot\|)$ is an inner product space.*

Proof. We will show the following chains of implications: *1* $\Rightarrow$ *3* $\Rightarrow$ *5* $\Rightarrow$ *7* $\Rightarrow$ *1* and *2* $\Rightarrow$ *4* $\Rightarrow$ *6* $\Rightarrow$ *7* $\Rightarrow$ *2*.

Implications *1* $\Rightarrow$ *3* and *2* $\Rightarrow$ *4* are obvious. To show *3* $\Rightarrow$ *5* and *4* $\Rightarrow$ *6* take $g = 0$. Then $f = c\|\cdot\|^2$ is strongly convex (resp. strongly midconvex) with modulus c. Consequently, $\frac{1}{c}f = \|\cdot\|^2$ is strongly convex (resp. strongly midconvex) with modulus 1.

To see that *5* $\Rightarrow$ *7* and *6* $\Rightarrow$ *7* also hold, observe that, by the strong convexity or strong midconvexity with modulus 1 of $\|\cdot\|^2$ we have

$$\left\|\frac{x+y}{2}\right\|^2 \le \frac{\|x\|^2 + \|y\|^2}{2} - \frac{1}{4}\|x-y\|^2 ,$$

and hence

$$\|x+y\|^2 + \|x-y\|^2 \le 2\|x\|^2 + 2\|y\|^2 \tag{2.1}$$

for all $x, y \in X$. Now, putting $u = x + y$ and $v = x - y$ in (2.1), we get

$$2\|u\|^2 + 2\|v\|^2 \le \|u+v\|^2 + \|u-v\|^2, \qquad u, v \in X. \tag{2.2}$$

Conditions (2.1) and (2.2) mean that the norm $\|\cdot\|$ satisfies the parallelogram law, which implies that $(X, \|\cdot\|)$ is an inner product space.

Implications *7* $\Rightarrow$ *1* and *7* $\Rightarrow$ *2* follow by Lemma 2.1. $\square$

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